On Some Discrete Subgroups of the Lorentz Group

A.N. Tarakanov *
Institute of Informational Technologies,
Belarusian State University of Informatics and Radioelectronics
Kozlov str. 28, 220037, Minsk, Belarus

Abstract

Some discrete subgroups of the Lorentz group are found using Fedorov's parametrization by means of complex vector-parameter. It is shown that the discrete subgroup of the Lorentz group, which have not fixed points, are contained in boosts along a spatial direction for time-like and space-like vectors and are discrete subgroups of the group $SO(1,1)$, whereas discrete subgroups of isotropic vector are subgroups of $SO(1,1) \times E(1,1)$.

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From the physical point of view discrete subgroups of the Lorentz group arise when one attempts to construct a theory of quantized space-time with some discrete symmetry going over to Lorentz symmetry at continual limit (see, e.g., [1]). Under such discrete transformation the space-time, represented as some 1+3-dimensional lattice, should go over into itself. Thus, the problem is to find these discrete transformations, which, obviously, should belong to discrete subgroup of the Lorentz group. Despite numerous approaches to construction of 1+3-dimensional lattices, this problem remains unresolved till now though there is some advancement in this direction (see, e.g., [2]- [6]). Works [7], [8] should also be noted where some discrete subgroups of the Lorentz group are constructed starting from homomorphism between $SO(1,3)$ and $SL(2,C)$. The invariance principle under such subgroups, which act independently on the particle states with various momenta, allows defining all elements of the S-matrix [9].

*E-mail: tarak-ph@mail.ru

The purpose of this work is to set an example of construction of discrete subgroups of the Lorentz group on the basis of chosen parametrization. For construction of discrete subgroups we will use Fedorov's parametrization of the Lorentz group by means of complex vector-parameter $\mathbf{q} = \mathbf{a} + i\mathbf{b}$ [10]. Let us give the underlying information.

As it is known, the Lorentz group is a group of motions of the Minkowski space $\mathbf{E}^R_{1,3}$. It will be a discrete point group of symmetry if two conditions are satisfied: a) there exists at least one point called singular one, which is invariant under all transformations of group; b) the orbit of any nonsingular point is discrete ([6], p. 94). If $\mathbf{L}(q) \in SO(1,3)$, $\mathrm{x} \in \mathbf{E}^R_{1,3}$, then the first condition implies $\mathbf{L}(q)\mathrm{x} = \mathrm{x}$, which thus selects the little Lorentz group from the whole group. The second condition specifies lattice in the Minkowski space.

The matrix of the Lorentz transformation is given by

$$\mathbf{L}(\mathrm{q}) = \frac{(1+\mathrm{q})(1+\mathrm{q}^*)}{|1+\mathbf{q}^2|} \equiv \mathbf{L}(\mathbf{q}) =$$

$$= \frac{1}{|1+\mathbf{q}^2|}\begin{pmatrix} 1+|\mathbf{q}|^2 & i(\mathbf{q}-\mathbf{q}^*+[\mathbf{q}\mathbf{q}^*]) \\ i(\mathbf{q}-\mathbf{q}^*-[\mathbf{q}\mathbf{q}^*]) & 1-|\mathbf{q}|^2+(\mathbf{q}+\mathbf{q}^*)^\times+\mathbf{q}\circ\mathbf{q}^*+\mathbf{q}^*\circ\mathbf{q} \end{pmatrix}, \quad (1)$$

where 4×4-matrix q has the form

$$\mathrm{q} = \begin{pmatrix} 0 & i\mathbf{q} \\ i\mathbf{q} & \mathbf{q}^\times \end{pmatrix}, \quad \mathrm{q}^* = \begin{pmatrix} 0 & -i\mathbf{q}^* \\ -i\mathbf{q}^* & \mathbf{q}^{*\times} \end{pmatrix}, \quad (2)$$

3×3-matrix $\mathbf{q}^\times$ has components $(\mathbf{q}^\times)_{ij} = \varepsilon_{ijk}q_k$, and the sign $\circ$ implies dyadic product: $(\mathbf{q}\circ\mathbf{q}^*)_{ij} = q_i q_j^*$. Matrix (1) satisfies to condition of pseudo-orthogonality

$$\mathbf{L}(\mathrm{q})\eta\widetilde{\mathbf{L}}(\mathrm{q}) = \eta\,, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \quad (3)$$

The composition law of vector-parameters looks like

$$\mathbf{q}'' = <\mathbf{q},\mathbf{q}'> \equiv \frac{\mathbf{q}+\mathbf{q}'+[\mathbf{q}\mathbf{q}']}{1-\mathbf{q}\mathbf{q}'}\,, \quad (4)$$

and $\mathbf{L}(\mathbf{q}'') = \mathbf{L}(\mathbf{q})\mathbf{L}(\mathbf{q}')$. Let's write down an action of $\mathbf{L}(\mathbf{q})$ on a vector $\mathrm{x} = \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix}$: $\mathrm{x}' = \mathbf{L}(\mathbf{q})\mathrm{x}$, or

$$\begin{cases} {}'x_0 = \dfrac{1}{|1+\mathbf{q}^2|}\{(1+|\mathbf{q}|^2)x^0 + i(\mathbf{q}-\mathbf{q}^*+[\mathbf{q}\mathbf{q}^*])\mathbf{x}\}\,, \\ {}'\mathbf{x} = \dfrac{1}{|1+\mathbf{q}^2|}\{i(\mathbf{q}-\mathbf{q}^*-[\mathbf{q}\mathbf{q}^*])x^0 + (1-|\mathbf{q}|^2)\mathbf{x} + [(\mathbf{q}+\mathbf{q}^*),\mathbf{x}] + \mathbf{q}(\mathbf{q}^*\mathbf{x}) + \mathbf{q}^*(\mathbf{q}\mathbf{x})\}\,. \end{cases} \quad (5)$$

Matrix $\mathbf{L}(\mathbf{q})$ can be represented also as

$$\mathbf{L}(\mathbf{q}) = \frac{1+\boldsymbol{\alpha}}{1-\boldsymbol{\alpha}} = \frac{1+\frac{1}{2}(\mathbf{q}^2+\mathbf{q}^{*2})+2(\boldsymbol{\beta}+\boldsymbol{\beta}^2)}{|1+\mathbf{q}^2|}\,, \quad (6)$$

where $\boldsymbol{\alpha}$ is anti-Hermitian matrix: $\boldsymbol{\alpha}^\dagger = -\boldsymbol{\alpha}$. Its relation with vector-parameter looks as follows:

$$\boldsymbol{\alpha} = \xi \boldsymbol{\beta} + \zeta \boldsymbol{\beta}^3 \;, \tag{7}$$

$$\boldsymbol{\beta} = \frac{1}{2}(\mathbf{q} + \mathbf{q}^*) = \begin{pmatrix} 0 & -\mathbf{b} \\ -\mathbf{b} & \mathbf{a}^\times \end{pmatrix} \;,\; \boldsymbol{\beta}^\dagger = -\boldsymbol{\beta} \;, \tag{8}$$

$$\xi = 1 - \Delta_\beta \frac{\sqrt{(1-\Delta_\beta)^2 - 4|\boldsymbol{\beta}|} - (1-\Delta_\beta)}{2|\boldsymbol{\beta}|} \;,\; \zeta = \frac{\sqrt{(1-\Delta_\beta)^2 - 4|\boldsymbol{\beta}|}}{2|\boldsymbol{\beta}|} \;, \tag{9}$$

where

$$\Delta_\beta = \frac{1}{2}\mathrm{Sp}\boldsymbol{\beta}^2 = -\frac{1}{2}(\mathbf{q}^2 + \mathbf{q}^{*2}) \;,\; |\boldsymbol{\beta}| = \det\boldsymbol{\beta} = \frac{1}{16}(\mathbf{q}^2 - \mathbf{q}^{*2})^2 \;. \tag{10}$$

Further we will need to know the structure of small Lorentz group leaving a vector x fixed: $\mathbf{L}(\mathbf{q})\mathrm{x} = \mathrm{x}$. Here we have three cases that we will consider below.

1. x is *a time-like vector*, $(x^0)^2 - \mathbf{x}^2 > 0$. In this case by means of some transformation $\mathbf{L}(\mathbf{c})$ one may obtain a vector $\overset{\circ}{\mathrm{x}} = (x^0, \mathbf{0})$

$$\overset{\circ}{\mathrm{x}} = \mathbf{L}(\mathbf{c})\mathrm{x} \;, \tag{11}$$

Using properties of the little Lorentz group we obtain from Eq.(11)

$$\overset{\circ}{\mathrm{x}} = \mathbf{L}(\mathbf{c})\mathrm{x} = \mathbf{L}(\mathbf{c})\mathbf{L}(\mathbf{q})\mathrm{x} = \mathbf{L}(\mathbf{c})\mathbf{L}(\mathbf{q})\mathbf{L}(-\mathbf{c})\,\overset{\circ}{\mathrm{x}} = \mathbf{L}(<\mathbf{c},\mathbf{q},-\mathbf{c}>)\,\overset{\circ}{\mathrm{x}} = \mathbf{L}(O(\mathbf{c})\mathbf{q})\,\overset{\circ}{\mathrm{x}} \;, \tag{12}$$

where

$$O(\mathbf{c}) = \mathbf{1} + 2\frac{\mathbf{c}^\times + (\mathbf{c}^\times)^2}{1 + \mathbf{c}^2} \tag{13}$$

is a matrix from the complex rotation group. Thus, the vector $\overset{\circ}{\mathrm{x}}$ will not change under Lorentz transformation $\mathbf{L}(\mathbf{q}')$ with the vector-parameter

$$\mathbf{q}' = O(\mathbf{c})\mathbf{q} \;. \tag{14}$$

The possible structure of the vector $\mathbf{q}'$ is specified from equations (5), where $\mathbf{q}$ and x' should be replaced by $\mathbf{q}'$ and $\overset{\circ}{\mathrm{x}}$, respectively. Then these equations give

$$\overset{\circ}{x}{}^0 = \frac{1 + |\mathbf{q}'|^2}{|1 + \mathbf{q}'^2|} x^0 \;, \tag{15}$$

$$\mathbf{q}' - \mathbf{q}'^* - [\mathbf{q}'\mathbf{q}'^*] = \mathbf{0} \;. \tag{16}$$

Hence it follows $\mathbf{q}' = \mathbf{q}'^*$ and $\mathbf{L}(\mathbf{q}')$ takes the form

$$\mathbf{L}(\mathbf{q}') = \begin{pmatrix} 1 & 0 \\ 0 & O(\mathbf{q}') \end{pmatrix} \;, \tag{17}$$

i.e. it is a 3-dimensional rotation.

Using for $\mathbf{L}(\mathbf{q})$ a representation (6), we obtain for the little group of the vector x

$$\boldsymbol{\alpha}\mathrm{x} = (\xi\boldsymbol{\beta} + \zeta\boldsymbol{\beta}^3)\mathrm{x} = \begin{pmatrix} 0 & -\mathbf{B} \\ -\mathbf{B} & \mathbf{A}^{\times} \end{pmatrix} \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} -\mathbf{B}\mathbf{x} \\ -\mathbf{B}x^0 + [\mathbf{A}\mathbf{x}] \end{pmatrix} = 0 , \tag{18}$$

where

$$\mathbf{A} = (\xi - \zeta a^2 + \zeta b^2)\mathbf{a} - \zeta(\mathbf{ab})\mathbf{b} , \tag{19}$$

$$\mathbf{B} = (\xi - \zeta a^2 + \zeta b^2)\mathbf{b} + \zeta(\mathbf{ab})\mathbf{a} . \tag{20}$$

It follows from Eq.(18) that vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal: $(\mathbf{AB}) = 0$, but it is fulfilled only at $|\boldsymbol{\beta}| = 0$. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}$ and $\mathbf{A} = \mathbf{a}$, $\mathbf{B} = \mathbf{b}$. As it is shown in [5], $|\boldsymbol{\beta}| = 0$ is necessary and sufficient condition for matrix $\mathbf{L}(\mathbf{q})$ to have eigenvalue 1 that takes place for the little Lorentz group. Thus, vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, and vector $\mathbf{b}$ is orthogonal to $\mathbf{x}$

$$(\mathbf{ab}) = 0 , \ (\mathbf{bx}) = 0 . \tag{21}$$

The first condition means the vector $\mathbf{q}$ to be canonical. When the vector $\overset{\circ}{\mathrm{x}}$ is time-like, Eq.(21) is fulfilled automatically due to $\mathbf{b}' = 0$. In general case of time-like vector x a transformation from the little Lorentz group is given by

$$\mathbf{L}(\mathbf{q}) = \mathbf{L}(O(-\mathbf{c})\mathbf{a}') , \tag{22}$$

where $O(-\mathbf{c}) = O^{-1}(\mathbf{c})$ is specified in Eq.(13). The vector $\mathbf{q} = \mathbf{a} + i\mathbf{b}$ satisfies to conditions (21), implying $\mathbf{q}$ to be represented as

$$\mathbf{q} = \mathbf{a} + i\varepsilon[\mathbf{e}_1\mathbf{a}] = \mathbf{a} - \frac{i\mathbf{x}}{x^0}[\mathbf{e}_1\mathbf{a}] , \tag{23}$$

where $\mathbf{e}_1$ is a unit vector in the $\mathbf{x}$-direction: $\mathbf{e}_1 = \frac{\mathbf{x}}{|\mathbf{x}|}$, $\varepsilon = -\frac{|\mathbf{x}|}{x^0}$. The structure of vectors $\mathbf{q}$ and $\mathbf{q}'$ determines the structure of the vector $\mathbf{c}$:

$$\mathbf{c} = \mu(\mathbf{q} + \mathbf{q}') + \frac{2[\mathbf{q}\mathbf{q}']}{(\mathbf{q} + \mathbf{q}')^2} . \tag{24}$$

From $\mathbf{q}^2 = \mathbf{q}'^2$ it follows $(\mathbf{e}_1\mathbf{a})^2 = \mathbf{a}'^2$. Substituting $\mathbf{q}' = \mathbf{a}'$ in Eqs.(23) and (24), we obtain

$$\mathbf{c} = \mu\left(\mathbf{a} + \mathbf{a}' - \frac{i[\mathbf{x}\mathbf{a}]}{x^0}\right) + \frac{[\mathbf{a}\mathbf{a}'] + \frac{i}{x^0}(\mathbf{x}[\mathbf{a}\mathbf{a}'] - \mathbf{a}'(\mathbf{x}\mathbf{a}))}{\mathbf{a}'(\mathbf{a} + \mathbf{a}') - \frac{i(\mathbf{x}[\mathbf{a}\mathbf{a}'])}{x^0}} , \tag{25}$$

where μ is arbitrary complex number.

2. x is *a space-like vector*, $(x^0)^2 - \mathbf{x}^2 < 0$. In this case some transformation $\mathbf{L}(\mathbf{d})$ can give a vector $\overset{\circ}{\mathrm{x}} = (0, \overset{\circ}{\mathbf{x}})$

$$\overset{\circ}{\mathrm{x}} = \mathbf{L}(\mathbf{d})\mathrm{x} . \tag{26}$$

For such $\overset{\circ}{\mathrm{x}}$ we also have relations (12)-(14), where $\mathbf{c}$ is replaced by $\mathbf{d}$. For arbitrary space-like vector x conditions (18), where $\mathbf{A} = \mathbf{a}$ and $\mathbf{B} = \mathbf{b}$, and (21) are fulfilled. For the vector $\overset{\circ}{\mathrm{x}}$ Eqs.(18), (21) transform into

$$[\mathbf{a}' \overset{\circ}{\mathbf{x}}] = \mathbf{0} , \ (\mathbf{a}'\mathbf{b}') = 0 , \ (\mathbf{b}' \overset{\circ}{\mathbf{x}}) = 0 . \tag{27}$$

i.e. vector $\mathbf{a}'$ is parallel to $\mathring{\mathbf{x}}$, and vector $\mathbf{b}'$ is orthogonal to $\mathring{\mathbf{x}}$ and $\mathbf{a}'$. Replacing $\mathbf{q}$ by $\mathbf{q}'$ in Eq.(5), and x$'$ by space-like $\mathring{\mathrm{x}}$, we obtain

$$(\mathbf{q}' - \mathbf{q}'^{*} + [\mathbf{q}'\mathbf{q}'^{*}])\,\mathring{\mathbf{x}} = 0\ . \tag{28}$$

$$\mathring{\mathbf{x}} = \frac{1}{|1+\mathbf{q}'^{2}|}\left\{(1-|\mathbf{q}'|^{2})\,\mathring{\mathbf{x}} + [(\mathbf{q}'+\mathbf{q}'^{*}),\mathring{\mathbf{x}}] + \mathbf{q}'(\mathbf{q}'^{*}\,\mathring{\mathbf{x}}) + \mathbf{q}'^{*}(\mathbf{q}'\,\mathring{\mathbf{x}})\right\}\ , \tag{29}$$

or

$$\left\{\begin{array}{c} (\mathbf{b}'\,\mathring{\mathbf{x}}) - (\mathring{\mathbf{x}}\,[\mathbf{a}'\mathbf{b}']) = 0\ , \\ \mathbf{a}'^{2}\,\mathring{\mathbf{x}} - [\mathbf{a}'\,\mathring{\mathbf{x}}] - \mathbf{a}'(\mathbf{a}'\,\mathring{\mathbf{x}}) - \mathbf{b}'(\mathbf{b}'\,\mathring{\mathbf{x}}) = \mathbf{0}\ . \end{array}\right. \tag{30}$$

The first condition is fulfilled identically, whereas from the second one it follows

$$\mathbf{a}' = a'\frac{\mathring{\mathbf{x}}}{|\,\mathring{\mathbf{x}}\,|} = a'\,\mathring{\mathbf{e}}_1\ \ . \tag{31}$$

Let us choose as unit vectors of the basis the vectors

$$\mathring{\mathbf{e}}_1\ ,\ \mathring{\mathbf{e}} = \frac{1}{\sqrt{2}}(\mathring{\mathbf{e}}_2 + i\,\mathring{\mathbf{e}}_3)\ ,\ \mathring{\mathbf{e}}^{*} = \frac{1}{\sqrt{2}}(\mathring{\mathbf{e}}_2 - i\,\mathring{\mathbf{e}}_3)\ , \tag{32}$$

satisfying to algebra

$$[\mathring{\mathbf{e}}_1,\mathring{\mathbf{e}}] = -i\,\mathring{\mathbf{e}}\ ,\ [\mathring{\mathbf{e}}_1,\mathring{\mathbf{e}}^{*}] = i\,\mathring{\mathbf{e}}^{*}\ ,\ [\mathring{\mathbf{e}},\mathring{\mathbf{e}}^{*}] = -i\,\mathring{\mathbf{e}}_1\ ; \tag{33}$$

$$(\mathring{\mathbf{e}}_1)^2 = 1\ ,\ (\mathring{\mathbf{e}})^2 = (\mathring{\mathbf{e}}^{*})^2 = 0\ ,\ \mathring{\mathbf{e}}\mathring{\mathbf{e}}^{*} = 1\ ; \mathring{\mathbf{e}}_1\mathring{\mathbf{e}} = \mathring{\mathbf{e}}_1\mathring{\mathbf{e}}^{*} = 0\ . \tag{34}$$

Then the vector-parameter $\mathbf{q}'$ can be represented as

$$\mathbf{q}' = a'\,\mathring{\mathbf{e}}_1 + i(\gamma'\,\mathring{\mathbf{e}} + \gamma'^{*}\,\mathring{\mathbf{e}}^{*})\ , a' = a'^{*}\ , \tag{35}$$

and $\mathbf{L}(\mathbf{q}')$ looks like

$$\mathbf{L}(\mathbf{q}') = \frac{1}{1+a'^{2}-2\gamma'\gamma'^{*}}\begin{pmatrix} 1+a'^{2}+2\gamma'\gamma'^{*} & \begin{array}{c} -2\gamma'^{*}(1-ia')\,\mathring{\mathbf{e}}^{*} - \\ -2\gamma'(1-ia')\,\mathring{\mathbf{e}} \end{array} \\ \begin{array}{c} -2\gamma'(1-ia')\,\mathring{\mathbf{e}} - \\ -2\gamma'^{*}(1+ia')\,\mathring{\mathbf{e}}^{*} \end{array} & \begin{array}{c} 1-a'^{2}-2bb^{*}+ \\ +2a'\,\mathring{\mathbf{e}}_1^{\times} + 2a'^{2}\,\mathring{\mathbf{e}}_1\circ\mathring{\mathbf{e}}_1 + \\ +2(\gamma'\,\mathring{\mathbf{e}} + \gamma'^{*}\,\mathring{\mathbf{e}}^{*})\circ(\gamma'\,\mathring{\mathbf{e}} + \gamma'^{*}\,\mathring{\mathbf{e}}^{*}) \end{array} \end{pmatrix}. \tag{36}$$

In the general case of space-like vector x a transformation from the little Lorentz group is given by

$$\mathbf{L}(\mathbf{q}) = \mathbf{L}(O(-\mathbf{d})\mathbf{q}')\ , \tag{37}$$

where vector $\mathbf{q}$, specified in the same way as in time-like case in Eq.(23), can be written down also in the form

$$\mathbf{q} = a\mathbf{e}_1 + \left(1 - \frac{|\mathbf{x}|}{x^0}\right)b\mathbf{e} + \left(1 + \frac{|\mathbf{x}|}{x^0}\right)b^{*}\mathbf{e}^{*}\ , \tag{38}$$

and $\mathbf{e}_1$, $\mathbf{e}$, $\mathbf{e}^*$ satisfy the same relations as Eqs.(33), (34), and $\mathbf{a} = a\mathbf{e}_1 + b\mathbf{e} + b^*\mathbf{e}^*$. The connection between $\mathbf{q}$, $\mathbf{q}'$, and $\mathbf{d}$ looks like

$$
a\mathbf{e}_1 + \left(1 - \frac{|\mathbf{x}|}{x^0}\right) b\mathbf{e} + \left(1 + \frac{|\mathbf{x}|}{x^0}\right) b^*\mathbf{e}^* =
$$

$$
= \left(1 + 2\frac{-\mathbf{d}^\times + \mathbf{d}\circ\mathbf{d} - \mathbf{d}^2}{1+\mathbf{d}^2}\right)\left(a' \overset{\circ}{\mathbf{e}}_1 + i\gamma\, \overset{\circ}{\mathbf{e}} + i\gamma^*\, \overset{\circ}{\mathbf{e}}{}^*\right) , \tag{39}
$$

where it should be keep in mind that $\mathbf{e} = \overset{\circ}{\mathbf{e}}$, $\mathbf{e}^* = \overset{\circ}{\mathbf{e}}{}^*$, and $\overset{\circ}{\mathbf{e}}_1 = \frac{\overset{\circ}{\mathbf{x}}}{|\overset{\circ}{\mathbf{x}}|}$ is determined from Eq.(26), or

$$
\overset{\circ}{\mathbf{x}} = \frac{(\mathbf{d} - \mathbf{d}^* - [\mathbf{d}\mathbf{d}^*])((\mathbf{d} - \mathbf{d}^* + [\mathbf{d}\mathbf{d}^*])\mathbf{x})}{(1+|\mathbf{d}|^2)|1+\mathbf{d}^2|} +
$$

$$
+ \frac{(1-|\mathbf{d}|^2)\mathbf{x} + [(\mathbf{d}+\mathbf{d}^*), \mathbf{x}] + \mathbf{d}(\mathbf{d}^*\mathbf{x}) + \mathbf{d}^*(\mathbf{d}\mathbf{x})}{|1+\mathbf{d}^2|} . \tag{40}
$$

It is seen from here, that sought connection between $\mathbf{q}$, $\mathbf{q}'$ and $\mathbf{d}$ is not so simple as in Eq.(24).

3. x is *isotropic vector*, $(x^0)^2 - \mathbf{x}^2 = 0$. In this case an explicit form of the vector-parameter $\mathbf{q}$ follows from Eq.(23):

$$
\mathbf{q} = \mathbf{a} - i[\mathbf{e}_1\mathbf{a}] . \tag{41}
$$

Hence a transformation from the little Lorentz group takes the form

$$
\mathbf{L}(\mathbf{q}') = \frac{1}{1+(\mathbf{e}_1\mathbf{a})^2}\begin{pmatrix} 1 + 2a^2 - (\mathbf{e}_1\mathbf{a})^2 & -2(a^2\mathbf{e}_1 - \mathbf{a}(\mathbf{e}_1\mathbf{a}) - [\mathbf{e}_1\mathbf{a}]) \\ 2[a^2\mathbf{e}_1 - \mathbf{a}(\mathbf{e}_1\mathbf{a}) + [\mathbf{e}_1\mathbf{a}]] & \begin{matrix} 1 - (\mathbf{e}_1\mathbf{a})^2 + 2\mathbf{a}^\times - 2a^2\mathbf{e}_1\circ\mathbf{e}_1 + \\ +2(\mathbf{e}_1\mathbf{a})(\mathbf{e}_1\circ\mathbf{a} + \mathbf{a}\circ\mathbf{e}_1) \end{matrix} \end{pmatrix} . \tag{42}
$$

If vector $\mathbf{a}$ is orthogonal to $\mathbf{x}$-direction, i.e. $(\mathbf{e}_1\mathbf{a}) = 0$, we have

$$
\mathbf{L}(\mathbf{q}') = \begin{pmatrix} 1 + 2a^2 & -2[a^2\mathbf{e}_1 - [\mathbf{e}_1\mathbf{a}]] \\ 2[a^2\mathbf{e}_1 + [\mathbf{e}_1\mathbf{a}]] & 1 + 2\mathbf{a}^\times - 2a^2\mathbf{e}_1\circ\mathbf{e}_1 \end{pmatrix} . \tag{43}
$$

When $\mathbf{a}$ is parallel to $\mathbf{x}$, then $\mathbf{q} = \mathbf{a}$ and $\mathbf{L}(\mathbf{q}')$ looks like Eq.(17).

Let us consider now discrete Lorentz transformations. Let components of the vector-parameter $\mathbf{q}$ be rational complex numbers, i.e. numbers with real and imaginary parts looking like m/n, where m and n are integers. Then a composition law (4) of two rational vector-parameters generates rational vector-parameters as well. Identity and inverse elements correspond to $\mathbf{q} = \mathbf{0}$ and $\mathbf{q}' = -\mathbf{q}$, respectively. It is obvious they are also rational. Matrix $\mathbf{L}(\mathbf{q})$ specified in Eq.(1) is not rational, for $|1+\mathbf{q}^2| = \sqrt{(1+\mathbf{q}^2)(1+\mathbf{q}^{*2})}$ is irrational in general. Nevertheless its components accept discrete set of values, specified by discreteness of rational values of the vector-parameter $\mathbf{q}$. Thus, rational $\mathbf{q}$'s give discrete subgroups of the Lorentz group. Setting some initial coordinates $\mathrm{x} = (x^0, \mathbf{x})$ which are not necessarily possessing property of rationality, by means of Lorentz transformations we will obtain new coordinates $\mathrm{x}' = \mathbf{L}(\mathbf{q})\mathbf{x}$, and, enumerating all possible rational values of $\mathbf{q}$, we will obtain some discrete set of points. Obviously, we cannot obtain such set if $\mathbf{L}(\mathbf{q})$ belongs to the little Lorentz group leaving points x immovable. Thus, *in order that the discrete subgroup of the Lorentz group did not contain elements which leave vectors immovable, it should not contain discrete subgroups of the little Lorentz group.*

The little Lorentz group is $SO(3)$ for time-like vectors, $SO(1,2)$ for space-time vectors, and a group isomorphic to group $E(2)$ of flat motions, for vector (41) can be represented in the form

$$\mathbf{q} = a_1\mathbf{e}_1 + 2a^*\mathbf{e}^* =< a_1\mathbf{e}_1, \frac{2a^*}{1+ia_1}\mathbf{e}^* > , \tag{44}$$

with

$$\mathbf{a} = a_1\mathbf{e}_1 + a\ \mathbf{e} + a^*\mathbf{e}^*\ ,\ a_1 = a_1^*\ . \tag{45}$$

The vector-parameter $a_1\mathbf{e}_1$ corresponds to rotation on an angle $\varphi = 2\arctan a_1$ round the $\mathbf{x}$-direction, and vector-parameter $\frac{2a^*}{1+ia_1}\mathbf{e}^*$ corresponds to translation in the plane which is orthogonal to $\mathbf{x}$. Hence, subgroups of the Lorentz group not having immovable points are contained in boosts, generating groups $SO(1,1)$, along the $\mathbf{x}$-direction for time-like and space-time vectors. In the case of isotropic vectors such subgroups are contained in the group generated by the vector-parameter

$$\mathbf{q} = ib\mathbf{e}_1 + c\ \mathbf{e} =< ib\mathbf{e}_1, \frac{c}{1+b}\mathbf{e} >\ . \tag{46}$$

The vector-parameter $ib\mathbf{e}_1$ corresponds to boosts, generating group $SO(1,1)$, in the $\mathbf{x}$-direction, and vector $\frac{c}{1+b}\mathbf{e}$ gives rise simultaneously to (various) dilatations of temporal coordinate and vector $\mathbf{x}$ and translations in the plane which is orthogonal to $\mathbf{x}$. Denoting this group as $E(1,1)$ one may assert that discrete subgroups of isotropic vector are subgroups of the group $SO(1,1) \times E(1,1)$, where symbol $\times$ means semidirect product.

Let us consider boosts $SO(1,1)$ in the $\mathbf{x}$-direction, which are inherent to all kinds of vectors and specified by the vector-parameter

$$\mathbf{q} = ib\mathbf{e}_1\ . \tag{47}$$

A composition law (4) reduces to composition of parameter b:

$$b'' = \frac{b+b'}{1+bb'}\ . \tag{48}$$

Here one can see at least two types of discrete subgroups.

1) b is rational number, $b = m/n$;

2) $b = \tanh(\mu r)$, where $r = m/n$ is an integer or rational number, which composition law is trivial: $r'' = r + r'$; $0 < \mu < \infty$ is a fixed real number, determining continuum of discrete subgroups of such kind. Here the subgroup is extracted corresponding to integer r.

For the group $E(1,1)$ specified by the vector-parameter

$$\mathbf{q} = d\mathbf{e} = \frac{c}{1+b}\mathbf{e}\ , \tag{49}$$

a composition law (4) reduces to

$$d'' = \frac{d + d'}{1 - dd'} \ , \ \text{ or } \ \frac{c''}{1 + b''} = \frac{c(1 + b') + c'(1 + b)}{(1 + b)(1 + b') - cc'} \ . \tag{50}$$

Here we also obtain two two types of discrete subgroups.

1) d is rational number, $d = m/n$;

2) $d = \tan(\mu r)$, where $r = m/n$ is an integer or rational number, which composition law is trivial: $r'' = r + r'$; $0 < \mu < \infty$ is a fixed real number, determining continuum of discrete subgroups of such kind. Here the subgroup is also extracted corresponding to integer r.

For the group $SO(1,1) \times E(1,1)$, taking into account the law (48) in Eq.(50), we obtain a composition law for c:

$$c'' = \frac{c(1 + b') + c'(1 + b)}{(1 + bb')\left[1 - \frac{cc'}{(1+b)(1+b')}\right]} \ . \tag{51}$$

From here it follows

1) if $b = m/n$, then c is also rational: $c = p/q$, with $d = \frac{k}{l} = \frac{pn}{q(m+n)}$ and

$$\frac{p''}{q''} = \frac{[pq'(n' + m') + p'q(n + m)](n + m)(n' + m')}{[qq'(n + m)(n' + m') - nn'pp'](nn' + mm')} \ ; \tag{52}$$

2) if $b = \tanh(\mu m/n)$, $d = \tanh(\nu k/l)$, then $c = \tanh(\nu k/l)[1 + \tanh(\mu m/n)]$.

In conclusion it should be noted that the Lorentz group may be parametrized by various ways, which determine range of parameters. Hence, discrete subgroups may be obtained in various parametrizations. A question about whether arbitrary parametrization admits existence of discrete subgroups of the Lorentz group, by now remains opened.

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