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4 authors, including:


Frank Werner
Otto-von-Guericke-Universität Magdeburg


Tsung-Chyan Lai
Harbin Engineering University

Measures of problem uncertainty for total weighted flow time scheduling on a single machine

Yu.N. Sotskov[a], **F. Werner**[b1] , **T.-C. Lai**[c], **N.G. Egorova**[a]

[a] *United Institute of Informatics Problems, The National Academy of Sciences of Belarus, Minsk, Belarus*
[b] *Otto-von-Guericke-University, Faculty of Mathematics, Magdeburg, Germany*
[c] *National Taiwan University, Department of Business Administration, Taipei, Taiwan*

A R T I C L E   I N F O

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A B S T R A C T

A single-machine scheduling problem is investigated under the assumption that the processing time of a job can take any real value from a given closed interval. The criterion is to minimize the total weighted completion time for a set of given jobs. As a measure of uncertainty for such a scheduling problem, it is reasonable to consider the cardinality of a minimal dominant set of job permutations containing an optimal permutation for each possible realization of the job processing times. We show that a minimal dominant set may be uniquely determined and demonstrate how to select a suitable solution method for the individual problem using the value of an uncertainty measure.

1 Introduction

In a real-life scheduling problem, a job processing time may remain uncertain till job completion. Due to this reason, it is often not possible to implement a *deterministic method* [19, 31] for constructing a practically useful schedule which is really optimal or rather close to a really optimal schedule.

In the OR literature, there are presented several methods for solving scheduling problems involving some forms of uncertainty [11, 19, 23, 28]. In the well-developed *stochastic method* [1, 19, 20, 33], it is assumed that numerical parameters are random variables with given probability distributions. If the processing times or other numerical parameters may be defined as fuzzy numbers (i.e., as fuzzy sets with known membership functions), *a method based on fuzzy logic* is used for scheduling under fuzziness [4, 6, 11, 30, 34].

If the processing times can be represented neither as random variables with known probability distributions nor as fuzzy numbers, other methods are needed to solve a scheduling problem under uncertainty [3, 21]. In particular, the *robust method* [6, 11, 34] assumes that the decision-maker prefers a schedule hedging against the worst-case scenario among all the possible realizations of the job processing times. The *stability method* [12, 13, 14, 16, 22, 26] combines a stability analysis with a multi-stage scheduling decision framework on the basis of the information obtained while some jobs have been completed.

Each of the above methods has a specific field for implementation, a method suitable for one class of problems may fail for another one. A decision on selecting a method for solving a concrete scheduling problem influences both the quality of a schedule obtained and the time needed for constructing a practically appropriate schedule.

In this paper, we propose to use a measure of problem uncertainty for selecting a suitable

[1]Corresponding author.
E-mail address: frank.werner@ovgu.de (F. Werner).

solution method. As an example of an uncertain scheduling problem, we consider the problem of minimizing total weighted flow time on a single machine with interval job processing times. A measure of problem uncertainty is based on the cardinality of a minimal dominant set of job permutations. For any fixed scenario, a minimal dominant set contains at least one permutation which is optimal. We show that a minimal dominant set may be uniquely determined and demonstrate how to select suitable solution methods using the measure of problem uncertainty.

2 Problem setting and preliminaries

A set of n jobs $\mathcal{J} = \{J_1, J_2, ..., J_n\}$, $n \geq 2$, has to be processed on a single machine, a weight $w_i > 0$ being given for each job $J_i \in \mathcal{J}$. The processing time p_i of a job J_i can take any real value between a lower bound $p_i^L \geq 0$ and an upper bound $p_i^U \geq p_i^L$, both real bounds being given before scheduling. It is assumed that the exact value p_i of the job processing time may remain unknown until the completion of job J_i. This assumption is rather realistic for a real-life scheduling problem since at least rough bounds p_i^L and p_i^U may be defined for any practical job $J_i \in \mathcal{J}$ (indeed, for the most uncertain job $J_i \in \mathcal{J}$, one can let $p_i^L = 0$ and p_i^U to be equal to the length of the planning horizon).

Let T denote the set of all vectors $p = (p_1, p_2, \dots, p_n)$ of the possible job processing times (scenarios)

$$T = \{p \mid p \in R_+^n,\ p_i^L \leq p_i \leq p_i^U,\ i \in \{1, 2, \dots, n\}\}, \tag{1}$$

and let $S = \{\pi_1, \pi_2, \dots, \pi_{n!}\}$ denote the set of all permutations $\pi_k = (J_{k_1}, J_{k_2}, \dots, J_{k_n})$ of the jobs $\mathcal{J} = \{J_{k_1}, J_{k_2}, \dots, J_{k_n}\} = \{J_1, J_2, ..., J_n\}$.

If one fixes both the permutation π_k of the jobs $\mathcal{J}$ and the scenario $p \in T$, then it is easy to determine the completion time $C_i = C_i(\pi_k, p)$ of job $J_i \in \mathcal{J}$ in a semi-active schedule defined by permutation π_k. (A schedule is called semi-active if no job $J_i \in \mathcal{J}$ can start earlier without delaying the completion time of another job from set $\mathcal{J}$ and without altering the processing permutation of the jobs $\mathcal{J}$ [19].)

The criterion $\sum w_i C_i$ under consideration is the minimization of the sum of the weighted completion times:

$$\sum_{J_i \in \mathcal{J}} w_i C_i(\pi_t, p) = \min_{\pi_k \in S} \left\{ \sum_{J_i \in \mathcal{J}} w_i C_i(\pi_k, p) \right\},$$

where permutation $\pi_t = (J_{t_1}, J_{t_2}, \dots, J_{t_n}) \in S$ is optimal. By adopting the three-field notation $\alpha|\beta|\gamma$ from [7], the above scheduling problem is denoted as $1|p_i^L \leq p_i \leq p_i^U| \sum w_i C_i$.

Since the scenario $p \in T$ may remain unknown before the completion of the jobs $\mathcal{J}$, the completion time C_i of each job $J_i \in \mathcal{J}$ cannot be exactly calculated before the time of the schedule realization. Therefore, problem $1|p_i^L \leq p_i \leq p_i^U| \sum w_i C_i$ of finding an optimal permutation (an optimal semi-active schedule) is not mathematically correct: The value of the objective function $\gamma = \sum_{J_i \in \mathcal{J}} w_i C_i(\pi_k, p)$ for a permutation $\pi_k \in S$ may remain uncertain until the time of the schedule realization.

If a scenario $p \in T$ is fixed before scheduling (i.e., equalities $p_i^L = p_i^U = p_i$ hold and segment $[p_i^L, p_i^U]$ is degenerated into one point $p_i \in [p_i, p_i]$ for each job J_i, $i \in \{1, 2, \dots, n\}$), then problem $1|p_i^L \leq p_i \leq p_i^U| \sum w_i C_i$ reduces to the classical problem $1||\sum w_i C_i$, i.e., to that with deterministic job processing times. The latter problem is mathematically correct and can be solved in $O(n \log n)$ time due to Smith [24].

In what follows, problem $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$ with the objective function $\gamma = f(C_1, C_2, \dots, C_n)$ is called an uncertain problem in contrast to its deterministic counterpart, problem $\alpha||\gamma$, which is called a deterministic problem.

Note that there is a principal difference between a deterministic problem and its uncertain counterpart. While for solving any individual deterministic problem (any instance) of the same mass problem $\alpha||\gamma$, one can apply the same appropriate method and the optimal solution will be definitely obtained, any two individual uncertain problems may need different methods to be solved well. First, there may be no information available for implementing this or that method. Second, a method may be not efficient or not effective for a concrete instance of the uncertain problem. Furthermore, it is easy to convince that, while considering a multi-stage scheduling framework (like in the stability method), the same individual uncertain problem may need a different method at different stages of the decision-making. At least, the need of using different methods may follow from changing available information while some jobs have been completed.

The next section contains a brief survey of methods available for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ under consideration.

3 Literature review

While an optimal sequencing rule for the deterministic problem $1||\sum w_i C_i$ has been known since 1956 [24], its uncertain counterpart $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ continues to attract the attention of the researchers who develop different approaches for correcting the uncertain problem and apply appropriate methods for solving it in this or that sense (see [4–7, 11, 17, 18] among others).

3.1 Robust method

For problem $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$, there usually does not exist a permutation of the jobs $\mathcal{J}$ that remains optimal for all scenarios of set T. Therefore, an additional criterion may be introduced for problem $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$, e.g., a robust schedule minimizing the worst-case deviation from optimality was introduced in [6, 11] to hedge against data uncertainty.

In a robust approach, set T could contain a continuum of scenarios (i.e., T is defined according to (1)) or this set could contain a finite number of scenarios as follows:

$$T = \{p^j = (p_1^j, p_2^j, \dots, p_n^j) \mid p^j \in R_+^n, j \in \{1, 2, \dots, m\}\}.$$

For a scenario $p \in T$, let γ_p^t denote the optimal value of the objective function $\gamma = f(C_1, C_2, \dots, C_n)$ for problem $\alpha||\gamma$ with the fixed scenario p. Let

$$f(C_1(\pi_t, p), \dots, C_n(\pi_t, p)) = \gamma_p^t = \min_{\pi_k \in S} \gamma_p^k = \min_{\pi_k \in S} f(C_1(\pi_k, p), \dots, C_n(\pi_k, p)),$$

where permutation $\pi_t \in S$ is optimal. For permutation $\pi_k \in S$ and any scenario $p \in T$, the difference $\gamma_p^k - \gamma_p^t = r(\pi_k, p)$ is called the regret for permutation π_k with the objective function value equal to γ_p^k. The value $Z(\pi_k) = \max\{r(\pi_k, p) \mid p \in T\}$ is called the worst-case absolute regret. The worst-case relative regret is defined as follows:

$$Z'(\pi_k) = \max\left\{\frac{r(\pi_k, p)}{\gamma_p^t} \mid p \in T\right\}$$

provided that $\gamma_p^t \neq 0$.

In [6, 34], the problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$ of minimizing the total completion time (when $w_i = 1$ for each job $J_i \in \mathcal{J}$) has been considered. For a given specific scenario $p^j \in T$, the deterministic problem $1||\sum C_i$ arises which can be solved using the shortest processing time (SPT) rule [24]: Process the jobs of set $\mathcal{J}$ in non-decreasing order of their processing times $p_i^j,\ J_i \in \mathcal{J}$.

While the deterministic problem $1||\sum C_i$ is solvable in $O(n \log n)$ time, finding a permutation $\pi_t \in S$ of minimizing either the worst-case absolute regret $Z(\pi_t)$ or the worst-case relative regret $Z'(\pi_k)$ for the uncertain counterpart $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$ is binary NP-hard even for two scenarios, $m = 2$. The NP-hardness proofs have been published in [6] and [34], respectively.

In [10], a 2-approximation algorithm has been developed to minimize the worst-case regret for problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$. In [6, 17, 34], exact and heuristic algorithms have been tested to minimize the worst-case regret for problem $1|p_i^L \leq p_i \leq p_i^U|\sum C_i$.

Only a few very special cases [9] are known to be polynomially solvable for minimizing the worst-case regret for the problems $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$.

3.2 Stochastic method

The second part of monograph [19] is totally devoted to the stochastic method, allowing the scheduler to minimize the mathematical expectation $\gamma = E(f(C_1, C_2, \ldots, C_n))$ of the objective function $f(C_1, C_2, \ldots, C_n)$ for processing jobs with random processing time provided that the probability distributions of all random values are known before scheduling.

In particular, a *stochastically optimal schedule* for problem $1|p_i^L \leq p_i \leq p_i^U|\gamma$ with the criterion $\gamma = E(\sum w_i C_i)$ in the class of *non-preemptive static list policies* [19] (i.e., a schedule minimizing the expected sum of the weighted completion times provided that the jobs are ordered at time zero according to a chosen priority list) may be constructed by the weighted shortest processing time (WSEPT) rule: Process the jobs in non-increasing order of the ratios

$$\frac{w_i}{E(p_i)},$$

where $E(p_i)$ is the expected value of the random processing time p_i (see [33] or page 232 in [19]).

3.3 Fuzzy method

If it is possible to present the processing times of the jobs $\mathcal{J}$ as fuzzy numbers with known membership functions, then the fuzzy method is available for solving an uncertain scheduling problem. The book [23] is devoted to the fuzzy method used in a scheduling environment.

In [8], the authors proposed an n-job, one machine scheduling model, where the due-dates for the jobs are fuzzy random variables. In this model, job processing times are script, and one assigns satisfaction levels to the job completion times according to membership functions. The membership functions are non-increasing, and their support positions depend on the expected due-dates, which are exponentially distributed random variables. The fuzzy algorithm from [32] is applicable to this scheduling model as well.

3.4 Stability method

If no additional criterion is introduced (contrary to the robust method), there is no information on the probability distributions of the random processing times (contrary to the stochastic method) and there are no membership functions for non-script processing times (contrary to the fuzzy method), then the stability method [12, 14, 22] may be used for solving problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ either exactly or heuristically. This method combines a stability analysis [15, 22, 25, 27, 29], a multi-stage scheduling decision framework (the off-line planning stage and the on-line scheduling stages) [16, 22], and the solution concept of a minimal dominant set [12, 13, 14, 22, 26].

Definition 1 *A set of permutations $S(T) \subseteq S$ is a minimal dominant set for problem $\alpha|p_i^L \leq p_i \leq p_i^U|\gamma$, if for any fixed scenario $p \in T$, set $S(T)$ contains at least one permutation (semi-active schedule), which is optimal for the deterministic counterpart $\alpha||\gamma$ associated with scenario p, this property being lost for any proper subset of set $S(T)$.*

Due to Definition 1, set $S(T)$ is a minimal dominant set of job permutations with respect to inclusion. The set $S(T)$ has been investigated in [2, 12, 14, 22] for the makespan criterion, and in [2, 13, 22] for the total completion time criterion.

Before introducing a measure of uncertainty for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$ under consideration, we recall known results for this uncertain problem along with those for its deterministic counterpart $1||\sum w_iC_i$.

In [24], it was proven that problem $1||\sum w_iC_i$ can be solved in $O(n \log n)$ time due to the sufficient condition for the optimality of permutation $\pi_k = (J_{k_1}, J_{k_2}, \dots, J_{k_n}) \in S$ as follows:

$$\frac{w_{k_1}}{p_{k_1}} \geq \frac{w_{k_2}}{p_{k_2}} \geq \dots \geq \frac{w_{k_n}}{p_{k_n}}, \tag{2}$$

where inequality $p_{k_i} > 0$ must hold for each job $J_{k_i} \in \mathcal{J}$. Due to the sufficient condition (2), problem $1||\sum w_iC_i$ can be solved to optimality by the weighted shortest processing time (WSPT) rule: Process the jobs in non-increasing order of their weight-to-process ratios

$$\frac{w_{k_i}}{p_{k_i}}.$$

The system of inequalities (2) provides also a necessary condition for the optimality of permutation $\pi_k \in S$ (see [5]).

Theorem 1 [5, 24] *Permutation $\pi_k = (J_{k_1}, J_{k_2}, \dots, J_{k_n}) \in S$ is optimal for the deterministic problem $1||\sum w_iC_i$ if and only if inequalities (2) hold.*

A minimal dominant set $S(T)$ may be determined by using a dominance relation on the set $\mathcal{J}$.

Definition 2 *Job J_u dominates job J_v with respect to T (it is denoted by $J_u \mapsto J_v$), if there exists a minimal dominant set $S(T)$ for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$ such that job J_u precedes job J_v in each permutation of the set $S(T)$.*

Note that a minimal dominant set constructed for problem $1||\sum w_iC_i$ associated with a scenario $p \in T$ is a singleton, $S(T) = \{\pi_k\}$, where $T = \{p\}$.

The following claim has been proven in [26].

Theorem 2 [26] *For problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$, job J_u dominates job J_v with respect to T if and only if*

$$\frac{w_u}{p_u^U} \geq \frac{w_v}{p_v^L}. \tag{3}$$

In the simplest case of an uncertain problem, a minimal dominant set for an individual problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$ may be a singleton, $\{\pi_k\} = S(T)$, which is also a solution to the deterministic counterpart $1||\sum w_iC_i$ associated with any scenario $p \in T$.

Theorem 3 [26] *For the existence of a dominant singleton $S(T) = \{\pi_k\} = \{(J_{k_1}, J_{k_2}, \dots, J_{k_n})\}$ for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$, inequalities (4) are necessary and sufficient:*

$$\frac{w_{k_1}}{p_{k_1}^U} \geq \frac{w_{k_2}}{p_{k_2}^L}; \quad \frac{w_{k_2}}{p_{k_2}^U} \geq \frac{w_{k_3}}{p_{k_3}^L}; \quad \dots; \quad \frac{w_{k_{n-1}}}{p_{k_{n-1}}^U} \geq \frac{w_{k_n}}{p_{k_n}^L}. \tag{4}$$

4 Maximal cardinality $|S(T)| = n!$ of a minimal dominant set

The cardinality of a minimal dominant set $S(T)$ may be used to measure the uncertainty of the individual problem (the instance) $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$. In particular, equality $|S(T)| = 1$ (see Theorem 3) means that there is no uncertainty for the instance of problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ at all. Indeed, such an instance may be solved exactly similarly as its deterministic counterpart $1||\sum w_iC_i$. If $S(T) = \{\pi_k\}$, then permutation π_k provides an exact solution to the instance $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ since this permutation is optimal for any instance $1||\sum w_iC_i$ with every scenario $p \in T$. Such a permutation π_k may be constructed in $O(n \log n)$ time using the constructive proof of Theorem 3 given in [26].

On the other hand, the most difficult case of problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ (in the sense of problem uncertainty) is that with $|S(T)| = n!$. A theorem characterizing the most uncertain individual problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ uses the following notations.

Let $a = \min\left\{\frac{w_i}{p_i^U} \mid J_i \in \mathcal{J}\right\}$ and $b = \max\left\{\frac{w_i}{p_i^L} \mid J_i \in \mathcal{J}\right\}$. Moreover, let $r \in [a, b]$ be a real number. The following subsets $\mathcal{J}_r$ of set $\mathcal{J}$ are crucial for establishing the uniqueness of a minimal dominant set and for a criterion for its largest cardinality $|S(T)| = n!$:

$$\mathcal{J}_r = \left\{J_i \in \mathcal{J} \mid r = \frac{w_i}{p_i^U} = \frac{w_i}{p_i^L}\right\}. \quad (5)$$

In the following auxiliary claim, we use the notation $1|p|\sum w_iC_i$ for indicating an instance (an individual problem) of the mass deterministic problem $1||\sum w_iC_i$ associated with a specific scenario p.

Lemma 1 *In any optimal permutation for the instance $1|p|\sum w_iC_i$, job J_u precedes job J_v if and only if inequality*

$$\frac{w_u}{p_u} > \frac{w_v}{p_v} \quad (6)$$

holds.

Proof. *Sufficiency.* By contradiction, we assume that there exists an optimal permutation $\pi_m \in S$ for the instance $1|p|\sum w_iC_i$ such that inequality (6) holds, however, job J_u follows job J_v in permutation π_m. Since the necessity of condition (2) given in Theorem 1 implies the inequalities

$$\frac{w_v}{p_v} \ge \frac{w_{v+1}}{p_{v+1}} \ge \ldots \ge \frac{w_u}{p_u}$$

(where $v < u$), we obtain inequality

$$\frac{w_v}{p_v} \ge \frac{w_u}{p_u}$$

contradicting inequality (6). Sufficiency has been proven.

Necessity. Let job J_u precede job J_v in all optimal permutations for the instance $1|p|\sum w_iC_i$. By contradiction we assume

$$\frac{w_u}{p_u} \le \frac{w_v}{p_v}.$$

Thus, due to Theorem 1, job J_v must precede job J_u in any optimal permutation existing for the instance $1|p|\sum w_iC_i$. This contradiction shows that our assumption is wrong which completes the proof. ■

Theorem 4 *Assume that there does not exist a real $r \in [a, b]$ such that $|\mathcal{J}_r| \ge 2$. For the existence of a minimal dominant set $S(T)$ for the instance $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ attaining* *the maximum cardinality $|S(T)| = n!$, it is necessary and sufficient that the following inequality holds:*

$$\max\left\{\frac{w_i}{p_i^U} \mid J_i \in \mathcal{J}\right\} < \min\left\{\frac{w_i}{p_i^L} \mid J_i \in \mathcal{J}\right\}. \tag{7}$$

Proof. *Sufficiency.* Let inequality (7) hold: $a < b$.

We choose any permutation $\pi_k = (J_{k_1}, J_{k_2}, \ldots, J_{k_n}) \in S$ of the n jobs and show that this permutation has to belong to any minimal dominant set $S(T)$ constructed for the instance $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ under consideration.

To this end, we show that there exists a vector $p^* = (p_1^*, p_2^*, \ldots, p_n^*) \in T$ such that the following inequalities hold:

$$\frac{w_{k_1}}{p_{k_1}^*} > \frac{w_{k_2}}{p_{k_2}^*} > \ldots > \frac{w_{k_n}}{p_{k_n}^*}. \tag{8}$$

Indeed, due to the strict inequality (7), the length of the segment $[a,\ b]$, $a < b$, is strictly positive, and moreover, the segment $[a,\ b]$ is equal to the intersection of the n segments

$$\left[\frac{w_i}{p_i^U},\ \frac{w_i}{p_i^L}\right],\ i \in \{1, 2, \ldots, n\},\ \text{namely:}\quad [a,\ b] = \bigcap_{i=1}^{n}\left[\frac{w_i}{p_i^U},\ \frac{w_i}{p_i^L}\right].$$

Therefore, since the segment $[a,\ b]$, $a < b$, of non-negative real numbers is everywhere dense, it is possible to find n real numbers $p_{k_i}^*$, $J_{k_i} \in \mathcal{J}$, which satisfy all inequalities (8). Thus, due to Lemma 1, in any optimal permutation for the deterministic problem $1||\sum w_iC_i$ associated with the vector p^* of the job processing times, job J_{k_1} must precede job J_{k_2}, job J_{k_2} must precede job J_{k_3}, and so on, job $J_{k_{n-1}}$ must precede job J_{k_n}.

Thus, permutation π_k is the unique optimal permutation for the problem $1||\sum w_iC_i$ associated with the vector p^* of the job processing times. Hence, according to Definition 1, any minimal dominant set $S(T)$ constructed for the uncertain problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ necessarily contains permutation π_k.

Since permutation π_k has been chosen arbitrarily in set S, any minimal dominant set $S(T)$ contains all permutations of set S, i.e., set $S(T)$ coincides with the whole set S. As a result, we obtain the equalities $|S(T)| = |S| = n!$. Sufficiency has been proven.

Necessity. Let $|S(T)| = n!$. Hence $|S(T)| = |S|$.

We have to show that inequality (7) holds.

By contradiction, we assume that inequality (7) does not hold. Hence, there exist at least two jobs $J_u \in \mathcal{J}$ and $J_v \in \mathcal{J}$ such that inequality

$$\frac{w_u}{p_u^U} \ge \frac{w_v}{p_v^L}$$

holds. Then, from the sufficiency of condition (3) in Theorem 2, it follows that job J_u dominates job J_v with respect to T, i.e., there exists a minimal dominant set $S'(T)$ for problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ such that all permutations in set $S'(T)$ look like

$$(\ldots, J_u, \ldots, J_v, \ldots) \text{ or } (\ldots, J_u, J_v, \ldots).$$

We obtain a contradiction: The above set $S(T) = S$ is not a minimal dominant set for the uncertain problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ under consideration since $S(T)$ is not a minimal set with respect to inclusion (it is possible to remove a permutation $\pi_k \in S \setminus S'(T) \ne \emptyset$ from set $S(T) = S$, and the remaining proper subset $S(T) \setminus \{\pi_k\}$ still satisfies Definition 1).

Theorem 4 has been proven. ∎

The above proof of Theorem 4 implies the following claim.

Corollary 1 *Assume that there does not exist a real $r \in [a,b]$ such that $|\mathcal{J}_r| \geq 2$. For any permutation $\pi_k \in S$ there exists a vector $p \in T$ such that permutation π_k is the unique optimal permutation for problem $1||\sum w_i C_i$ associated with the vector p of the job processing times if and only if inequality (7) holds.*

Proof. The proof of the *sufficiency* of condition (7) in Corollary 1 is entirely contained in that of Theorem 4.

For the proof of the *necessity* of condition (7) in Corollary 1, we note that, if for any permutation $\pi_k \in S$, there exists a vector $p \in T$ such that permutation π_k is the unique optimal permutation for the deterministic problem $1||\sum w_i C_i$ associated with the vector p of the job processing times, then (due to Definition 1) equality $S(T) = S$ holds for any minimal dominant set $S(T)$ constructed for the uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$.

Hence, $|S(T)| = n!$ and the necessity of condition (7) in Corollary 1 follows from the necessity of condition (7) in Theorem 4. ∎

It takes $O(n)$ time to check condition (7) of Theorem 4. Indeed, $O(n)$ is the complexity of finding a maximum in the set

$$\left\{\frac{w_i}{p_i^U} \mid J_i \in \mathcal{J}\right\}$$

of real numbers and a minimum in the set

$$\left\{\frac{w_i}{p_i^L} \mid J_i \in \mathcal{J}\right\}$$

of real numbers.

5 The uniqueness of a minimal dominant set

Let Z denote the set of all individual problems $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$. We consider a mapping $\varphi : Z \to R_+$, where

$$\varphi(z) = 1 - \frac{n! - |S(T)|}{n! - 1}, \quad z \in Z. \tag{9}$$

We remind that the value $\varphi(z)$ may be called a measure (in the mathematical sense) for the individual problem $z \in Z$, if the mapping $\varphi : Z \to R_+$ is an additive non-negative function.

It is clear that inequality $\varphi(z) \geq 0$ must hold for any problem $z \in Z$. The additivity of mapping $\varphi : Z \to R_+$ will be substituted by setting that the measure of uncertainty of the set of all problems $z_k \in Z$ of the sequence

$$z_1 \in Z,\ z_2 \in Z,\ \ldots,\ z_k \in Z, \ldots \tag{10}$$

is equal to

$$\sum_{i=1}^{\infty} \varphi(z_i).$$

Next, we shall show how to get the uniqueness of the mapping $\varphi : Z \to R_+$.

For the extreme cases characterized by Theorem 3 for the simplest case, $|S(T)| = 1$, and by Theorem 4 for the hardest case, $|S(T)| = n!$, mapping $\varphi : Z \to R_+$ is clearly unique.

In the general case, a mapping $\varphi : Z \to R_+$ will be unique if and only if the minimal dominant set for an individual problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ will be unique. Before presenting the criterion for the uniqueness of a set $S(T)$, we prove two simple lemmas about the deterministic problem $1||\sum w_i C_i$.

Lemma 2 *For the instance $1|p|\sum w_iC_i$, there exist both an optimal permutation with job J_u preceding job J_v and an optimal permutation with job J_v preceding job J_u if and only if equality*

$$\frac{w_u}{p_u} = \frac{w_v}{p_v}. \tag{11}$$

holds.

Proof. *Sufficiency.* Since set S is finite, there exists a permutation π_l of the form $\pi_l = (\ldots, J_u, \ldots, J_v, \ldots) \in S$ or permutation π_m of the form $\pi_m = (\ldots, J_v, \ldots, J_u, \ldots) \in S$ which is optimal for the instance $1|p|\sum w_iC_i$.

Since equality (11) holds, a part of the necessary and sufficient condition (2) of the optimality of permutation π_l looks as follows:

$$\ldots \geq \frac{w_u}{p_u} = \ldots = \frac{w_v}{p_v} \geq \ldots, \tag{12}$$

and that of permutation π_m looks as follows:

$$\ldots \geq \frac{w_v}{p_v} = \ldots = \frac{w_u}{p_u} \geq \ldots. \tag{13}$$

If condition (12) holds, then condition (13) also holds, and vice versa.

Due to Theorem 1, for the instance $1|p|\sum w_iC_i$, there exist both an optimal permutation of the form $\pi_l = (\ldots, J_u, \ldots, J_v, \ldots)$ and one of the form $\pi_m = (\ldots, J_v, \ldots, J_u, \ldots)$. Sufficiency has been proven.

Necessity. We assume that there exist both an optimal permutation of the form π_l and one of the form π_m for the instance $1|p|\sum w_iC_i$.

Due to the necessity of condition (2) for the permutation optimality (Theorem 1), this is possible only if equality (11) holds. ■

The following claim directly follows from Lemma 2.

Lemma 3 *For the instance $1|p|\sum w_iC_i$, an optimal permutation is unique if and only if for any pair of jobs $J_u \in \mathcal{J}$ and $J_v \in \mathcal{J}$, equality (11) does not hold.*

Next, we prove a criterion for the uniqueness of a minimal dominant set $S(T)$.

Theorem 5 *For problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$, a minimal dominant set $S(T)$ is unique if and only if there is no a real number $r \in [a, b]$ such that $|\mathcal{J}_r| \geq 2$.*

Proof. For any pair of jobs $J_t \in \mathcal{J}$ and $J_v \in \mathcal{J}$, we shall examine all possible relative arrangements of the segment

$$\left[\frac{w_t}{p_t^U}, \frac{w_t}{p_t^L}\right]$$

and the segment

$$\left[\frac{w_v}{p_v^U}, \frac{w_v}{p_v^L}\right].$$

W.l.o.g. we can assume that inequality

$$\frac{w_v}{p_v^U} \leq \frac{w_t}{p_t^U}$$

holds. Due to the symmetry of jobs J_t and J_v, it is sufficient to examine the following nine possible cases *(a)–(i)*.

Case *(a)*:

$$\frac{w_t}{p_t^U} < \frac{w_t}{p_t^L}, \ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L}, \ \frac{w_v}{p_v^L} < \frac{w_t}{p_t^U}.$$

Inequality

$$\frac{w_v}{p_v} < \frac{w_t}{p_t}$$

holds for each scenario $p \in T$. Due to Lemma 1, in all optimal permutations for the instances $1|p|\sum w_i C_i$ with all scenarios $p \in T$, job J_t precedes job J_v. Due to Definition 1, *in each permutation of any minimal dominant set $S(T)$ job J_t precedes job J_v.*

Case *(b)*:

$$\frac{w_t}{p_t^U} < \frac{w_t}{p_t^L}, \ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L}, \ \frac{w_v}{p_v^L} \leq \frac{w_t}{p_t^U}.$$

If for the scenario $p \in T$ at least one of the conditions

$$\frac{w_v}{p_v} \neq \frac{w_v}{p_v^L}$$

or

$$\frac{w_t}{p_t} \neq \frac{w_t}{p_t^U}$$

holds, then arguing in the same way as in case *(a)*, we obtain that *in each permutation of any minimal dominant set $S(T)$, job J_t precedes job J_v.*

For the remaining vector $p' = (p'_1, p'_2, \ldots, p'_n) \in T$ for which both equalities

$$\frac{w_v}{p'_v} = \frac{w_v}{p_v^L}$$

and

$$\frac{w_t}{p'_t} = \frac{w_t}{p_t^U}$$

hold, we obtain equality

$$\frac{w_v}{p'_v} = \frac{w_t}{p'_t}.$$

Due to Lemma 2, for the instance $1|p'|\sum w_i C_i$, there exist both an optimal permutation of the form $\pi_l = (\ldots, J_t, \ldots, J_v, \ldots) \in S$ and one of the form $\pi_m = (\ldots, J_v, \ldots, J_t, \ldots) \in S$.

However, no permutation of the form π_m may be contained in a minimal dominant set, since such a permutation will be redundant. Indeed, a permutation of the form π_l will be definitely contained in any set $S(T)$ because of scenario $p \in T$ with $p \neq p'$ (due to Definition 1). The permutation π_l provides an optimal solution for the instance $1|p'|\sum w_i C_i$. If set $S(T)$ also contains a permutation of the form π_m, then $S(T)$ is not a minimal dominant set (contradicting to Definition 1).

Thus, we can conclude that *in each permutation of any minimal dominant set $S(T)$, job J_t precedes job J_v.*

Case *(c)*:

$$\frac{w_t}{p_t^U} < \frac{w_t}{p_t^L}, \ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L}, \ \frac{w_v}{p_v^L} > \frac{w_t}{p_t^U}.$$

Due to the strictness of the above three inequalities, the length of the intersection of the segment

$$\left[\frac{w_t}{p_t^U}, \frac{w_t}{p_t^L}\right]$$

and the segment

$$\left[\frac{w_v}{p_v^U}, \frac{w_v}{p_v^L}\right]$$

must be strictly positive. There exist both a scenario $p \in T$ and a scenario $p' \in T$ such that

$$\frac{w_t}{p_t} > \frac{w_v}{p_v}$$

and

$$\frac{w_t}{p'_t} < \frac{w_v}{p'_v},$$

respectively. Due to Lemma 1, in all optimal permutations for the instance $1|p|\sum w_iC_i$, job J_t precedes job J_v, and in all optimal permutations for the instance $1|p'|\sum w_iC_i$, job J_v precedes job J_t.

Thus, due to Definition 1, *any minimal dominant set constructed for problem* $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ *contains both a permutation of the form* $\pi_l = (\dots, J_t, \dots, J_v, \dots) \in S$ *and a permutation of the form* $\pi_m = (\dots, J_v, \dots, J_t, \dots) \in S$.

Case *(d)* with

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^L} < \frac{w_t}{p_t^U}$$

is examined similarly as case *(a)*.

Case *(e)* with

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^L} = \frac{w_t}{p_t^U}$$

is examined similarly as case *(b)*.

Case *(f)* with

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^L} < \frac{w_t}{p_t^U} < \frac{w_v}{p_v^U}$$

is examined similarly as case *(c)*.

Case *(g)* with

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} < \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^U} = \frac{w_t}{p_t^U}$$

is examined similarly as case *(b)*.

Case *(h)*:

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} = \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^L} < \frac{w_t}{p_t^U}.$$

The processing times of both jobs J_t and J_v are fixed and the strict inequality

$$\frac{w_v}{p_v} < \frac{w_t}{p_t}$$

holds. Due to Lemma 1, in all optimal permutations for the instance $1|p|\sum w_iC_i$, job J_v precedes job J_t.

Hence, due to Definition 1, *in each permutation of any minimal dominant set* $S(T)$, *job* J_t *precedes job* J_v.

Case *(i)*:

$$\frac{w_t}{p_t^U} = \frac{w_t}{p_t^L},\ \frac{w_v}{p_v^U} = \frac{w_v}{p_v^L},\ \frac{w_v}{p_v^L} = \frac{w_t}{p_t^U}.$$

We see that the processing times of both jobs J_t and J_v are fixed: $p_t^L = p_t^U = p_t$, $p_v^L = p_v^U = p_t$, and equality

$$\frac{w_v}{p_v} = \frac{w_t}{p_t}$$

holds. Due to Lemma 3 (Lemma 2, respectively) for each instance $1|p|\sum w_iC_i$ with the scenario $p \in T$, the optimal permutation is not unique (there exist both an optimal permutation $\pi_l \in S$

with job J_t preceding job J_v and an optimal permutation $\pi_m \in S$ with job J_v preceding job J_t). Due to Definition 1, the pair of jobs J_t and J_v generates two different minimal dominant sets $S(T)$ and $S'(T)$. Set $S(T)$ contains a permutation of the form π_l and does not contain a permutation of the form π_m, while set $S'(T)$ is the other way around.

Thus, *at least two minimal dominant sets exist for problem* $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$.

It is easy to convince that, *if case (i) occurs, then inequality* $|\mathcal{J}_r| \geq 2$ *holds, where*

$$r = \frac{w_t}{p_t^U} = \frac{w_t}{p_t^L} = \frac{w_v}{p_v^U} = \frac{w_v}{p_v^L},$$

and vice versa.

Since the above treatment in each of all possible cases *(a)–(i)* is applicable to any pair of jobs of set $\mathcal{J}$, we can complete the proof as follows.

Sufficiency (if). We assume that there does not exist a real number $r \in [a, b]$ such that $|\mathcal{J}_r| \geq 2$.

This means that there is no pair of jobs J_t and J_v in the set $\mathcal{J}$ with their weight-to-process ratios satisfying case *(i)*. Thus, case *(i)* does not occur. In each of the remaining cases from *(a)* to *(h)*, the order of the jobs $J_t \in \mathcal{J}$ and $J_v \in \mathcal{J}$ is well defined in permutation π_l (in permutations π_l and π_m) of a minimal dominant set $S(T)$.

Namely: one order (job J_t preceding job J_v) is realized in a permutation $\pi_l \in S(T)$ in cases *(a)*, *(b)*, *(d)*, *(e)* and *(h)*; one order (job J_v preceding job J_t) is realized in permutation $\pi_m \in S(T)$ in case *(g)*; both orders of these two jobs are realized in two permutations $\pi_l \in S(T)$ and $\pi_m \in S(T)$ which are contained in any minimal dominant set in each of the two cases *(c)* and *(f)* (in one permutation, job J_t precedes job J_v while in the other permutation, job J_v precedes job J_t).

Thus, a minimal dominant set $S(T)$ is unique for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ if there is no real number $r \in [a, b]$ such that $|\mathcal{J}_r| \geq 2$.

Necessity (only if). We assume that for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$, the minimal dominant set $S(T)$ is uniquely determined.

By contradiction, we assume that there exists a real number $r \in [a, b]$ for which inequality $|\mathcal{J}_r| \geq 2$ holds.

Inequality $|\mathcal{J}_r| \geq 2$ holds only if case *(i)* occurs for a pair of jobs from set $\mathcal{J}_r$. Thus, there exist at least two minimal dominant sets. This contradiction completes the proof. ■

The next claim directly follows from Lemma 3 and the above proof of Theorem 5.

Corollary 2 *Let $S(T)$ be a minimal dominant set for problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$. For any permutation $\pi_k \in S(T)$, there exists a scenario $p \in T$ such that π_k is the unique optimal permutation for the instance $1|p|\sum w_i C_i$ if and only if there is no real number $r \in [a, b]$ such that $|\mathcal{J}_r| \geq 2$.*

Thus, if the condition of Theorem 5 holds (i.e., there is no real number $r \in [a, b]$ such that $|\mathcal{J}_r| \geq 2$), then value $\varphi(z)$ may be considered as an uncertainty measure of the problem $z \in Z$. If there exists a set $\mathcal{J}_{r_q}$ which is not a singleton, then a minimal dominant set is not uniquely determined.

Next, we propose a modification of the mapping $\varphi(z) : Z \to R_+$ which is a function for any individual problem $z \in Z$. Moreover, we show that the size n of the problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ may be reduced by the quantity $|\mathcal{J}_{r_q}| - 1$ for each non-singleton $\mathcal{J}_{r_q}$ via identifying a set of jobs $\mathcal{J}_{r_q}$ by one job.

Due to Theorem 1, in any optimal permutation $\pi_l \in S$ generated by the instance $1|p|\sum w_i C_i$, all the jobs of set $\mathcal{J}_{r_q} \subseteq \mathcal{J}$ must be adjacently located one by one:

$$\pi_l = (\ldots, \pi(\mathcal{J}_{r_q}), \ldots).$$

Hereafter, $\pi(\mathcal{J}_{r_q})$ is a permutation of the jobs $\mathcal{J}_{r_q}$. Moreover, the order of the jobs

$$\{J_{q(1)}, J_{q(2)}, \ldots, J_{q(|\mathcal{J}_{r_q}|)}\} = \mathcal{J}_{r_q}$$

in permutation $\pi(\mathcal{J}_{r_q})$ does not influence the value of the objective function

$$\gamma = \sum_{i=1}^{n} w_i C_i$$

calculated for any permutation $\pi_k \in S$ of the form $\pi_k = (\ldots, \pi(\mathcal{J}_{r_q}), \ldots)$ (indeed, the processing time of each job $J_{q(v)} \in \mathcal{J}_{r_q}$ is fixed and the weight-to-process ratios are the same for all jobs of set $\mathcal{J}_{r_q}$).

Therefore, while looking for an optimal permutation for any instance $1|p|\sum w_i C_i$ generated by the uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ via fixing a scenario $p \in T$, one can treat all the jobs

$$\{J_{q(1)}, J_{q(2)}, \ldots, J_{q(|\mathcal{J}_{r_q}|)}\} = \mathcal{J}_{r_q}$$

as one job with the numerical parameters (weight and processing time) equal to those of any job of set $\mathcal{J}_{r_q}$.

By choosing only one job from each of such sets $\mathcal{J}_{r_q}$, $r_q \in \{r_1, r_2, \ldots, r_m\}$, $|\mathcal{J}_{r_q}| \geq 2$, the original instance of the uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ can be transformed into an equivalent instance (let this instance be denoted as $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$) with a smaller cardinality of the set of jobs to be scheduled (let this set be denoted as $\mathcal{J}^*$):

$$|\mathcal{J}^*| = |\mathcal{J}| - \sum_{q=1}^{m}(|\mathcal{J}_{r_q}| - 1) = n + m - \sum_{q=1}^{m} |\mathcal{J}_{r_q}|.$$

By summarizing, we obtain Theorem 6, where $1^*|p|\sum w_i C_i$ denotes the deterministic instance generated by the uncertain instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ via fixing a scenario $p \in T$.

Theorem 6 *An instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ is equivalent to the original instance of the uncertain problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ in the sense that for any fixed scenario $p \in T$, an optimal permutation π_k of the instance $1^*|p|\sum w_i C_i$ is obtained from the corresponding optimal permutation π_t of the instance $1|p|\sum w_i C_i$ of the original uncertain problem via deleting all jobs of set $\mathcal{J} \setminus \mathcal{J}^*$ from permutation π_t.*

Along with a smaller size, the equivalent instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ has a unique minimal dominant set $S(T)$ (due to Theorem 5). Consequently, set $S(T)$ constructed for the instance $1^*|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$ is a minimal dominant set with respect to both inclusion and cardinality.

In the general case of problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_i C_i$, one can use the following modification of the mapping $\varphi : Z \to R_+$:

$$\varphi^*(z) = 1 - \frac{n^*! - |S(T)|}{n! - 1}, \ \ z \in Z, \tag{14}$$

in order to measure the uncertainty of this problem.

6 Calculation of the uncertainty measures

Since the cardinality of a minimal dominant set could range from 1 (Theorem 3) to $n!$ (Theorem 4), it is impossible to generate all the elements of set $S(T)$ in polynomial time, and so there is no polynomial algorithm for enumerating all permutations of a set $S(T)$. Fortunately, due to Theorem 2, one can obtain a compact presentation of a minimal dominant set $S(T)$ for a problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ in the form of a dominance digraph $(\mathcal{J},\mathcal{A})$ with the vertex set $\mathcal{J}$ and the arc set $\mathcal{A}$. To this end, one can check condition (3) for each pair of jobs J_u and J_v from set $\mathcal{J}$ and construct a digraph $(\mathcal{J},\mathcal{A})$ of the dominance relation on the set $\mathcal{J}$ as follows: Arc (J_u, J_v) belongs to set $\mathcal{A}$ if and only if $J_u \mapsto J_v$. The construction of the dominance digraph $(\mathcal{J},\mathcal{A})$ takes $O(n^2)$ time.

Theorem 7 *The digraph $(\mathcal{J},\mathcal{A})$ constructed for problem $1|p_i^L \le p_i \le p_i^U|\sum w_iC_i$ defines a strict order relation on the set $\mathcal{J}$ if and only if there is no real $r \in [a,b]$ such that $|\mathcal{J}_r| \ge 2$.*

It is clear that dominance digraph $(\mathcal{J},\mathcal{A})$ is uniquely determined for any problem $z \in Z$. We consider the mapping $\mu : Z \to R_+$, where

$$\mu(z) = 1 - \frac{2|\mathcal{A}|}{n(n-1)}, \ \ z \in Z. \tag{15}$$

The uniqueness of dominance digraph $(\mathcal{J},\mathcal{A})$ implies that mapping $\mu : Z \to R_+$ is a function. This function is non-negative. The additivity of the function $\mu : Z \to R_+$ may be settled similarly as that of the mapping $\varphi : Z \to R_+$. Therefore, value $\mu(z)$ may be considered as a measure of uncertainty for the problem $z \in Z$.

Problem $z \in Z$ has the largest value $\mu(z) = 1$ of the measure $\mu(z)$ if equality $|S(T)| = n!$ holds, i.e., the set of arcs of digraph $\mathcal{G}$ is empty: $|\mathcal{A}| = 0$. In such a case, a minimal dominant set for problem $z \in Z$ has the maximal cardinality (for the corresponding problem with size n) and $\mu(z) = 1 = \varphi(z)$.

A zero value $\varphi(z) = 0$ corresponds to a zero value $\mu(z) = 0$. In such a case, equality $|S(T)| = 1$ holds for the individual problem $z \in Z$, i.e., the minimal dominant set is a singleton: $\{\pi_k\} = S(T)$, and permutation π_k is a solution to the deterministic counterpart $1|p|\sum w_iC_i$ with any scenario $p \in T$.

7 Concluding remarks

In today's competitive marketplace, an enterprise needs to use optimal scheduling decisions as much as possible in spite of usual data uncertainty. A schedule minimizing the worst-case regret (such a schedule may be constructed due to robust scheduling [6]) is useful generally for the worst-case scenario. The worst-case scenario may be practically realized rather seldom (it is unlikely that all the processing times assume their worst values just for the factual schedule). Consequently, a schedule which is optimal for the worst-case regret may be not competitive for the actual scenario being often far away from the worst-case one.

A stochastically optimal schedule is actually efficient, if the probability distribution of each random processing time will be known before scheduling and if a sufficiently large number of scenarios will be realized in a close scheduling environment. A stochastically optimal schedule may appear not competitive for a unique scenario which is factually realized and an enterprise may have not enough chances to compensate a loss caused by using a stochastically optimal schedule but which is not optimal for the factual scenario. Using stochastically optimal schedules for a sufficiently long time (and for a large number of similar scenarios) may be impossible for an enterprise since another competitor may be more productive via achieving better results on the market due to better scheduling policies.

In Sections 5 and 6, we proposed to use measures of problem uncertainty for selecting a suitable solution method. These measures of problem uncertainty are based on the cardinality of a minimal dominant set of job permutations and on the cardinality of a set of arcs in the dominance digraph.

Due to the assumption of reasonable restrictions on the job set $\mathcal{J}$, set $S(T)$ turns out to be a unique minimal dominant set for an instance of problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$. Consequently, a dominant set $S(T)$ being minimal with respect to *inclusion* (Definition 1) becomes minimal with respect to *cardinality*. A restriction on set $\mathcal{J}$ providing the singularity of a set $S(T)$ implies the identification of appropriate jobs without a loss of potentially optimal schedules and with decreasing the size $n = |\mathcal{J}|$ of the original problem $1|p_i^L \leq p_i \leq p_i^U|\sum w_iC_i$.

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