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I.V. DAINIAK, Ph. D. in Eng., S.E. KARPOVICH, D. Sc. in Eng.
Belarusian State University of Informatics and Radioelectronics, Minsk

R. SZCZEBIOT, Ph. D. in Eng.
Lomza State University of Applied Sciences, Poland

L. GOLDYN, Ph. D. in Eng.
Bialystok University of Technology, Poland

M.M. FORUTAN
Belarusian State University of Informatics and Radioelectronics, Minsk

PROGRAM MOTION PLANNING AND CONTROL FOR MULTICOORDINATE MECHANICAL SYSTEM

The problem of motion planning for multicoordinate mechanical system based on linear stepping motors was considered. The solution of inverse dynamic problem was used for the receiving of analytical control functions expressed through kinematic parameters of motion program with optimization on different criteria. The cases of program motion with constant speed, motion with constant acceleration and constant speed, and motion optimization by speed were considered, and the combined equations and control functions were obtained. The speed optimization of a program motion with sites of acceleration and constant speed was proposed.

Keywords: *program motion, control function, mechanical system, linear stepping motor*

Introduction. Let us consider the problem of motion planning for multicoordinate mechanical system with moving parts on the base of linear stepping motors (LSM). Approach to control of this system is presented using a solution of inverse dynamic problem with possible optimization on different criteria. Control functions satisfied the dynamic conditions are obtained using developed algorithms as analytical functions expressed through kinematic parameters of motion program.

Motions of multicoordinate drive based on LSM can be described by the second order system of differential equations obtained from Lagrange principle. Generally, this system is defined as

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n, u_1, \ldots, u_r),\ i = 1, \ldots, n, \tag{1}$$

where $\mathbf{x} = (x_1, ..., x_n)$ is vector of phase coordinates of device; $\mathbf{u} = (u_1, ..., u_r)$ is vector of control fucntions. System (1) may be rewritten in vector notation as

$$\dot{x}_i = f_i(t, \mathbf{x}, \mathbf{u}),\ i = 1, \ldots, n, \text{ or } \dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}),$$

where $\boldsymbol{f} = (f_1, ..., f_n)$.

The problem of program motion synthesis for multicoordinate drive consists in forming of controls $u = u(t, \mathbf{x})$ satisfied some technical requirements

$$u \in \tilde{U},$$

where $\tilde{U}$ is given set in $\mathbf{R}^r$, and a solution $x = x(t)$ of system (1) corresponding to this $\tilde{U}$ satisfies the additional conditions

$$\begin{gathered} \omega_k(t, x_1, \ldots, x_n) = 0;\ k = 1, \ldots, m,\ m < n, \\ \text{or } \omega_k(t, \mathbf{x}) = 0;\ k = 1, \ldots, m,\ m < n. \end{gathered} \tag{2}$$

The motion is performed along curve or surface described by equations (2). The system (2) is called program system, which includes holonomic and nonholonomic constrains.

As the problem of program motion synthesis is generally solved without uniqueness, the problem of optimal program motion synthesis can be considered. Controls $u = u(t, x)$ realizing the program motion and minimizing a functional must be obtained.

Therefore, the problem of optimal program motion synthesis can be considered as a problem of minimization of criterion function $J = t_1 \to \min$ on the set of solutions of system (1) with phase constraints

$$x(0) = 0;\ x(t_1) = x_1;\ \omega_k(t, \mathbf{x}) = 0,\ k = 1, \ldots, m.$$

The problem of motion with minimal spending of control resources can be considered in the same way.

These problems deal with optimal control with phase constraints on segment $[0; t_1]$. Necessary optimal conditions in form of maximum principle are known for such problems, i.e. these problems can be principally solved. However, a practical solution is quite difficult because the phase constraints are imposed on whole segment $[0; t_1]$, and the optimal conditions in general and coupled system in particular contain a regular degree. The last objection makes it necessary to consider different approaches to solution of the problem, in particular the approach to solution of inverse dynamic problem.

Differential equations of motions of multicoordinate mechanical systems based on LSM can be represented as

$$\dot{x}_i = p_i(x_1, \ldots, x_n) + u_i b_i(x_1, \ldots, x_n),\ i = 1, \ldots, n, \tag{3}$$

where $\mathbf{x} = (x_1, ..., x_n)$ are generalized device coordinates; $\mathbf{u} = (u_1, ..., u_r)$ is control vector.

System (3) may be rewritten in vector form

$$\dot{x}_i = p_i(\mathbf{x}) + u_i b_i(\mathbf{x}),\ i = 1, \ldots, n.$$

The problem consists in forming of control functions $u = u(t, \mathbf{x})$ which belong to $\mathbf{R}^r$, and corresponding solution of the system (3) satisfies the additional conditions

$$\omega_k(t, \mathbf{x}) = 0,\ k = 1, \ldots, r. \tag{4}$$

However, if $x = x(t)$ is a solution satisfied the program (4) then

$$\omega_k\left(t, x(t)\right) \equiv 0;\ k = 1, \ldots, r.$$

Whence

$$\frac{d}{dt}\omega_k\left(t,x(t)\right) \equiv 0,\ k=1,\ldots,r \ \text{ or}$$

$$\sum_{i=1}^{n}\left(\frac{\partial\omega_k(t,\mathbf{x})}{\partial x_i}\left(p_i(\mathbf{x})+u_ib_i(\mathbf{x})\right)+\frac{\partial\omega_k(t,\mathbf{x})}{\partial t}\right)\equiv 0,$$

when $x = x(t)$ satisfies (4).

The last expression is equivalent to the condition

$$\sum_{i=1}^{n}\left(\frac{\partial\omega_k(t,\mathbf{x})}{\partial x_i}\left(p_i(\mathbf{x})+u_ib_i(\mathbf{x})\right)+\frac{\partial\omega_k(t,\mathbf{x})}{\partial t}\right)= \\ = R_k(t,\mathbf{x},\omega_k),k=1,\ldots,r, \qquad (5)$$

where R_k is arbitrary function with $R_k(t,\mathbf{x},0)\equiv 0$.

Therefore, the condition (5) is necessary and sufficient for implementing the program (4) along solution $x = x(t)$ of the system (3). It can be used for calculating the necessary controls $u_i = u_i(t, \mathbf{x})$, $i = 1, \ldots, r$.

As $r < n$, the system (5) defines the controls ambiguously, and a functional must be minimized on free controls additionally. E.g. the control optimization problem with constraints

$$u(t,\mathbf{x})\in\tilde{U};\ \omega_k(t,\mathbf{x})=0,\ k=1,\ldots,r,$$

may be considered for each time moment.

Different problems solving of program motions planning for multicoordinate systems based on LSM is described next. As example we use a 3-coordinate system, realizing parabola motion.

Program motion with constant speed. Three-coordinate mechanical system with LSM (figure 1) and coordinate alternation φ–x–x moves along a parabola with constant speed V. Masses m_1, m_2, m_3 and moments of inertia J_1, J_2, J_3 of movable parts are given. Optimal controls u_1, u_2, u_3 have to be obtained.

Using Lagrange method, motion equations can be derived as follows

$$\begin{cases} J_z\ddot{\varphi}_1+m_3S_3^2\ddot{\varphi}_1+2m_3S_3\dot{\varphi}_1\dot{S}_3=u_1-b_1\dot{\varphi}_1; \\ (m_2+m_2)\ddot{S}_2=u_2-b_2\dot{S}_2; \\ m_3\ddot{S}_3-m_3S_3\dot{\varphi}_1^2=u_3-b_3\dot{S}_3, \end{cases}$$

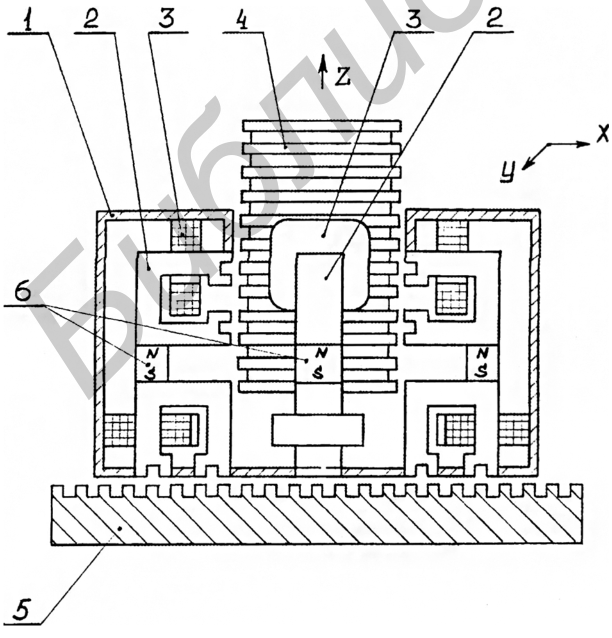

Figure 1 — Three-coordinate LSM:
1 — inductor; 2 — electromagnetic blocks; 3 — field windings; 4 — anchor; 5 — stator; 6 — permanent magnets

where $J_z = J_1 + J_2 + J_3$.

This system may be defined in standard form. We denote

$$\varphi_1=x_1;\quad \dot{\varphi}_1=x_2;\quad \ddot{\varphi}_1=\dot{x}_2;$$
$$S_2=x_3;\quad \dot{S}_2=x_4;\quad \ddot{S}_2=\dot{x}_4;$$
$$S_3=x_5;\quad \dot{S}_3=x_6;\quad \ddot{S}_3=\dot{x}_6,$$

whence

$$\begin{cases} \dot{x}_1=x_2; \\ \dot{x}_2=\dfrac{u_2-b_1x_2-2m_3x_5x_6}{J_z+m_3x_5^2}; \\ \dot{x}_3=x_4; \\ \dot{x}_4=\dfrac{u_4-b_2x_4}{m_2+m_3}; \\ \dot{x}_5=x_6; \\ \dot{x}_6=\dfrac{u_6-b_3x_6+m_3x_3x_2^2}{m_3}. \end{cases} \qquad (6)$$

We assume that the motion needs to be performed along parabola (figure 2):

$$x_3=h-k(x_5-b)^2.$$

with constant speed $V^2=\dot{S}_2^2+\dot{S}_3^2$, where $\varphi_1\equiv 0$.

Therefore, the motion program is defined by combined equations

$$\begin{cases} \omega_1=x_3+k(x_5-b)^2-h=0; \\ \omega_2=x_4^2+x_6^2-V^2=0. \end{cases} \qquad (7)$$

Holonomic constrain is determined by the first equation of (7), nonholonomic constrain is determined by the second equation.

As $\varphi = 0$ the motion equations are simplified and can be written in the form

$$\begin{cases} \dot{x}_1=\dot{x}_2=0; \\ \dot{x}_3=x_4; \\ \dot{x}_4=\alpha_1x_4+\beta_1u_4; \\ \dot{x}_5=x_6; \\ \dot{x}_6=\alpha_2x_6+\beta_2u_6, \end{cases} \qquad (8)$$

where

$$\alpha_1=\frac{-b_2}{m_2+m_3};\ \alpha_2=\frac{b_3}{m_3};$$
$$\beta_1=\frac{1}{m_2+m_3};\ \beta_2=\frac{1}{m_3}.$$

According to the condition $x_2 = 0$ and (6), control u_2 is defined unambiguously:

$$u_2=b_1x_2+2m_3x_5x_6.$$

In the sequel, we will assume that set of controls $\tilde{U}$ is considerably large.

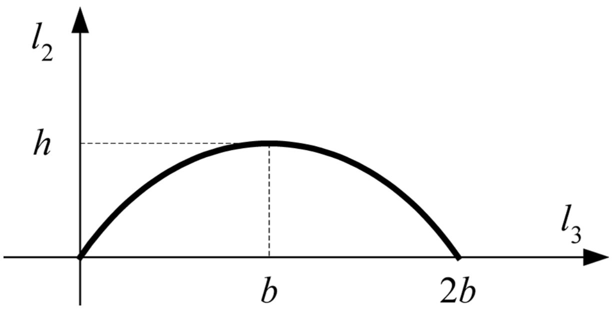

Figure 2 — Geometric condition of parabola program motion

By virtue of (5), in order to a solution $x = x(t)$ of the system (8) would satisfy the equation $\omega_1(\mathbf{x}) = 0$ it is sufficient that

$$\omega_3 = \sum_{i=1}^{6} \frac{\partial \omega_1}{\partial x_i} f_i = 0, \qquad (9)$$

where f_i are the right parts of (1).

However, for the condition (9) would be satisfied, it is necessary and sufficient that

$$\sum_{i=1}^{6} \frac{\partial \omega_3}{\partial x_i} f_i = R_1(t, \mathbf{x}, \omega_3), \qquad (10)$$

where $R_1(t, \mathbf{x}, 0) \equiv 0$.

For fulfillment of $\omega_2(\mathbf{x}) = 0$ along trajectory $x = x(t)$, it is necessary and sufficient that

$$\sum_{i=1}^{6} \frac{\partial \omega_2}{\partial x_i} f_i = R_2(t, \mathbf{x}, \omega_2), \qquad (11)$$

where $R_2(t, \mathbf{x}, 0) \equiv 0$.

Therefore, the fulfillment of conditions (10) and (11) is sufficient to implement the program (7)

$$\begin{cases} \alpha_1 x_4 + \beta_1 u_4 + 2kx_6^2 + 2k(x_5 - b)(\alpha_2 x_6 + \beta_2 u_6) = R_1; \\ x_4(\alpha_1 x_4 + \beta_1 u_4) + x_6(\alpha_2 x_6 + \beta_2 u_6) = R_2, \end{cases} \qquad (12)$$

where

$$R_1 = R_1\left(\mathbf{x}, x_3 + 2k(x_5 - b)^2 - h\right);$$

$$R_2 = R_2\left(\mathbf{x}, x_4^2 + x_6^2 - V^2\right).$$

Solving the system (12) for u_4 and u_6 we obtain

$$u_4 = \frac{R_1 - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2 x_6 (x_5 - b)}{\beta_1} -$$

$$-\frac{2k(x_5 - b)\left(x_4 R_1 - 2kx_6^2 - 2k\alpha_2 x_4 x_6 (x_5 - b) - R_2 + \alpha_2 x_6^2\right)}{\beta_1 \beta_2 \left(2kx_4(x_5 - b) - x_6\right)};$$

$$u_6 = \frac{x_4 R_1 - 2kx_4 x_6^2 - 2k\alpha_2 x_4 x_6 (x_5 - b) - R_2 + \alpha_2 x_6^2}{\beta_2 \left(2kx_4(x_5 - b) - x_6\right)}.$$

Presented approach allows solving a number of problems in realizing optimal program motions in aspects of maximal speed and minimal spending of motion resources. Required control functions satisfied the given conditions were obtained. Other program motion problems can be solved analogously for various schemes of multicoordinate mechanical systems based on LSM and requirements for contour motion of actuator.

Program motion with constant acceleration and constant speed. We shall consider system (6), for which it is required to realize a motion on a parabola

$$x_3 = h - k(x_5 - b)^2,$$

with speed, varied on the law of a trapeze (figure 3)

$$V(t) = \begin{cases} at, & 0 \le t \le t_1; \\ V_1, & t_1 \le t \le \tau - t_1; \\ a(t - \tau), & \tau - t_1 \le t \le \tau, \end{cases} \qquad (13)$$

where V_1 — constant speed; a — constant acceleration at a motion along parabola; τ — time, for which the point runs along parabola with parameters

$$x_5(\tau) = 2b; \; x_3(\tau) = x_4(\tau) = x_6(\tau) = 0.$$

Thus the program of a motion is set by the equations

$$\begin{cases} \omega_1 = x_3 + k(x_5 - b)^2 - h = 0; \\ \omega_2 = x_4^2 + x_6^2 - V^2(t) = 0. \end{cases} \qquad (14)$$

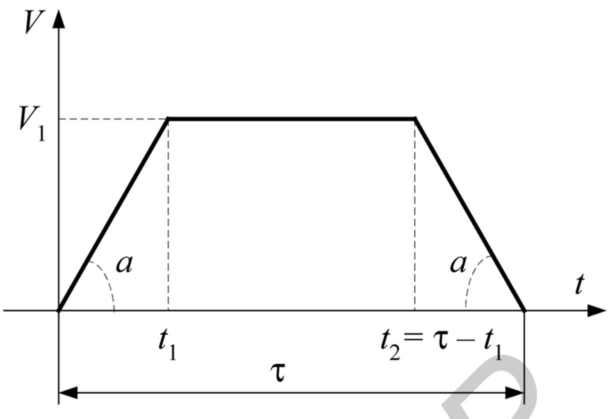

Figure 3 — Kinematic conditions of program motions with variable speed on trapezium rule

It's easy to see, that $t_1 = \dfrac{V_1}{a}$ is determined by value of given speed V_1. We shall find length of a site run by a point of a parabola:

$$l = \int_0^{2b} \sqrt{1 + (\varphi_x')^2}\, dx = \int_0^{2b} \sqrt{1 + 4k^2 (x_5 - b)^2}\, dx_5 =$$

$$= \left| \begin{matrix} 2k(x_5 - b) = \varphi \\ 2k dx_5 = d\varphi \end{matrix} \right| = \frac{1}{2k} \int_{-2bk}^{2bk} \sqrt{1 + \varphi^2}\, d\varphi =$$

$$= \frac{1}{k} \int_0^{2bk} \sqrt{1 + \varphi^2}\, d\varphi = \frac{1}{2k}\left(\varphi\sqrt{1 + \varphi^2} + \ln\left|\varphi + \sqrt{1 + \varphi^2}\right|\right)\Bigg|_0^{2bk} = \qquad (15)$$

$$= \frac{1}{2k}\left(2bk\sqrt{1 + 4b^2k^2} + \ln\left(2bk + \sqrt{1 + 4b^2k^2}\right)\right).$$

Length of a site, on which a motion occurs to acceleration a

$$l_1 = \frac{at^2}{2}\Bigg|_0^{t_1} = \frac{a}{2}\left(\frac{V_1}{a}\right)^2 = \frac{V_1^2}{2a}.$$

From reasons of symmetry the site of delay will be equal:

$$l_k = l_1 = \frac{V_1^2}{2a}.$$

Thus, basic site of a parabola, on which the motion occurs to constant speed, has length

$$l_2 = l - l_k - l_1 = l - \frac{V_1^2}{a}.$$

In a result it is possible to find motion time along parabola:

$$\tau = \frac{V_1}{a} + \frac{l_2}{V_1} + \frac{V_1}{a} = \frac{2V_1}{a} + \frac{l}{V_1} - \frac{V_1}{a} = \frac{l}{V_1} + \frac{V_1}{a}. \qquad (16)$$

We shall find managing influence at meanings given for a considered case of speed V_1 and acceleration a. Obviously, the first equation (11) remains without changes. In too time, as

$$\frac{\partial \omega_2}{\partial x_4} = 2x_4; \; \frac{\partial \omega_2}{\partial x_6} = 2x_6; \; \frac{\partial \omega_2}{\partial x_i} = 0, \; i = 4,6;$$

$$\frac{\partial \omega_2}{\partial t} = \begin{cases} -2a^2 t, & 0 \le t \le t_1; \\ 0, & t_1 \le t \le \tau - t_1; \\ 2a^2(t - \tau), & \tau - t_1 \le t \le \tau, \end{cases}$$

in the equation (12) to the right part it is necessary to add composed, equal $-\dfrac{\partial \omega_2}{\partial t}$.

As result the combined equations for definition u_4 and u_6 takes the form

$$\begin{cases} \alpha_1 x_4 + \beta_1 u_4 + 2kx_6^2 + 2k(x_5 - b)(\alpha_2 x_6 + \beta_2 u_6) = 0; \\ x_4(\alpha_1 x_4 + \beta_1 u_4) + x_6(\alpha_2 x_6 + \beta_2 u_6) = \gamma(t), \end{cases}$$

where

$$\gamma(t) = \begin{cases} 2a^2 t, & 0 \le t \le \dfrac{V_1}{a}; \\ 0, & \dfrac{V_1}{a} \le t \le \dfrac{l}{V_1}; \\ -2a^2\left(\dfrac{l}{V_1} + \dfrac{V_1}{a} - t\right), & \dfrac{l}{V_1} \le t \le \dfrac{l}{V_1} + \dfrac{V_1}{a}. \end{cases} \qquad (17)$$

From here we shall receive required control functions:

$$\begin{cases} u_2 = b_1 x_2 + 2m_3 x_5 x_6 \equiv 0; \\ u_4 = \dfrac{-\alpha_1 x_4 - 2kx_6^2 - 2k(x_5 - b)\alpha_2 x_6}{\beta_1} - \\ -\dfrac{2k(x_5 - b)\left(\alpha_2 x_6^2 - 2kx_6^2 - 2k\alpha_2 x_4 x_6 (x_5 - b) + \gamma(t)\right)}{\beta_1 \beta_2 \left(2kx_4(x_5 - b) - x_6\right)}; \\ u_6 = \dfrac{\alpha_2 x_6 - 2kx_4 x_6^2 - 2k\alpha_2 x_4 x_6 (x_5 - b) + \gamma(t)}{\beta_2\left(2kx_4(x_5 - b) - x_6\right)}. \end{cases} \qquad (18)$$

Thus, motion on a parabola $x_3 = h - k(x_5 - b)$, $x_1 = 0$ with speed (13) from point $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$ to point $x_1 = x_2 = x_3 = x_4 = x_6 = 0$, $x_5 = 2b$ is realized by control functions (18), and $t_1 = \dfrac{V_1}{a}$, $\tau = \dfrac{l}{V_1} + \dfrac{V_1}{a}$,

$$l = \frac{1}{2k}\left(2bk\sqrt{1 + 4b^2k^2} + \ln\left(2bk + \sqrt{1 + 4b^2k^2}\right)\right).$$

As the parameters b and a get out arbitrarily, indefinitely many control functions, realizing the given program turn out. Thus, it is possible to consider a problem of optimum speed concerning parameters V and a, which has the following form:

$$\tau = \frac{l}{V_1} + \frac{V_1}{a} \to \min.$$

Speed optimum of a program motion. We shall consider the mechanical system with three degrees of freedom. It is necessary to determine managing influence of drives, which are realizing speed optimum on a parabola.

The problem is reduced to realization of a motion given by the combined equations

$$\begin{cases} \dot{x}_1 = \dot{x}_2 = 0; \\ \dot{x}_3 = x_4; \\ \dot{x}_4 = \alpha_1 x_4 + \beta_1 u_4; \\ \dot{x}_5 = x_6; \\ \dot{x}_6 = \alpha_2 x_6 + \beta_2 u_6, \end{cases} \qquad (19)$$

at fulfillment of the program

$$\omega_1 = x_3 + k(x_5 - b)^2 - h = 0. \qquad (20)$$

Control functions u_4 and u_6 for realizing the program are possible to find from a condition

$$\alpha_1 x_4 + \beta_1 x_4 + 2kx_6^2 + 2k(x_5 - b)(\alpha_2 x_4 - \beta_2 u_6) = R_1, \qquad (21)$$

where $R_1 = R_1(t, \mathbf{x}, \omega_1)$ — any function with $R_1(t, \mathbf{x}, 0) = 0$.

As the equation (21) insolubly is unequivocal, we shall consider it under an additional condition

$$J = \beta_1^2 u_4^2 + u_6^2 \to \min.$$

The last problem is equal to

$$J = \left(R_1 - \alpha_1 x_4 - 2kx_6^2 - 2k(x_5 - b)(\alpha_2 x_6 + \beta_2 u_6)\right)^2 + u_6^2 \to \min$$

$$\to \frac{\partial J}{\partial u_6} = 2u_6 + 2P(Q + Pu_6) = 0,$$

where

$$\begin{cases} P(x_2) = 2k\beta_2(x_5 - b); \\ Q(\mathbf{x}) = R_1 - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2(x_5 - b)x_6. \end{cases} \qquad (22)$$

From here it is possible to receive

$$u_4 = \frac{1}{\beta_1}(Q + Pu_6) = \frac{1}{\beta_1}\left(Q - \frac{QP}{1 + P^2}\right) = \frac{Q}{\beta_1(1 + P^2)};$$

$$u_6 = -\frac{QP}{1 + P^2},$$

where P and Q are determined in (22).

Further we shall finally get

$$\begin{cases} u_4 = \dfrac{Q}{\beta_1(1 + P^2)}; \\ u_6 = \dfrac{QP}{1 + P^2}. \end{cases}$$

We shall present function R_1 as $R_1 = \sum_{k=1}^{m} \overline{u}_k(t)\omega_1^k$, where $\overline{u}_k(t)$ — piecewise continuous function.

Substituting meaning u_4, u_6 and R_1 in (19), we shall receive system of equations

$$\begin{cases} \dot{x}_3 = x_4; \\ \dot{x}_4 = \alpha_1 x_4 + \dfrac{\sum_{k=1}^{m} u_k(t)\omega_1^k - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2(x_5 - b)x_6}{1 + 4\beta_2^2(x_5 - b)^2}; \\ \dot{x}_5 = x_6; \\ \dot{x}_6 = \alpha_2 x_6 + \beta_2 \dfrac{\sum_{k=1}^{m} u_k(t)\omega_1^k - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2(x_5 - b)x_6}{1 + 4\beta_2^2(x_5 - b)^2} \times \\ \times 2k\beta_2(x_5 - b). \end{cases} \qquad (23)$$

In view of this problem of optimum speed (OS) we shall generate as follows: on trajectories of system (23) to proceed from a point (0; 0; 0; 0) in a point (0; 0; 2; 0) for the minimum time.

This problem is a typical problem of the theory of optimum processes and can be solved using any available computing methods, based on a principle of Pontryagin maximum. The case of a problem of optimum speed for a motion on a parabola with sites of acceleration, constant speed and deceleration is decided below.

Speed optimization of a program motion with sites of acceleration and constant speed. Is considered same, as manipulator with three degrees of freedom is higher. It is necessary to determine managing influence of drives, at which motion on a parabola with sites of acceleration, constant speed and braking is realized. As follows from previous, the problem is reduced to realization of a motion, given by the equations

$$\begin{cases} \dot{x}_1 = \dot{x}_2 = 0; \\ \dot{x}_3 = x_4; \\ \dot{x}_4 = \alpha_1 x_4 + \beta_1 u_4; \\ \dot{x}_5 = x_6; \\ \dot{x}_6 = \alpha_2 x_6 + \beta_2 u_6, \end{cases} \quad (24)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constant under additional conditions, determined by the program of a motion

$$\begin{cases} \omega_1 = x_3 + k(x_5 - b)^2 - h = 0; \\ \omega_2 = x_4^2 + x_6^2 - V^2(t) = 0, \end{cases} \quad (25)$$

with speed, varied on the trapeze law (13).

To determine the control functions u_4, u_6 for realizing a required motion, we use the system of equation

$$\begin{cases} \alpha_1 x_4 + \beta_1 u_4 + 2kx_6^2 + 2k(x_5 - b)(\alpha_2 x_6 + \beta_2 u_6) = R_1; \\ x_4(\alpha_1 x_4 + \beta_1 u_4) + x_6(\alpha_2 x_6 + \beta_2 u_6) = \gamma(t) + R_2, \end{cases} \quad (26)$$

where

$$\gamma(t) = \begin{cases} 2a^2 t, & 0 \le t \le \dfrac{V_1}{a}; \\ 0, & \dfrac{V_1}{a} \le t \le \dfrac{l}{V_1}; \\ -2a^2\left(\dfrac{l}{V_1} + \dfrac{V_1}{a} - t\right), & \dfrac{l}{V_1} \le t \le \dfrac{l}{V_1} + \dfrac{V_1}{a}; \end{cases}$$

$$l = \frac{1}{2k}\left(2bk\sqrt{1+4b^2k^2} + \ln\left(2bk + \sqrt{1+4b^2k^2}\right)\right);$$

$R_1 = R_1(t, \mathbf{x}, \omega_1)$, $R_2 = R_2(t, \mathbf{x}, \omega_2)$ — any functions with $R_1(t, \mathbf{x}, 0) \equiv 0$, $R_2(t, \mathbf{x}, 0) \equiv 0$.

For example,

$$R_1 = \sum_{k=1}^{m} \overline{U}_{k1}(t)\omega_1^k; \; R_2 = \sum_{k=1}^{m} \overline{U}_{k2}(t)\omega_2^k,$$

where $\overline{U}_{k1}(t), \overline{U}_{k2}(t)$ — piecewise continuous functions, for simplicity we accept $R_1(t, \mathbf{x}, 0) \equiv 0$, $R_2(t, \mathbf{x}, 0) \equiv 0$.

From system (26) following meanings of control functions u_4, u_6 turn out

$$U_6 = \frac{R_1 - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2(x_5 - b)x_6}{\beta_1} - \frac{2k(x_5-b)\left(x_4 R_1 - 2kx_6^2 - 2k\alpha_2 x_4(x_5-b)x_6 + \alpha_2 x_6^2 - R_2 - \gamma(t)\right)}{\beta_1\beta_2\left(2kx_4(x_5-b) - x_6\right)};$$

$$U_6 = \frac{\alpha_2 x_6^2 - 2kx_4 x_6^2 - 2k\alpha_2 x_4(x_5-b)x_6 + x_4 R_1 - R_2 - \gamma(t)}{\beta_2\left(2kx_4(x_5-b) - x_6\right)}.$$

Substituting in (24) we receive system of equations

$$\begin{cases} \dot{x}_3 = x_4; \\ \dot{x}_4 = \alpha_1 x_4 + \sum\limits_{k=1}^{m} \overline{U}_k^{(1)}(t)\omega_1^k - \alpha_1 x_4 - 2kx_6^2 - 2k\alpha_2(x_5 - b)x_6 - \\ \quad - \dfrac{2k(x_5-b)}{\beta_2\left(2kx_4(x_5-b) - x_6\right)}\left(\alpha_2 x_6^2 - 2kx_6^2 - 2k\alpha_2 x_4(x_5-b)x_6\right) + \\ \quad + \dfrac{2k(x_5-b)}{\beta_2\left(2kx_4(x_5-b) - x_6\right)}\left(x_4\sum\limits_{k=1}^{m}\overline{U}_k^{(1)}(t)\omega_1^k - \sum\limits_{k=1}^{m}\overline{U}_k^{(2)}(t)\omega_2^k - \gamma(t)\right); \\ \dot{x}_5 = x_6; \\ \dot{x}_6 = \alpha_2 x_6 + \dfrac{\alpha_2 x_6^2 - 2x_4 x_6^2 - 2k\alpha_2 x_4(x_5-b)x_6}{2kx_4(x_5-b) - x_6} + \\ \quad + \dfrac{x_4\sum\limits_{k=1}^{m}\overline{U}_k^{(1)}(t)\omega_1^k - \sum\limits_{k=1}^{m}\overline{U}_k^{(2)}(t)\omega_2^k - \gamma(t)}{2kx_4(x_5-b) - x_6}. \end{cases} \quad (27)$$

As the initial conditions are determined at $t = 0$, $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$; $t = T$, $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$, $x_5 = 2b$

Thus, it is possible to formulate the following problem of optimum speed (OS) for system (27) to choose control functions $\overline{U}_{k1}(t), \overline{U}_{k2}(t)$, where $k = 1, 2, \ldots, m$, and parameters a and V such to proceed from point (0; 0; 0; 0; 0; 0) to point (0; 0; 0; 0; 2; 0) for minimum time T.

On the other hand it was shown, that in considered task

$$T = \frac{l}{V_1} + \frac{V_1}{a_1}. \quad (28)$$

It is obvious values of parameters V, a are limited:

$$0 \le a \le a_1; \quad 0 \le V \le V_1.$$

From (28) follows, that min T is reached at $a = a_1$

Thus $T = \dfrac{l}{V_1} + \dfrac{V_1}{a_1}$ — function of one variable on $[0; V_1]$ and we have

$$T_V' = -\frac{l}{V_1^2} + \frac{1}{a_1} = 0 \;\Rightarrow\; V = \sqrt{l a_1}.$$

The minimum is reached in a point

$$V = \begin{cases} \sqrt{l a_1}, & \sqrt{l a_1} \le V_1; \\ V_1, & \sqrt{l a_1} > V_1. \end{cases} \quad (29)$$

From here the minimum time T is calculated as follows

$$T^* = \frac{1}{\sqrt{l a_1}} + \frac{\sqrt{l a_1}}{a_1} = 2\sqrt{\frac{l}{a_1}}, \text{ if } \sqrt{l a_1} \le V_1.$$

If $\sqrt{l a_1} > V_1$, then $T^* = \dfrac{l}{V_1} + \dfrac{V_1}{a_1}$.

Thus, two optimal solutions are possible in dependence from whether a condition to $\sqrt{l a_1} \le V_1$ is carried out.

When $\sqrt{l a_1} \le V_1$ we have $T^* = 2\sqrt{\dfrac{l}{a_1}}$ and the motion consists of two equal segments: acceleration and braking (figure 4).

When $\sqrt{l a_1} > V_1$ we have $T^* = \dfrac{l}{V_1} + \dfrac{V_1}{a_1}$ and the motion consists of three segments: acceleration, motion with constant speed and braking (figure 5).

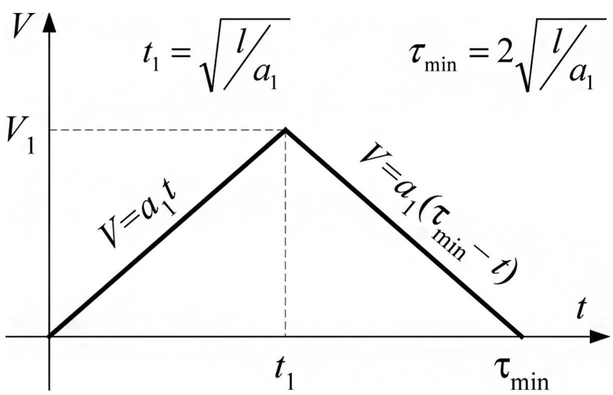

Figure 4 — Optimal program motion with acceleration and deceleration

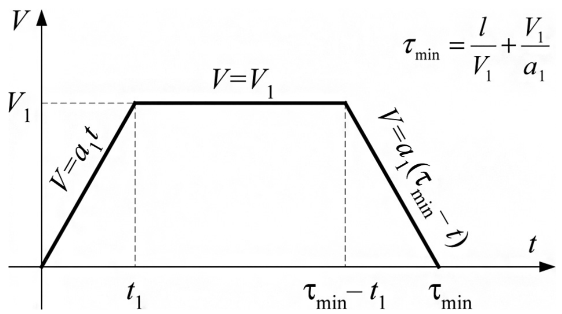

Figure 5 — Optimal program motion with acceleration, constant speed and deceleration

Using optimal values of parameters a and V we can solve OS problem independently of vector of control functions $\overline{U}_{k1}(t), \overline{U}_{k2}(t)$. Thus, we can accept

$$\begin{cases} \overline{U}_{k1}(t) \equiv 0;\ \overline{U}_{k2}(t) \equiv 0; \\ k = 1,2,\ldots,m; \\ a = a_1; \\ V = \begin{cases} \sqrt{l\,a_1},\ \sqrt{l\,a_1} \le V_1; \\ V_1,\ \sqrt{l\,a_1} > V_1. \end{cases} \end{cases}$$

The last expression includes two cases of the law of speed for solving of OS problem.

List of symbols

OS — optimal speed
LSM — linear stepping motor

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Дайняк И.В., Карпович С.Е., Szczebiot R., Goldyn L., Форутан М.М.

Построение программного движения и управление многокоординатной механической системой

Рассмотрен вопрос построения движения многокоординатной механической системы на основе линейных шаговых двигателей. Решение обратной задачи динамики было использовано для получения аналитических управляющих функций, выраженных через кинематические параметры программного движения с оптимизацией по различным критериям. Рассмотрены случаи программного движения с постоянной скоростью, движения с постоянным ускорением и постоянной скоростью, а также оптимизации движения, получены системы уравнений и управляющие функции. Предложена оптимизация скорости программного движения с участками ускорения и постоянной скорости.

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