Reductive homogeneous spaces and connections on them

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When a homogeneous space admits an invariant affine connection? If there exists at least one invariant connection then the space is isotropy-faithful, but the isotropy-faithfulness is not sufficient for the space in order to have invariant connections. If a homogeneous space is reductive, then the space admits an invariant connection. The purpose of the work is the classification of three-dimensional reductive homogeneous spaces and invariant affine connections on them.

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. The problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$. Since we are interested in only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. An *isotropic* $\mathfrak{g}$*-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [2] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [3]. We say that a homogeneous space $\overline{G}/G$ is *reductive* if the Lie algebra $\bar{\mathfrak{g}}$ may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{g}$ and an $\mathrm{ad}(G)$-invariant subspace $\mathfrak{m}$, that is, if $\bar{\mathfrak{g}} = \mathfrak{g} + \mathfrak{m}$, $\mathfrak{g} \cap \mathfrak{m} = 0$ and $\mathrm{ad}(G)\mathfrak{m} \subset \mathfrak{m}$. Last condition implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ and, conversely, if G is connected. If a homogeneous space is reductive, then the space always admits an invariant connection. In any of the following cases a homogeneous space $\overline{G}/G$ is reductive [3]: G is compact; G is connected and $\mathfrak{g}$ is reductive in $\bar{\mathfrak{g}}$; G is a discrete subgroup of $\overline{G}$. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas: $R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$, $(x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)] - \Lambda([x_1, x_2])$; $T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m}$, $(x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto \Lambda(x_1)(x_2+\mathfrak{g}) - \Lambda(x_2)(x_1+\mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}$.

We divide the solution of the problem of classification all three-dimensional reductive pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ into the following parts. We classify (up to isomorphism) faithful three-dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic. All there pairs are described in [1]. From all isotropy-faithful pairs we choose reductive pairs.

We describe all local three-dimensional reductive homogeneous spaces, it is equivalent to the description of effective pairs of Lie algebras, and all invariant affine connections on the spaces together with their curvature, torsion tensors and holonomy algebras. The results of work can be used in research work of the differential geometry, differential equations, topology, in the theory of representations, in the theoretical physics.

References

[1] N. P. Mozhey, Three-dimensional isotropically-faithful homogeneous spaces and connections on them. *Publisher University of Kazan, Kazan* (2015).

[2] K. Nomizu, Invariant affine connections on homogeneous spaces. *Amer. J. Math.* **76** (1954) 33-65.

[3] S. Kobayashi, K. Nomizu, Foundations of differential geometry. *John Wiley and Sons, New York.* **1** (1963); **2** (1969).

Graph complexity and tetrahedral complexity of compact 3-manifolds

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The graph complexity $c_g(M)$ of a compact 3-manifold M is the minimum order among all 4-colored graphs representing the manifold, while the tetrahedral complexity $c_{tet}(M)$ is the minimum number of tetrahedra in a (pseudo) triangulation of M. By construction $c_{tet}(M) \leq c_g(M)$ and in the closed case the inequality is always strict, but in the case of hyperbolic manifolds with toric boundary the two invariants often coincide. In this talk we describe an infinite family of 3-manifolds of this type and compute the value of their complexity. Moreover, we present the census of compact orientable prime 3-manifolds with toric boundary, up to graph complexity 14.