ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2017, Vol. 38, No. 6, pp. 1042–1049. © Pleiades Publishing, Ltd., 2017.

Three-Dimensional Non-Reductive Homogeneous Spaces of Solvable Groups Lie, Admitting Affine Connections

N. Mozhey*

(Submitted by M. A. Malakhaltsev)

Belarusian State University of Informatics and Radioelectronics, Minsk, 220013 Belarus

Received January 17, 2017

Abstract—When a homogeneous space admits an invariant affine connection? If there exists at least one invariant connection then the space is isotropy-faithful, but the isotropy-faithfulness is not sufficient for the space in order to have invariant connections. If a homogeneous space is reductive, then the space admits an invariant connection. The purpose of the work is the classification of three-dimensional non-reductive homogeneous spaces, admitting invariant affine connections. We concerned only case, when Lie group is solvable. The local classification of homogeneous spaces is equivalent to the description of effective pairs of Lie algebras. The peculiarity of techniques presented in the work is the application of purely algebraic approach, the compound of different methods of differential geometry, theory of Lie groups, Lie algebras and homogeneous spaces.

DOI: 10.1134/S1995080217060075

Keywords and phrases: *Affine connection, homogeneous space, transformation group, Lie algebra.*

1. INTRODUCTION

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a solvable Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. A large class of homogeneous spaces is spaces with solvable transformation group. In the study of homogeneous spaces, it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only effective actions of $\overline{G}$ on M. Since we are interested in only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\overline{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\overline{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\overline{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A pair $(\overline{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\overline{\mathfrak{g}}$, a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective. An *isotropic* $\mathfrak{g}$*-module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\overline{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y + \mathfrak{g}) = [x, y] + \mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\overline{\mathfrak{g}}, \mathfrak{g})$. The pair $(\overline{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. We divide the solution of the problem of classification all three-dimensional isotropically—faithful pairs $(\overline{\mathfrak{g}}, \mathfrak{g})$ into the following parts. We classify (up to isomorphism) faithful three-dimensional $\mathfrak{g}$-modules U. This is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classify (up to equivalence) all pairs $(\overline{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\overline{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic. All there

*E-mail: mozheynatalya@mail.ru

pairs are described in [1]. Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [2] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [3].

It appears that the isotropy-faithfulness is not sufficient for the pair in order to have invariant connections. The simplest example can be given for $\operatorname{codim}_{\bar{\mathfrak{g}}}\mathfrak{g} = 2$. The Lie algebra $\bar{\mathfrak{g}}$ has the following commutation table:

$$\begin{array}{c|cccc} & e_1 & e_2 & u_1 & u_2 \\ \hline e_1 & 0 & e_2 & 2u_1 & e_2 + u_2 \\ e_2 & -e_2 & 0 & 0 & u_1 \\ u_1 & -2u_1 & 0 & 0 & 0 \\ u_2 & -e_2 - u_2 & -u_1 & 0 & 0 \end{array},$$

and $\mathfrak{g}$ is spanned by e_1 and e_2. Then direct calculations show that there are no affine connections on this pair. Moreover, the complete one-by-one analysis of all isotropy-faithful effective pairs in codimension 2 (pairs can be found in [4]) shows that this is the only example for this codimension. Classification of such pairs in codimension 3 can be found in [5].

We say that a homogeneous space $\overline{G}/G$ is *reductive* if the Lie algebra $\bar{\mathfrak{g}}$ may be decomposed into a vector space direct sum of the Lie algebra $\mathfrak{g}$ and an $\operatorname{ad}(G)$-invariant subspace $\mathfrak{m}$, that is, if $\bar{\mathfrak{g}} = \mathfrak{g} + \mathfrak{m}$, $\mathfrak{g} \cap \mathfrak{m} = 0$ and $\operatorname{ad}(G)\mathfrak{m} \subset \mathfrak{m}$. Last condition implies $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ and, conversely, if G is connected. If a homogeneous space is reductive, then the space always admits an invariant connection. In any of the following cases a homogeneous space $\overline{G}/G$ is reductive [3]: (a) G is compact; (b) G is connected and $\mathfrak{g}$ is reductive in $\bar{\mathfrak{g}}$ in the sense that $\operatorname{ad}(g)$ is completely reducible (this is the case if G is connected and semi-simple); (c) G is a discrete subgroup of $\overline{G}$.

Let's find all three-dimensional non-reductive homogeneous spaces $\overline{G}/G$, admitting invariant affine connections. We define $(\bar{\mathfrak{g}}, \mathfrak{g})$ by the commutation table of the Lie algebra $\bar{\mathfrak{g}}$. Here by $\{e_1, \ldots, e_n\}$ we denote a basis of $\bar{\mathfrak{g}}$ ($n = \dim \bar{\mathfrak{g}}$). We assume that the Lie algebra $\mathfrak{g}$ is generated by $e_1, \ldots, e_{n-3}$. Let $\{u_1 = e_{n-2}, u_2 = e_{n-1}, u_3 = e_n\}$ be a basis of $\mathfrak{m}$. We describe affine connection by $\Lambda(e_{n-2})$, $\Lambda(e_{n-1})$, $\Lambda(e_n)$. To refer to the pair we use the notation $d.n.m$, where d is the dimension of the subalgebra, n is the number of the subalgebra of $\mathfrak{gl}(3, \mathbb{R})$, m is the number of $(\bar{\mathfrak{g}}, \mathfrak{g})$ in [1].

2. THE CLASSIFICATION OF PAIRS

Find non-reductive pairs, such that the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has invariant affine connections. The information about the non-reductive pairs and the affine connections is contained in the proof of the theorem.

Theorem 1. *If the non-reductive pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ has affine connections and $\bar{\mathfrak{g}}$ is solvable then $\mathfrak{g} \subset \mathfrak{gl}(3, \mathbb{R})$ is equivalent to one of the subalgebras:*

$$4.11.\ \boxed{\begin{matrix} x & z & u \\ & y & \\ & & 1/2y \end{matrix}};\ 4.21.\ \boxed{\begin{matrix} x & y & u \\ & x & z \\ & & \end{matrix}};\ 3.19.\ \boxed{\begin{matrix} y & & z \\ x & & \\ & & \lambda x \end{matrix}}, \lambda = 0, 1/2;\ 3.25.\ \boxed{\begin{matrix} & x & z \\ & & y \\ & & \end{matrix}};$$

$$3.20.\ \boxed{\begin{matrix} x & y & z \\ & \lambda x & \\ & & \mu x \end{matrix}},\ \begin{matrix} \lambda = 1/2\mu; \\ \lambda = 2\mu; \\ \lambda\mu = 0; \end{matrix}\ \wedge \lambda \le \mu;\ 3.27.\ \boxed{\begin{matrix} y & x & z \\ & & y \\ & & \end{matrix}};\ 2.9.\ \boxed{\begin{matrix} y & & x \\ & & \\ & & y \end{matrix}};$$