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SPATIAL EQUATIONS OF THE LINEAR KELVIN-VOIGT VISCOELASTICITY, BASED ON DEVIATORS

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I. INTRODUCTION

The aim of this paper was the theoretically substantiate derivation of a complete spatial system of equations using deviators for one of the simplest models of creep-the Kelvin-Voigt model, for which one-dimensional differential equation is well known [1]. In addition, it was necessary to determine the constraints on the physical parameters under which the constructed spatial generalization can be used to solve three-dimensional creep problems of a rigid body.

In modern scientific literature, it was sometimes mentioned that this spatial model was used [2, 3]. However, the systems of equations of state cited in these works (by means of which some spatial applied problems were solved) indicate that the authors incorrectly interpret this model and, accordingly, use incorrect equations of state in solutions.

II. ONE-DIMENSIONAL LINEAR MODELS OF KELVIN-VOIGT VISCOELASTICITY

By a one-dimensional linear viscoelastic Kelvin-Voigt model we mean the equation of state of an uniaxially loaded rod, which can be written in the form [1]:

$$\sigma(t)=E\cdot\varepsilon(t)+\eta\cdot\dot{\varepsilon}(t) \qquad (1)$$

where E is predetermined modulus of elasticity of the rod material, η is predetermined viscosity of the material, $\sigma(t)$ is the priori given (control) average stress on the rod, $\varepsilon(t)$ is the required average rod deformation, $\dot{\varepsilon}(t)=\frac{d\varepsilon(t)}{dt}$. The expression "average over the rod" explains the absence of an axial coordinate in the recording of the equation of state (1). Equation (1) is also called the law of deformation of a non-relaxing body [1].

III. SPATIAL GENERALIZATION OF THE LINEAR KELVIN-VOIGT MODELS WITH HELP OF DEVIATORS

It was considered a three-dimensional space with a Cartesian coordinate system $\mathbf{x}=(x_1,x_2,x_3)$. Let $D_\sigma(\mathbf{x},t)$ is deviator of stresses [4], $D_\varepsilon(\mathbf{x},t)$ is deviator of deformations [4], $G=\frac{E}{2\cdot(1+\nu)}$ is shear modulus, ν is Poisson ratio of a body material, $K=\frac{E}{3\cdot(1-2\cdot\nu)}$ is bulk modulus [4].

By analogy with the one-dimensional case (1), the three-dimensional linear Kelvin-Voigt model in terms of deviators can be written in the form:

$$D_\sigma = 2 \cdot G \cdot D_\varepsilon + \eta \cdot \dot{D}_\varepsilon, \tag{2}$$

where $\dot{D}_\varepsilon = \frac{d}{dt} D_\varepsilon$ is deviator of strain rates. Equation (1) must be supplemented by an equation connecting the mean normal stresses with volume deformations in the form:

$$\frac{1}{3} \sum_i \sigma_{ii}(\mathbf{x}, t) = K \cdot \sum_i \varepsilon_{ii}(\mathbf{x}, t) + \eta \cdot \sum_i \dot{\varepsilon}_{ii}(\mathbf{x}, t) \tag{3}$$

where $\sigma_{ii}(\mathbf{x}, t)$, $\varepsilon_{ii}(\mathbf{x}, t)$, $\dot{\varepsilon}_{ii}(\mathbf{x}, t)$ are normal stresses, deformations, rates of normal deformations.

IV. SEPARATION OF VARIABLES FOR THE GENERALIZED LINEAR KELVIN-VOIGT EQUATIONS

It was assumed that in (2) and (3) holds:

$$D_\sigma(\mathbf{x}, t) = D_\sigma^0(\mathbf{x}) \cdot \psi_D(t) \tag{4}$$

where $D_\sigma^0(\mathbf{x})$ is deviator of stress, depending only on the coordinates, the values of which are treated as an instantaneous solution of the elastic problem, $\psi_D(t)$ is a priori given deviator function of the external load variations at the boundary in time. It is used to describe variations in stresses at any point of a solid body due to the fact that the problem is quasistatic.

Further

$$D_\varepsilon(\mathbf{x}, t) = D_\varepsilon^0(\mathbf{x}) \cdot \varphi_D(t), \tag{5}$$

where $\varphi_D(t)$ is the desired deviator creep function, $D_\varepsilon^0(\mathbf{x})$ is deviator of deformations, depending only on the coordinates, whose values are also the solution of the instantaneous elastic problem:

$$D_\sigma^0(\mathbf{x}) = 2 \cdot G \cdot D_\varepsilon^0(\mathbf{x})$$

It should be noted that similar assumptions must obviously be satisfied for all stress and strain components:

$$\sigma_{ij}(\mathbf{x}, t) = \sigma_{ij}^0(\mathbf{x}) \cdot \psi_D(t), \; \varepsilon_{ij}(\mathbf{x}, t) = \varepsilon_{ij}^0(\mathbf{x}) \cdot \varphi_D(t), \tag{6}$$

The equations (2) and (3) should be rewritten taking into account (4) – (6):

$$D_\sigma^0(\mathbf{x}) \cdot \psi_D(t) = 2 \cdot G \cdot D_\varepsilon^0(\mathbf{x}) \cdot \left(\varphi_D(t) + \frac{\eta}{2 \cdot G} \cdot \dot{\varphi}_D(t) \right),$$
$$\frac{1}{3} \sum_i \sigma_{ij}^0(\mathbf{x}) \cdot \psi_D(t) = K \cdot \sum_i \varepsilon_{ii}^0(\mathbf{x}) \cdot \left(\varphi_D(t) + \frac{\eta}{K} \cdot \dot{\varphi}_D(t) \right). \tag{7}$$

It is obvious from (7) that the desired function $\varphi_D(t)$ should approximately satisfy two different equations:

$$\frac{\eta}{2 \cdot G} \cdot \dot{\varphi}_D(t) + \varphi_D(t) = \psi_D(t), \;\; \frac{\eta}{K} \cdot \dot{\varphi}_D(t) + \varphi_D(t) = \psi_D(t). \tag{8}$$

Adding the equations (8) it can be obtained that the equation with the average coefficients is necessary to solve:

$$A \cdot \dot{\varphi}_D(t) + \varphi_D(t) = \psi_D(t), \tag{9}$$

where $A = \frac{1}{2}\left(\frac{\eta}{2 \cdot G} + \frac{\eta}{K}\right) = \frac{4 - 5 \cdot \nu}{2} \cdot \frac{\eta}{E}$, and with help of subtraction of the equations (8) the theoretical relative accuracy δ of the generalized Kelvin-Voigt model can be expressed as:

$$\delta = \left| \frac{2 \cdot G - K}{2 \cdot G + K} \right| = \left| \frac{(1 + \nu) - 3 \cdot (1 - 2 \cdot \nu)}{(1 + \nu) + 3 \cdot (1 - 2 \cdot \nu)} \right| = \left| \frac{7 \cdot \nu - 2}{4 - 5 \cdot \nu} \right| \tag{10}$$

V. ON THE QUESTION OF THE INFLUENCE OF INITIAL CONDITIONS ON THE BEHAVIOR OF THE CREEP FUNCTION

The necessity of considering this question arises from the fact that, with the initial condition $\varphi_D(0) = 0$ the initial or unsteady creep of the material is considered and, accordingly, the deviators $D_\sigma^0(\mathbf{x})$ and $D_\varepsilon^0(\mathbf{x})$ in (7) have the physical meaning of a finite stress-strain state after the unsteady creep is completed.

Under the initial condition $\varphi_D(0) = \gamma_0 \neq 0$ the steady-state creep of the material is considered and the constant $\gamma_0 \cdot D_\varepsilon^0(\mathbf{x})$ (where $0 < \gamma_0$ dimensionless coefficient) has a physical meaning in how many times the deformation of the material, from which begins the process of steady creep $\gamma_0 \cdot D_\varepsilon^0(\mathbf{x})$ less than the final instant value $D_\varepsilon^0(\mathbf{x})$ corresponding to stresses $D_\sigma^0(\mathbf{x})$.

VI. DETERMINATION OF THE CREEP FUNCTION IN THE GENERALIZED LINEAR KELVIN-VOIGT VISCOELASTICITY MODEL FOR A CONSTANT LOAD

Note that for a constant load ($\psi_D(t) = 1$) from (9), for that the unsteady ($\varphi_{D,U}$) and steady-state ($\varphi_{D,S}$) deviator creep functions (5) it can be obtained:

$$\varphi_{D,U}(t) = 1 - e^{-\left(\frac{2\cdot E}{(4-5\nu)\cdot\eta}\right)\cdot t},\ \varphi_{D,S}(t) = 1 - (1-\gamma_0)e^{-\left(\frac{2\cdot E}{(4-5\nu)\cdot\eta}\right)\cdot t}.$$

VII. AN EXAMPLE OF A SOLUTION OF THE CONTACT PROBLEM FOR A PLANAR ROUND IN THE PLAN AND ABSOLUTELY RIGID STAMP ACTING ON THE BOUNDARY OF A WEIGHTLESS LINEAR VISCOELASTIC HALF-SPACE

In the case of a constant vertical load defined by value P acting along $0z$ axis on an axisymmetric stamp with a flat base, the solution of the instantaneous elastic problem for stresses has the form [5]:

$$\sigma_z(r,0) = -\frac{P}{2\cdot\pi}\sqrt{1-\frac{r^2}{a^2}}, \tag{11}$$

where a is a constant, the radius of the axisymmetric round stamp.

On the other hand, the displacements in the contact region in the instantaneous elastic problem for the distribution of contact stress (11) are determined by the expression [5]:

$$u_z(r,0) = -\frac{(1-\nu^2)}{E}\frac{P}{2\cdot\pi}.$$

Then, the contact displacement in the problem of unsteady creep ($u_{z,U}$) or for steady creep ($u_{z,S}$) of linear viscoelasticity (2) for a constant distribution of contact stresses (11) can be determined as:

$$u_{z,U}(r,0,t) = -\frac{(1-\nu^2)}{E}\frac{P}{2\cdot\pi}\left(1 - e^{-\left(\frac{2\cdot E}{(4-5\nu)\cdot\eta}\right)\cdot t}\right),$$
$$u_{z,S}(r,0,t) = -\frac{(1-\nu^2)}{E}\frac{P}{2\cdot\pi}\left(1 - (1-\gamma_0)e^{-\left(\frac{2\cdot E}{(4-5\nu)\cdot\eta}\right)\cdot t}\right).$$

It should be noted that this problem can be applied, for example, in ophthalmology, because describes the creep of the eyeball when determining the value of eye pressure.

VIII. GENERALIZATION OF THE MODEL FOR COMPOSITE BODY

The generalized Kelvin-Voigt viscoelasticity model for in average isotropic composite body was obtained in the form:

$$\langle D_\sigma\rangle = 2\cdot\langle G\rangle\cdot\langle D_\varepsilon\rangle + \langle\eta\rangle\cdot\langle\dot{D}_\varepsilon\rangle,$$
$$\frac{1}{3}\sum_i\langle\sigma_{ii}(\mathbf{x},t)\rangle = \langle K\rangle\cdot\sum_i\langle\varepsilon_{ii}(\mathbf{x},t)\rangle + \langle\eta\rangle\cdot\sum_i\langle\dot{\varepsilon}_{ii}(\mathbf{x},t)\rangle,$$

where $\langle D_\sigma(\mathbf{x},t)\rangle$, $\langle D_\varepsilon(\mathbf{x},t)\rangle$, $\langle \dot{D}_\varepsilon\rangle = \frac{d}{dt}\langle D_\varepsilon\rangle$ are average deviators of stresses, deformations and strain rates; $\langle G\rangle = \frac{\langle E\rangle}{2\cdot(1+\langle\nu\rangle)}$, $\langle\nu\rangle$, $\langle K\rangle = \frac{\langle E\rangle}{3\cdot(1-2\cdot\langle\nu\rangle)}$ are average shear modulus, Poisson ratio and bulk modulus of a composite material; $\langle\sigma_{ii}(\mathbf{x},t)\rangle$, $\langle\varepsilon_{ii}(\mathbf{x},t)\rangle$, $\langle\dot{\varepsilon}_{ii}(\mathbf{x},t)\rangle$ are average normal stresses, deformations, rates of normal deformations for composite body.

IX. CONCLUSIONS

For the first time, in the generalization of the one-dimensional Kelvin-Voigt model to the spatial case, deviators of stresses, deformations, and strain rates were used.

The equation (10) indicates a rigid dependence of the theoretical relative accuracy of the generalized three-dimensional linear Kelvin-Voigt model on the Poisson ratio of material. Using the expression obtained in the article, it is easy to establish that the spatial model is practically exact in the case when Poisson ratio is equal to 0.3. However, the error of the model sharply increases to 40% when a Poisson ratio turns to 0.4 (for example, for polymers and clay).

In the spatial case of creep deformation, in accordance with the Kelvin-Foig model, are finite.

The main hypothesis in the construction of steady-state creep is the determination how many times the finite creep deformations exceed the initial values.

A significant mistake in the Kelvin-Voigt model of steady linear viscoelasticity is that an instant elastic solution of the boundary value problem for a solid is the final state of the system, but in all other creep theories this is only the initial state of the system. Thus, the Kelvin-Voigt model repeatedly understates the real creep deformations of the system.

The comments on the Kelivin-Voigt model explain the conclusion for steady creep: this model is suitable only for a qualitative but not quantitative analysis of the change in the deformation of solids in time. Generalization of linear Kelvin-Voigt viscoelasticity model for composite body was created.

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