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# ON MATHEMATICAL MODELING OF PREADOLESCENTS AND ADOLESCENTS INTELLECT DEVELOPMENT

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**Abstract.** The problem of the mathematical modeling of the intellect development was considered. Analyzing the basic psychological concepts the limited growth stochastic model was selected. This model was presented by a stochastic differential equation with sub-fractional Brownian motion. Its parameters were estimated by the nonparametric identification method based on the parameter space investigation. The case study was done on 1351 boys and girls from Poland. It allowed proposing the models of intellect development for boys and girls.

**Key words:** intellect development, stochastic l modeling, sub-fractional Brownian motion

*Introduction.* The concepts of time, number, space, and relations of causality constitute the noetic categories of thought. Since substance, weight, and volume (these are parts of the noetic category) define all the objects, the understanding of noetic categories allows to get better achievements in education and to develop many useful skills explaining the world around us. The empirical studies proved that despite the growing tendency in the level of the intellectual development for older boys and girls from various ethnic groups, the level of the physical phenomenon understanding was still shallow. Before to search and to explain the psychological and educational reasons of the observed phenomenon (as well as data and their structures – the most important issues of BigData problem ), it is essential to find a trend, its speed, maximum value, etc... These tasks can be done by the methods of mathematical modeling. Therefore, the primary goal of this paper is to motivate a mathematical model selection and to develop the parameter estimation procedure.

*Basic psychological concepts.* Piaget showed that the concept of substance is built between 7– 8 years, the concept of weight between 8-9 years and the concept of volume between 11-12 years [1]. All works on the question of the order of mastery were carried out under the valid assumption of Piagetian stage theory. The sequence of topic mastery is as follows: first of all the conservation of substance, after the conservation of weight, and finally the volume conservation [2]. The question of the age at which the child masters the conservations was widely studied. If there is agreement on the age for both the substance conservation between 7-8 years and the weight conservation between 9-10 years [2], the discussion about the age of the volume conservation is still open. Research that shows that it takes the view adolescence adulthood to master the concepts underlying the volume conservation [3].

We can see that performance increases with age. This evolution corresponds to the different stages in the construction of the concept of volume. For Piaget and Inhelder [1] the concept of volume built in four stages. During the first stage, the child's thought is based on the following ideas: there is a confusion between weight and volume; the amount of a substance, the weight and the volume are proportional among themselves; the rising water level is caused by the weight of the submerged object. The biggest object (lack of differentiation between weight and volume) is stronger and therefore pushes the water to the top. If the child has to determine the water level rise as a result of the immersion of three solids of different colors but the same volume, we note that he encounters a logical-mathematical difficulty. The child will be able to say that object A has the same volume as object B, but he will not be able to conclude that object B has the same volume as object A. We find the same difficulty with heterogeneous objects. At that stage, the rising varies with the position of the object. A cylinder with an object horizontally submerged lets the level rise more than the same cylinder with an object vertically submerged. The last difficulty observed at this stage is the fact that the displaced water itself can be more or less compressed.

In the course of the second stage, thought develops in two ways: on a notional level, the child begins to dissociate the weight of the volume from the notion of "thickness" and, in logical-mathematical terms, we note that the child starts to compose with simple equalities by transduction reasoning. At this stage, some children estimate that the rising of the water level is proportional to the weight while others think that the weight is proportional to the volume. We observe at this stage the same difficulties as at the stage before when the object is horizontally or vertically submerged. In logical and mathematical terms, regarding objects of homogeneous form and weight, the subject accepts as true the equality between objects A and B and objects C and A. His discernment is based on their resemblance, and not on a transitive reasoning where $A = B$ , $B = C$ , therefore $A = C$ . When the comparisons are drawn between homogeneous objects and one heterogeneous object, then the equivalences are more disrupted. In the course of the third stage, we recognize visible progress in logical-mathematical terms. The logical-arithmetical conclusive understanding emerges, regarding weight and allows the child to dissociate weight from a form. The discovery of the law is only possible through observations made during the clinical interview and the evaluation and therefore remains inductive. In the course of the fourth stage, the child builds his understanding of the role of volume in the rising water level by deduction without having it "imposed" through experience as in the preceding stage. Every new additive composition is immediately successful. The clinical interview shows that the displaced volume is thought of as equivalent to the volume of the immersed solid. These four stages express as part of the Piagetian theory, the evolution of understanding concerning the weight volume dissociation task. Jamet observed the drop of the performances after 12 years [3]. How to explain it? Two explanations are possible: within the Piagetian framework or within the naive physics framework.

In the Piagetian theory, when a subject produced correct answers, this subject may be in stage III or stage IV. Only the explanation can distinguish two types of response. Only this, it's impossible to differentiate an answer of stage III and an answer of stage IV. The Piagetian tasks have dual functions, namely the diagnostic function the subject is at the stage II or III and the learning function. During the task, the subject learns something. He learns physical knowledge because he is in front of the experimental device. The experimenter asks him several questions on the same topic. The subject must explain his answers. Between the beginning and the end of the task, the subject has changed. He is not in the same state of knowledge. His knowledge has been modified by the interaction between the questions of the experimenter and the device. He is asked questions that did not arise before. In this condition, some subjects can give the correct answer at this end of the task. We have described the process that operates at stage III. At stage IV, the subject gives quickly the correct answer. The subject knows the law which explains the physic phenomena. His explanation is based on physical principles. The answer at stage IV is stable over time. It's not the case for the stage III. The answer at stage III is unstable. After some time, a subject classified in stage III may be able to give an answer classified in stage II. Our subjects performed four various tasks about the rising water level during 50 minutes.

The second explanation can be found within the naive physics framework. Research in naive physics shows that two types of knowledge coexist in the same subject. They are academic knowledge and naive knowledge. The academic knowledge is base on physical laws. Naive knowledge is based on our day-to-day experience [4], [5]. If a subject mobilizes naive knowledge for understanding the phenomenon of falling body, he will draw the falling body with a parabola curve. If the subject activates naive knowledge, he understands the failing body as in cartoons!

Two modalities of learning could be proposed: the first one involves interactions between two subjects and, the second one requires group discussion. The approach using interactions is a well-tried modality in the framework of the conservation of matter, for example, [6]. The principle is simple: the operational level of the subjects is evaluated individually. Then pairs of different operational levels are built, one non-retainer with one intermediary, one intermediary with one retainer. These pairs review the studied task a second time with the additional set-point to agree with the answer. The results show that the subjects make progress and that this development is stable in time. We have here a very quick modality of learning. The second modality could be to work individually instead of in a group and to select a wide range of homogeneous objects (same volume, same shape, same weight but different colors), and of heterogeneous objects (same volume but different weights, same weight but different volumes, etc). The subject is asked to predict every object concerning the rising of the water level followed by a systematic observation with verbalization.

*Mathematical model.* The intellectual development can be presented as a process with random walks between the increase and the drop of the performance. For any individual, this process has its unique realization. For the group of the individuals it can be characterized by speed of performance increase and maximal level of the development; the particular development depends on many internal and external factors; thus it has a random nature. Therefore a stochastic limited growth model can be used to describe and to understand the process of the intellect development in some population.

Let the probability space $(\Omega, \mathbf{F}, \mathbf{F}_{t\geq 0}, \mathsf{P})$ satisfies the usual conditions. The stochastic process, defined on this probability space and called sub-fractional Brownian motion, is given by means of fractional Brownian motion $B^H = \{B^H(t), t \in \mathsf{R}\}$ with Hurst parameter $0 < H < 1$, $B^H(0) = 0$ and $\mathtt{var} S(t) = (2 - 2^{2H-1}) t^{2H}$ as follows

$$S = \left\{ \frac{1}{\sqrt{2}} \left( B^H(t) + B^H(-t) \right), t \in \mathsf{R}_+ \right\}. \tag{1}$$

One of the properties of this process is self-similarity $\{S(at), t \in \mathsf{R}_+\} \overset{d}{=} \{a^H S^H(t), t \in \mathsf{R}_+\}$. The second moment of increments for any $t$ ¤ $s$ ([Bojdecki2004])

$$\mathsf{E}[S(t) - S(s)]^2 = -2^{2H-1}\left(t^{2H} + s^{2H}\right) + (t+s)^{2H} + (t-s)^{2H}, \tag{2}$$

$$(t-s)^{2H}\left(1, 2 - 2^{2H-1}\right)_- \leq \mathsf{E}[S(t) - S(s)]^2 \leq (t-s)^{2H}\left(1, 2 - 2^{2H-1}\right)_+ \tag{3}$$

that means sfBm does not have stationary increments. We suppose that the intellectual development satisfies a stochastic process $X = \{X(t), t \in \mathsf{R}_+\}$ which is the strong solution of the following SDE:

$$dX_t = \theta_1 X_t \left(1 - \frac{X_t}{\theta_2}\right) dt + \theta_3 dS_t, \; X(0) = x_0, \tag{4}$$

where: $\theta_1$, $\theta_2$, and $\theta_3$ are parameters, values of parameters are not known. Here $\theta_1$ is speed of performance increase, $\theta_2$ is maximal level of performance, $\theta_3$ is the diffusion of performance. Here we state that the solution of (4) exists and is unique for all $0 < H < 1$. It is assumed that

$$x_0^j, x_1^j, x_2^j, ..., x_N^j \tag{5}$$

are observed values of $X$ at the respective uniformly distributed times $t_i = i\Delta_t$ for $i = 0,1,..., N$ on the interval $[0,T]$, where $\Delta_t = \frac{T}{N}$ is the step size of the partition for $N$ observations and $T$ is the time span of the data. Index $j(j = 1,2,..., M)$ stands for the trajectory. The problem consists in determining parameters $H$, $\theta_1$, $\theta_2$, and $\theta_3$ of (4) by extracting information from observed panel data.

*The estimation procedure.* The progress in the field of stochastic processes was sufficient to introduce estimation methods for a linear SDE with additive noise. It was shown the consistency and the asymptotic behavior for the maximum likelihood estimator [7] as well as for the least squares of the estimator [8]. In [9] the task of parametric estimation was formulated as the constrained optimization problem and solved using a random search algorithm. We will use the ideas of [9] to adopt the estimation algorithm. It is possible to admit that

$$\mathrm{E}\left[X_t\right] = \frac{\theta_2 \mathrm{E}\left[X_{t_0}\right]}{\mathrm{E}\left[X_{t_0}\right] + \left(\theta_2 - \mathrm{E}\left[X_{t_0}\right]\right)\exp[-\theta_1 t]} \tag{6}$$

where: mean value $\mathrm{E}[X_t]$ can be estimated on the panel data available at the observation points $t_i$

$$\overline{x}(t_i) = \frac{1}{M}\sum_{j=1}^{M}\left(x^j(t_i)\right), \tag{7}$$

where: $t_i = i\Delta_t$, $i = 0,1,..., N$. The relations (6) and (7) allow to introduce the following goal function

$$\Phi_1(\theta_1, \theta_2) = \min_{\theta_1,\theta_2} \frac{1}{N}\sum_{i=0}^{N}\Big(\overline{x}(t_i) - \hat{x}(t_i)\Big)^2, \tag{8}$$

where: $\hat{x}$ denotes the estimate of $\overline{x}$. Moreover, the discrete model of the continuous model (4) can be presented by Euler scheme:

$$\tilde{X}_{i+1} = \tilde{X}_i + \theta_1 \tilde{X}_i \left(1 - \frac{\tilde{X}_i}{\theta_2}\right)\Delta_t + \theta_3\left(\tilde{S}_{i+1} - \tilde{S}_i\right), \tag{9}$$

for $i = 0,1,..., N-1$ on the time interval $[0,T]$ with $\tilde{X}_0 = X(0)$. It is obvious that

$$\mathrm{E}\left[\tilde{X}_{i+1} - \tilde{X}_i\right] = \theta_1 \tilde{X}_i \left(1 - \frac{\tilde{X}_i}{\theta_2}\right)\Delta_t, \tag{10}$$

and

$$\mathrm{E}\left[\tilde{S}_{i+1} - \tilde{S}_i\right]^2 = \frac{\mathrm{E}\left[\tilde{X}_{i+1} - \tilde{X}_i - \theta_1 \tilde{X}_i \left(1 - \frac{\tilde{X}_i}{\theta_2}\right)\Delta_t\right]^2}{\theta_3^2} \tag{11}$$

for $i = 0,1,..., N-1$. Since the model (4) is the additive one, quantities $K(H,\theta_3) := \theta_3^2 \mathsf{E}\left[\widetilde{S}_{i+1} - \widetilde{S}_i\right]^2$ can be calculated from the panel data (5) using estimates $\hat{\theta}_1$ and $\hat{\theta}_2$. Taking into account (2), the following goal function

$$\Phi_2(H,\theta_3) = \min_{H,\theta_3} \frac{1}{N} \sum_{i=0}^{N-1} \Big( K(H,\theta_3) - K(\hat{H},\hat{\theta}_3)) \Big)^2, \qquad (12)$$

where: $K(\hat{H},\hat{\theta}_3) = \hat{\theta}_3^2(-2^{2\hat{H}-1}((i+1)^{2\hat{H}} + (i)^{2\hat{H}}) + (2i+1)^{2\hat{H}})$. The optimization problem (8) and (12) can be solved by global optimization methods using the random search algorithms [10]. The analytical and experimental results proved the effectiveness and good speed of convergence of these algorithms ([9], [11]– [13]).

The case study. The intellect development of children can be described by the stochastic limited growth model (4). Moreover the speed and maximum level of intellect development depend on the gender of children. To verify the hypothesis we selected 1351 children of 11 years old and observed them during four years (twice per year from February 2012 to September 2016 - ten observation points). We took into account boys and girls from several provinces of Poland: G1 – Swietokrzyskie –126 persons (64 boys and 62 girls), G2 – Lubelskie – 274 persons (140 boys and 134 girls), G3 – Malopolskie – 414 persons (214 boys and 200 girls), G4 – Mazowieckie – 537 persons (271 boys and 266 girls). All these children were educated at the Polish schools and colleges with the same program on "Life and Nature Study" and "Physics". To estimate the parameters of the intellect development model we conducted ten Piagetian-type experiments. The procedures were analogically constructed to get three points in sum for the correct answers. The results of the experiments are listed on Fig.1. As it was possible to expect the intellect development of boys differs from that one of girls. Its significance was proved by t-test (see Tab.1).

*Table 1.*

The results of the experiments

| Experiment | Results | | | | Test | |
|---|---|---|---|---|---|---|
| | boys | | girls | | | |
| | mean | std | mean | std | value | p-value |
| 1 – Feb-2012 | 1.4552 | 0.0184 | 1.2206 | (0.0240) | 202.1060 | < 0.001 |
| 2 – Sep-2012 | 1.9620 | (0.0405) | 1.6304 | (0.0569) | 123.7809 | < 0.001 |
| 3 – Feb-2013 | 2.1905 | (0.0493) | 1.8522 | (0.0713) | 101.7717 | < 0.001 |
| 4 – Sep-2013 | 2.4684 | (0.0592) | 2.0955 | (0.0847) | 94.0930 | < 0.001 |
| 5 – Feb-2014 | 2.5250 | (0.0638) | 2.1759 | (0.0928) | 80.8424 | < 0.001 |
| 6 – Sep-2014 | 2.5305 | (0.0677) | 2.2079 | (0.0988) | 70.2455 | < 0.001 |
| 7 – Feb-2015 | 2.6642 | (0.0755) | 2.3482 | (0.1106) | 61.5432 | < 0.001 |
| 8 – Sep-2015 | 2.7228 | (0.0819) | 2.4221 | (0.1189) | 54.3144 | < 0.001 |
| 9 – Feb-2016 | 2.5527 | (0.0817) | 2.3082 | (0.1188) | 44.2238 | < 0.001 |
| 10 – Sep-2016 | 2.7237 | (0.0893) | 2.4446 | (0.1297) | 46.2217 | < 0.001 |

Denote by $X_1$ and $X_2$ the level of intellect development of boys and girls respectively. We suppose that the dynamics of the intellect development satisfies to (4). Applying the estimation procedure we got the following results, namely:

for the boys with $\hat{H}_1 = 0.6193$ and $\hat{x}_{10} = 1.504$

$$dX_{1t} = 0.2501X_{1t}\left(1 - \frac{X_{1t}}{2.7453}\right)dt + 0.0026dS_t,$$

for the girls with $\hat{H}_2 = 0.6703$ and $\hat{x}_{20} = 1.1918$

$$dX_{2t} = 0.2099X_{2t}\left(1 - \frac{X_{2t}}{2.4874}\right)dt + 0.0032dS_t .$$

These two models can be used in education, namely to estimate the effectiveness of the educational programs or to introduce new methods of teaching. However, these topics are not the interest of this work.

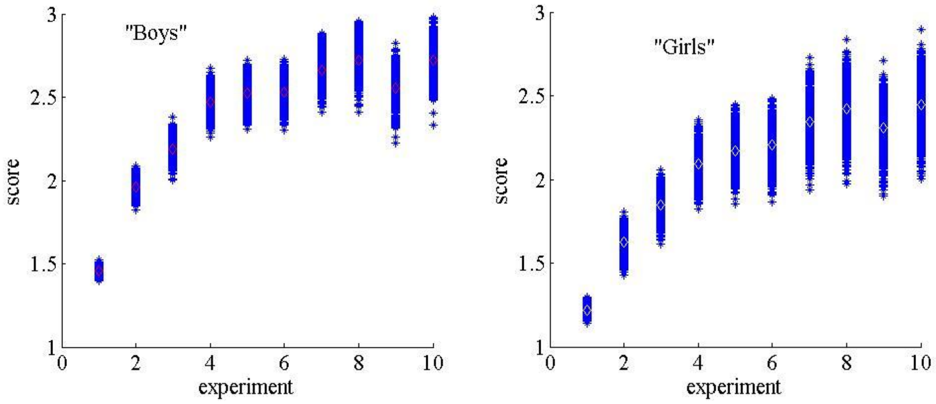


*Figure 1.* The intellect development data

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