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# Spin 1/2 particle with two mass states: interaction with external fields

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In the paper, a model for spin 1/2 particle with two mass states is developed on the base of Gel'fand–Yaglom approach in the theory of relativistic wave equations with extended sets of irreducible representations of the Lorentz group. In the end, the main generalized equation is presented in spin-tensor basis and with the use of the Dirac matrices. Besides 16-component wave function, we introduce two auxiliary bispinors, they determine initial 16-component wave function, and in absence of external field for these bispinor we derive two separate Dirac-like equations with masses $M_1$ and $M_2$. It is shown that in presence of external fields, electromagnetic one and gravitational non-Euclidean background, (with non-vanishing Ricci scalar curvature), the main wave equations is not split into separated wave equations, instead a quite definite mixing of two Dirac-like equations with additional Pauli interactions terms arises. This mixing also remains in presence only electromagnetic field, as well it remains in presence of gravitational field. It is shown that a generalized Majorana type wave equation with two mass states exists as well, such a generalized Majorana equation is not split into separated equations if Ricci scalar does not vanish. The procedure to solve the system of equations for two linked bispinors is discussed, which is based to of exclusion method. The problem ultimately reduces to a second order differential system for only one bispinor function.

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## 1. Spinor field with two mass states in Gel'fand–Yaglom approch

In the context of existence the similar neutrinos of different masses, in the present parer we examine existing in the frames of theory relativistic wave equation a possibility

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to describe a spin 1/2 particle with two mass states [1].

The model for particle with single value of spin $s = 1/2$ and two mass states is created on the base of the following linking scheme for Lorentz group representations (see notation in [2, 3])

$$\begin{array}{ccc} (\frac{1}{2},1) & - & (1,\frac{1}{2}) \\ | & & | \\ (0,\frac{1}{2}) & - & (\frac{1}{2},0). \end{array} \tag{1}$$

We numerate representations in (1) as follows

$$(0,\frac{1}{2}) \sim 1, \quad (\frac{1}{2},0) \sim 2, \quad (\frac{1}{2},1) \sim 3, \quad (1,\frac{1}{2}) \sim 4 \,. \tag{2}$$

The first order wave equation has the structure $(\Gamma_a \partial_a - M)\Psi = 0$. For spin blocks $C^{1/2}$ and $C^{3/2}$ of the matrix $\Gamma_4$

$$\Gamma_4 = (C^{1/2} \otimes I_2) \oplus (C^{3/2} \otimes I_4)$$

related to the scheme (1), we have the following structure in the Gel'fand–Yaglom basis [2, 3]

$$C^{1/2} = \left| \begin{array}{cccc} 0 & c_{12}^{1/2} & c_{13}^{1/2} & 0 \\ c_{21}^{1/2} & 0 & 0 & c_{24}^{1/2} \\ c_{31}^{1/2} & 0 & 0 & c_{34}^{1/2} \\ 0 & c_{42}^{1/2} & c_{43}^{1/2} & 0 \end{array} \right| , \quad c^{3/2} = \left| \begin{array}{cc} 0 & c_{34}^{3/2} \\ c_{43}^{3/2} & 0 \end{array} \right| , \tag{3}$$

where elements are not yet fixed. By reason of uniqueness of spin $S = 1/2$, we demand $c_{34}^{3/2} = c_{43}^{3/2} = 0$ ; due to relativistic invariance of the wave equation also it follows $c_{34}^{1/2} = c_{43}^{1/2} = 0$. Two last conditions means the break of link between representations $(\frac{1}{2},1)$ and $(1,\frac{1}{2})$ in the scheme (1), so that it transforms into slightly other one

$$(\frac{1}{2},1) - (0,\frac{1}{2}) - (\frac{1}{2},0) - (1,\frac{1}{2}) \,. \tag{4}$$

From invariance of the model under spatial reflection we obtain additional restriction

$$c_{21}^{1/2} = c_{12}^{1/2}, \quad c_{24}^{1/2} = c_{13}^{1/2}, \quad c_{42}^{1/2} = c_{31}^{1/2} \,. \tag{5}$$

Finally, existence of Lagrangian formulation for the model provides us with the next restrictions

$$c_{12}^{1/2} \quad \text{is} \quad \text{real}\,, \quad c_{12}^{1/2} = \frac{\eta_{34}^{1/2}}{\eta_{12}^{1/2}} (c_{34}^{1/2})^{*}, \tag{6}$$

where $\eta_{..}^{1/2}$ designate the elements of the block $\eta^{1/2}$ in the matrix of a bilinear form. Taking in mind all said and using the notations

$$c_{12}^{1/2} = c_1, \quad c_{13}^{1/2} = c_2, \quad \frac{\eta_{34}^{1/2}}{\eta_{12}^{1/2}} = \delta \,, \quad \delta = \pm 1 \,, \tag{7}$$

we obtain the following representation for the spin block $C^{1/2}$:

$$C^{1/2} = \left| \begin{array}{cccc} 0 & c_1 & c_2 & 0 \\ c_1 & 0 & 0 & c_2 \\ \delta c_2^* & 0 & 0 & 0 \\ 0 & \delta c_2^* & 0 & 0 \end{array} \right| . \tag{8}$$

Characteristic equation of 4-th degree for the matrix $C^{1/2}$ is bi-quadratic one

$$\Lambda^4 - \Lambda^2 \; ( c_1^2 + 2\delta|c_2|^2) + |c_2|^4 = 0 \; . \tag{9}$$

so we have the following four roots:

$$(\Lambda^2)_1 = \frac{(c_1^2 + 2\delta|c_2|^2) + c_1\sqrt{c_1^2 + 4\delta|c_2|^2}}{2},$$
$$(\Lambda^2)_2 = \frac{(c_1^2 + 2\delta|c_2|^2) - c_1\sqrt{c_1^2 + 4\delta|c_2|^2}}{2}. \tag{10}$$

These roots may be presented differently if one uses the quantities

$$\gamma_1 = \pm\frac{c_1 + \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} \; , \quad \gamma_2 = \pm\frac{c_1 - \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} \; ; \tag{11}$$

as easily verified that we have identities

$$\gamma_1^2 = \Lambda_1^2, \qquad \gamma_2^2 = \Lambda_2^2 \; . \tag{12}$$

From the general theory of the wave equations for particles with a spectrum of mass states [1–3] it is known that here we have models for a fermion with two positive and two negative masses:

$$M_1 = \frac{M}{\pm\sqrt{\Lambda_1^2}} = \frac{M}{\pm\sqrt{\gamma_1^2}} \; , \quad M_2 = \frac{M}{\pm\sqrt{\Lambda_2^2}} = \frac{M}{\pm\sqrt{\gamma_2^2}} \; . \tag{13}$$

Let us recall that for the ordinary Dirac equation the variant with negative mass may be transformed to the variant with positive mass by means of a simple linear transformation over the wave function:

$$(i\gamma^a\partial_a - M)\Psi = 0 \; , \quad M > 0; \quad \Psi' = \gamma^5\Psi, \quad \gamma'^a = \gamma^5\gamma^a\gamma^5 = -\gamma^a \; ,$$
$$[i\gamma'^a\partial_a - (-M)]\Psi' = 0, \quad (-M) < 0 \; . \tag{14}$$

As we see later, the variants $\delta = \pm 1$

$$\eta_{12}^{1/2} = \eta_{34}^{1/2} = +1 \; , \quad \eta_{12}^{1/2} = \eta_{34}^{1/2} = -1 \; . \tag{15}$$

corresponds to nonequivalent models, and the difference between is physically meaningful. The freedom in parameters $c_1, c_2$ must be agreed with real-valuedness of the both masses: $\Lambda_1^2 > 0, \quad \Lambda_2^2 > 0$.

## 2. The model in the modified Gel'fand–Yaglom basis

The modified Gel'fand–Yaglom basis for particles with half-integer spins is based on the use of special way of combining and enumeration of the basic vector of initial G–Y basis. In the case under consideration we should us the following four groups of states in modified basis:

$$\left|\begin{array}{r} \psi^{(0,1/2)}_{1/2,1/2} \\ \psi^{(0,1/2)}_{1/2,-1/2} \\ \\ \psi^{(1/2,0)}_{1/2,1/2} \\ \psi^{(1/2,0)}_{1/2,-1/2} \end{array}\right|, \qquad \left|\begin{array}{r} \psi^{(1,1/2)}_{1/2,1/2} \\ \psi^{(1,1/2)}_{1/2,-1/2} \\ \\ \psi^{(1/2,1)}_{1/2,1/2} \\ \psi^{(1/2,1)}_{1/2,-1/2} \end{array}\right|, \qquad \left|\begin{array}{r} \psi^{(1,1/2)}_{3/2,3/2} \\ \psi^{(1,1/2)}_{3/2,-3/2} \\ \\ \psi^{(1/2,1)}_{3/2,3/2} \\ \psi^{(1/2,1)}_{3/2,-3/2} \end{array}\right|, \qquad \left|\begin{array}{r} \psi^{(1,1/2)}_{3/2,1/2} \\ \psi^{(1,1/2)}_{3/2,-1/2} \\ \\ \psi^{(1/2,1)}_{3/2,1/2} \\ \psi^{(1/2,1)}_{3/2,-1/2} \end{array}\right|; \tag{16}$$

where in $\psi^{(l,l')}_{s,s_3}$: indices $(l,l')$ determine irreducible representations of the Lorentz group; $s$ denotes the values of spins; $s_3$ is a third projection of the spin.

In canonical Gel'fand–Yaglome basis, the vectors are enumerated as follows

$$\left|\begin{array}{r} \psi^{(0,1/2)}_{0,1/2} \\ \psi^{(0,1/2)}_{0,-1/2} \\ \psi^{(1/2,0)}_{1/2,0} \\ \psi^{(1/2,0)}_{-1/2,0} \end{array}\right|, \qquad \left|\begin{array}{r} \psi^{(1,1/2)}_{1,1/2} \\ \psi^{(1,1/2)}_{0,1/2} \\ \psi^{(1,1/2)}_{-1,1/2} \\ \psi^{(1,1/2)}_{1,-1/2} \\ \psi^{(1,1/2)}_{0,-1/2} \\ \psi^{(1,1/2)}_{-1,-1/2} \end{array}\right|, \qquad \left|\begin{array}{r} \psi^{(1/2,1)}_{1/2,1} \\ \psi^{(1/2,1)}_{1/2,0} \\ \psi^{(1/2,1)}_{1/2,-1} \\ \psi^{(1/2,1)}_{-1/2,1} \\ \psi^{(1/2,1)}_{-1/2,0} \\ \psi^{(1/2,1)}_{-1/2,-1} \end{array}\right|; \tag{17}$$

in $\psi^{(l,l')}_{l_3,l'_3}$ we use the notations: $l_3 = -l, -l+1, ..., +l$; $l'_3 = -l', -l'+1, ..., +l'$.

In spinor basis, the vectors are designated as follows

$$\left|\begin{array}{r} \psi^{\dot 1} \\ \psi^{\dot 2} \\ \psi_1 \\ \psi_1 \\ \psi^{\dot 1}_{(11)} \\ \psi^{\dot 1}_{(12)} \\ \psi^{\dot 1}_{(22)} \\ \psi^{\dot 2}_{(11)} \\ \psi^{\dot 2}_{(12)} \\ \psi^{\dot 2}_{(22)} \\ \psi^{\dot 1\dot 1}_{1} \\ \psi^{\dot 1\dot 2}_{1} \\ \psi^{\dot 2\dot 2}_{1} \\ \psi^{\dot 1\dot 1}_{2} \\ \psi^{\dot 1\dot 2}_{2} \\ \psi^{\dot 2\dot 2}_{2} \end{array}\right|. \tag{18}$$

Relationships between these tree bases are given by the formulas

$$\psi^{(l,l')}_{l_3,l'_3} = \sum_{s,m} (l l' l_3 l'_3 | s m) \psi^{\tau}_{s,m}, \tag{19}$$

$l_3, l'_3$ are fixed, $m = l_3 + l'_3$;

$$\psi^{\tau}_{s,m} = \sum_{l_3,l'_3} (l l' l_3 l'_3 \; s m) \psi^{(l,l')}_{(l_3,l'_3)}, \tag{20}$$

where $s, m$ are fixed, and $l_3 + l'_3 = m$;

$$\psi^{(l,l')}_{(l_3,l'_3)} = \left[ \frac{(2l)!}{(l+l_3)!(l-l_3)!} \right] \left[ \frac{(2l')!}{(l'+l'_3)!(l'-l'_3)!} \right] \psi^{\dot{1}\ldots\dot{1}\;\dot{2}\ldots\dot{2})}_{(1\ldots1\;2\ldots2)} , \tag{21}$$

where the number of indices of the type $\dot{1}$ equals $l' + l'_3$; the number of indices of the type $\dot{2}$ equals $l' - l'_3$; the number of indices of the type 1 equals $l + l_3$; and the number of indices of the type 2 equals $l - l_3$.

With respect to (2), four groups of vectors in modified basis are presented as

$$[\ \epsilon^1_{1/2,1/2}; \epsilon^1_{1/2,-1/2}; \epsilon^2_{1/2,1/2}; \epsilon^2_{1/2,-1/2}\ ], \quad [\ \epsilon^4_{1/2,1/2}; \epsilon^4_{1/2,-1/2}; \epsilon^3_{1/2,1/2}; \epsilon^3_{1/2,-1/2}\ ],$$

$$[\ \epsilon^4_{3/2,3/2}; \epsilon^4_{3/2,-3/2}; \epsilon^3_{3/2,3/2}; \epsilon^3_{3/2,-3/2}\ ], \quad [\ \epsilon^4_{3/2,1/2}; \epsilon^4_{3/2,-1/2}; \epsilon^3_{3/2,1/2}; \epsilon^3_{3/2,-1/2}\ ].$$

In this basis, the spin block $\Gamma_4^{(1/2)}$ has the structure

$$\Gamma_4^{(1/2)} = \left| \begin{array}{cccccccc}
0 & 0 & c_{12}^{(1/2)} & 0 & 0 & 0 & c_{13}^{(1/2)} & 0 \\
0 & 0 & 0 & c_{12}^{(1/2)} & 0 & 0 & 0 & c_{13}^{(1/2)} \\
c_{21}^{(1/2)} & 0 & 0 & 0 & c_{24}^{(1/2)} & 0 & 0 & 0 \\
0 & c_{21}^{(1/2)}0 & 0 & 0 & 0 & c_{24}^{(1/2)} & 0 & 0 \\
0 & 0 & c_{42}^{(1/2)} & 00 & 0 & 0 & c_{24}^{(1/2)} & 0 \\
0 & 0 & 0 & c_{42}^{(1/2)} & 0 & 0 & 0 & c_{43}^{(1/2)} \\
c_{31}^{(1/2)} & 0 & 0 & 0 & c_{34}^{(1/2)} & 0 & 0 & 0 \\
0 & c_{31}^{(1/2)} & 0 & 0 & 0 & c_{34}^{(1/2)} & 0 & 0
\end{array} \right| ;$$

from $P$-invariance it follows

$$c_{21}^{(1/2)} = c_{12}^{(1/2)}, \quad c_{24}^{(1/2)} = c_{13}^{(1/2)}, \quad c_{42}^{(1/2)} = c_{31}^{(1/2)}, \quad c_{34}^{(1/2)} = c_{43}^{(1/2)}$$

so the above spin block becomes simpler

$$\Gamma_4^{(1/2)} = \left| \begin{array}{cccccccc}
0 & 0 & c_{12}^{(1/2)} & 0 & 0 & 0 & c_{13}^{(1/2)} & 0 \\
0 & 0 & 0 & c_{12}^{(1/2)} & 0 & 0 & 0 & c_{13}^{(1/2)} \\
c_{12}^{(1/2)} & 0 & 0 & 0 & c_{13}^{(1/2)} & 0 & 0 & 0 \\
0 & c_{12}^{(1/2)}0 & 0 & 0 & 0 & c_{13}^{(1/2)} & 0 & 0 \\
0 & 0 & c_{42}^{(1/2)} & 00 & 0 & 0 & c_{43}^{(1/2)} & 0 \\
0 & 0 & 0 & c_{42}^{(1/2)} & 0 & 0 & 0 & c_{43}^{(1/2)} \\
c_{42}^{(1/2)} & 0 & 0 & 0 & c_{43}^{(1/2)} & 0 & 0 & 0 \\
0 & c_{42}^{(1/2)} & 0 & 0 & 0 & c_{43}^{(1/2)} & 0 & 0
\end{array} \right| \quad \Longrightarrow$$

$$\Gamma_4^{(1/2)} = \begin{vmatrix} c_{12}^{(1/2)} & c_{13}^{(1/2)} \\ c_{42}^{(1/2)} & c_{43}^{(1/2)} \end{vmatrix} \otimes \gamma_4 \,. \tag{22}$$

Requirement of existence of Lagrangian form for the model gives additional constraints

$$c_{42}^{(1/2)} = \frac{\eta_{34}^{(1/2)}}{\eta_{12}^{(1/2)}} \left( c_{13}^{(1/2)} \right)^* \;;$$

so with the notations

$$c_{12}^{(1/2)} = c_1, \quad c_{13}^{(1/2)} = c_2, \quad \frac{\eta_{34}^{(1/2)}}{\eta_{12}^{(1/2)}} = \delta, \quad \delta = \pm 1$$

we obtain

$$\Gamma_4^{(1/2)} = \begin{vmatrix} c_1 & c_2 \\ \delta c_2^* & c_{43}^{(1/2)} \end{vmatrix} \otimes \gamma_4 \,. \tag{23}$$

Let us consider the spin block $\Gamma_4^{(3/2)}$:

$$\Gamma_4^{(3/2)} = \begin{vmatrix} 0 & 0 & c_{43}^{(3/2)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{43}^{(3/2)} & 0 & 0 & 0 & 0 \\ c_{34}^{(3/2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c_{34}^{(3/2)} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{43}^{(3/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{43}^{(3/2)} \\ 0 & 0 & 0 & 0 & c_{34}^{(3/2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{34}^{(3/2)} & 0 & 0 \end{vmatrix} .$$

From $P$-invariance it follows

$$c_{34}^{(3/2)} = c_{43}^{(3/2)}, \qquad \Gamma_4^{(3/2)} = \begin{vmatrix} c_{43}^{(3/2)} & 0 \\ 0 & c_{43}^{(3/2)} \end{vmatrix} \otimes \gamma_4 \,.$$

Due to identity $c_{43}^{(3/2)} = 2c_{43}^{(1/2)}$, from the uniqueness of the spin value $S = 1/2$ we derive constraints $c_{43}^{(3/2)} = 0$, $c_{43}^{(1/2)} = 0$. Let us simplify the notation as follows

$$c_{1/2}^{(1/2)} = c_1, \qquad c_{13}^{(1/2)} = i\sqrt{3}c_{13} = c_2 \;; \tag{24}$$

so that

$$\Gamma_4^{(1/2)} = \begin{vmatrix} c_1 & c_2 \\ \delta c_2^* & 0 \end{vmatrix} . \tag{25}$$

Thus, the matrix $\Gamma_4$ has only one spin block

$$\Gamma_4^{(1/2)} = \begin{vmatrix} c_1 & c_2 \\ \delta c_2^* & 0 \end{vmatrix} \otimes \gamma_4 = C^{(1/2)} \otimes \gamma_4 . \tag{26}$$

Characteristic equation for $C^{(1/2)}$ is

$$\det \left| \begin{array}{cc} \lambda - c_1 & -c_2 \\ -\delta c_2^* & \lambda \end{array} \right| = \lambda(\lambda - c_1) - \delta |c_2|^2 = 0\,,$$

for the roots we get (compare them with (11)–(12))

$$\lambda_1 = \frac{c_1 + \sqrt{c_1^2 + 4\delta |c_2|^2}}{2}, \quad \lambda_2 = \frac{c_1 - \sqrt{c_1^2 + 4\delta |c_2|^2}}{2}\;; \tag{27}$$

note two identities

$$\lambda_1 \lambda_2 = -\delta |c_2|^2, \qquad \lambda_1 + \lambda_2 = c_1, \quad \lambda_1 - \lambda_2 = \sqrt{c_1^2 + 4\delta |c_2|^2}\,, \tag{28}$$

where $c_1, c_2$ are parameters of the model. The minimal equation for $C^{(1/2)}$ has the form

$$\left(C^{(1/2)} - \lambda_1\right)\left(C^{(1/2)} - \lambda_2\right) = 0\;; \tag{29}$$

whence with (28) in mind we get another representation for the minimal polynomial equation

$$\left(C^{(1/2)}\right)^2 - c_1 C^{(1/2)} - \delta |c_2|^2 = 0\;. \tag{30}$$

The minimal polynomial equation for the matrix $\Gamma_4^{(1/2)}$ is

$$\left[\left(\Gamma_4^{(1/2)}\right) - \lambda_1^2\right]\left[\left(\Gamma_4^{(1/2)}\right) - \lambda_2^2\right] = 0\;; \tag{31}$$

Correspondingly, the minimal polynomial equation for the matrix $\Gamma_4$ looks as follows

$$\Gamma_4 (\Gamma_4^2 - \lambda_1^2)\,(\Gamma_4^2 - \lambda_2^2) = 0\;. \tag{32}$$

## 3. Relationships between three bases

We start with the explicit form of the matrix $\Gamma_4$ in modified basis

$$\Gamma_4^{G-Y} = \left| \begin{array}{ccccccccc}
0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & \ldots\ldots\ldots \\
c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & \delta c_2^* & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & \delta c_2^* & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
\delta c_2^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & \delta c_2^* & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots\ldots
\end{array} \right| .$$

Using the formula (21), we can examine transition to canonical basis

$$\Psi_{canon} = B^{+}\Psi_{G-Y}$$

$$B^{+} = \begin{vmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{\sqrt{3}} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\frac{1}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot
\end{vmatrix} .$$

We are to find the matrix $\Gamma_4$ in canonical basis

$$\Gamma_4^{canon} = B^{+}\Gamma_4^{G-Y}B =$$

$$= \begin{vmatrix}
\cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{c_2}{\sqrt{3}} & \cdot & -c_2\sqrt{\frac{2}{3}} & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2\sqrt{\frac{2}{3}} & \cdot & -\frac{c_2}{\sqrt{3}} & \cdot \\
c_1 & \cdot & \cdot & \cdot & \cdot & -\frac{c_2}{\sqrt{3}} & \cdot & c_2\sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_1 & \cdot & \cdot & \cdot & \cdot & -c_2\sqrt{\frac{2}{3}} & \cdot & \frac{c_2}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\frac{\delta c_2^*}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\delta\sqrt{\frac{2}{3}}c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \delta\sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{\delta c_2^*}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{\delta c_2^*}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \delta\sqrt{\frac{2}{3}}c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\delta\sqrt{\frac{2}{3}}c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\frac{\delta c_2^*}{\sqrt{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{vmatrix} .$$

No we relate canonical and spinor bases:

$$\Psi_{canon} = A\psi_{spin},$$

$$A = \left| \begin{array}{cccccccccccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{2} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array} \right| .$$

For $\Gamma_4$ in spinor basis we get the form

$$\Gamma_4^{spin} = A^{-1}\Gamma_4^{canon}A =$$

$$\left| \begin{array}{cccccccccccccccc}
\cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & c_2\sqrt{\frac{2}{3}} & \cdot & -\sqrt{\frac{2}{3}}c_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & c_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{\frac{2}{3}}c_2 & \cdot & -\sqrt{\frac{2}{3}}c_2 & \cdot \\
c_1 & \cdot & \cdot & \cdot & \cdot & -c_2\sqrt{\frac{2}{3}} & \cdot & \sqrt{\frac{2}{3}}c_2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & c_1 & \cdot & \cdot & \cdot & \cdot & -\sqrt{\frac{2}{3}}c_2 & \cdot & c_2\sqrt{\frac{2}{3}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -\frac{\delta c_2^*}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\sqrt{\frac{2}{3}}\delta c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \sqrt{\frac{2}{3}}\delta c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{\delta c_2^*}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{\delta c_2^*}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \sqrt{\frac{2}{3}}\delta c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\sqrt{\frac{2}{3}}\delta c_2^* & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & -\frac{\delta c_2^*}{\sqrt{6}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \right| .$$

Thus, in spinor form the wave equation has the structure

$$c_1\,\partial^{\dot a b}\,\Psi_b+\sqrt{\frac{2}{3}}c_2\partial^c_{\dot b}\,\psi^{(\dot a\dot b)}_c+M\,\Psi^{\dot a}=0\,,\quad c_1\,\partial_{a\dot b}\,\psi^{\dot b}+\sqrt{\frac{2}{3}}c_2\partial^b{}_{\dot c}\,\psi^{\dot c}_{(ab)}+M\,\psi_a=0\,,$$
$$-\frac{\delta}{\sqrt6}c_2^*(\partial^{\dot b}_a\,\psi^{\dot c}+\partial^{\dot c}_a\,\psi^{\dot b})+M\,\psi^{\dot b\dot c}_a=0\,,\quad -\frac{\delta}{\sqrt6}c_2^*(\partial^{\dot a}_b\,\psi_c+\partial^{\dot a}_c\,\psi_b)+M\,\psi^{\dot a}_{(bc)}=0\,,\ (33)$$

where the derivative operator is spinor form is determined by the formula $\partial_{\dot a b}=-i\partial_\mu(\sigma^\mu)_{\dot a b}$, $\sigma^j$ stands for the Pauli matrices, $\sigma^4=iI_2$.

## 4. The wave equation in spin-tensor form

It is convenient to re-write eq. (33) differently

$$c_1\,\partial^{\dot a b}\,\Psi_b+\beta_2\,\partial^c_{\dot b}\,\psi^{(\dot a\dot b)}_c+M\,\Psi^{\dot a}=0\,,\quad c_1\,\partial_{a\dot b}\,\psi^{\dot b}+\beta_2\partial^b{}_{\dot c}\,\psi^{\dot c}_{(ab)}+M\,\psi_a=0\,,$$
$$\frac{\beta_3}{2}(\partial^{\dot b}_a\,\psi^{\dot c}+\partial^{\dot c}_a\,\psi^{\dot b})+M\,\psi^{\dot b\dot c}_a=0\,,\quad \frac{\beta_3}{2}(\partial^{\dot a}_b\,\psi_c+\partial^{\dot a}_c\,\psi_b)+M\,\psi^{\dot a}_{(bc)}=0\,,\qquad(34)$$

where

$$\beta_2=\sqrt{2/3}\;c_2,\quad \beta_3=-f\sqrt{2/3}\;c_2^*.\qquad(35)$$

To translate equations in (34) to spin-vector form, we are to use the known formulas

$$\Psi^{(\dot b\dot c)}_a=\frac12(\sigma^{\mu\dot b}{}_a\Psi^{\dot c}_\mu+\sigma^{\mu\dot c}{}_a\Psi^{\dot b}_\mu),\quad \Psi^{\dot a}_{(bc)}=\frac12(\sigma^{\mu\dot a}{}_b\Psi_{\mu c}+\sigma^{\mu\dot a}{}_c\Psi_{\mu b})$$
$$\Psi^{\dot a}=\sigma^{\mu\dot a b}\Psi_{\mu b},\quad \Psi_a=\sigma^\mu_{a\dot b}\Psi^{\dot b}_\mu\,.\qquad(36)$$

Instead of the first equation in (34), we obtain

$$c_1\partial^{\dot a b}\sigma^\mu_{b\dot c}\Psi^{\dot c}_\mu+\frac12\beta_2\partial^c_{\dot b}(\sigma^{\mu\dot a}{}_c\Psi^{\dot b}_\mu+\sigma^{\mu\dot b}{}_c\Psi^{\dot a}_\mu)+M\,\sigma^{\mu\dot a b}\Psi_{\mu b}=0\,,$$

$$c_1\partial_{a\dot b}\sigma^{\mu\dot b c}\Psi_{\mu c}+\frac12\beta_2\partial^b{}_{\dot c}\,(\sigma^{\mu\dot c}{}_a\Psi_{\mu b}+\sigma^{\mu\dot c}{}_b\Psi_{\mu a})+M\,\sigma^\mu_{a\dot b}\Psi^{\dot b}_\mu=0\,.$$

Whence it follows

$$c_1\partial^{\dot a b}\sigma^\mu_{b\dot c}\Psi^{\dot c}_\mu+\frac{\beta_2}{2}(-\sigma^{\mu\dot a c}\partial_{c\dot b}\Psi^{\dot b}_\mu+\frac{2}{i}\partial_\mu\Psi^{\dot a}_\mu)+M\,\sigma^{\mu\dot a b}\Psi_{\mu b}=0\,,$$

$$c_1\partial_{a\dot b}\sigma^{\mu\dot b c}\Psi_{\mu c}+\frac{\beta_2}{2}(-\sigma^\mu_{a\dot c}\partial^{\dot c b}\Psi_{\mu b}+\frac{2}{i}\partial_\mu\Psi_{\mu a})+M\,\sigma^\mu_{a\dot b}\Psi^{\dot b}_\mu=0\,.$$

In new representation, the 16-component wave function makes up the vector-bispinor

$$\Psi_\mu=\left|\begin{array}{c}\Psi^{\dot a}_\mu\\ \Psi_{\mu b}\end{array}\right|,\qquad(37)$$

and the last two equations are written with the use of the Dirac matrices as follows

$$c_1\partial_\nu\gamma_\nu\gamma_\mu\Psi_\mu+\frac{\beta_2}{2}(-\gamma_\mu\partial_\nu\gamma_\nu\Psi_\mu-2\partial_\mu\psi_\mu)+M\gamma_\mu\Psi_\mu=0\,.$$

Introducing the notation $\hat{\partial} = \gamma_\mu \Psi_\mu$ and taking into account the identity $\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = 2\delta_{\nu\mu}$, we re-write the previous equation as

$$(c_1 + \frac{\beta_2}{2})\hat{\partial}(\gamma_\mu \Psi_\mu) - 2\beta_2(\partial_\mu \psi_\mu) + iM(\gamma_\mu \Psi_\mu) = 0 \; . \qquad (38)$$

Now, we consider third and fourth equations in (34): taking in mind the formulas (36), we get

$$\frac{\beta_3}{2}(\partial^{\dot{a}}_{b} \sigma^{\mu\dot{c}d}\Psi_{\mu d} + \partial^{\dot{c}}_{b} \sigma^{\mu\dot{a}d}\Psi_{\mu d}) + \frac{M}{2}(\sigma^{\mu\dot{a}}_{\;\;\;b}\Psi^{\dot{c}}_{\mu} + \sigma^{\mu\dot{c}}_{\;\;\;b}\Psi^{\dot{a}}_{\mu}) = 0 \; ,$$

$$\frac{\beta_3}{2}(\partial^{\dot{b}}_{a} \sigma^{\mu}_{\;c\dot{d}}\Psi^{\dot{d}}_{\mu} + \partial^{\dot{b}}_{c} \sigma^{\mu}_{\;a\dot{d}}\Psi^{\dot{d}}_{\mu} + \frac{M}{2}(\sigma^{\mu\dot{b}}_{a}\Psi_{\mu c} + \sigma^{\mu\dot{b}}_{c}\Psi_{\mu a}) = 0 \; .$$

We multiply the first equation by$\sigma^{\lambda b}_{\;\;\;\dot{a}}$, and the second one – by $\sigma^{\lambda a}_{\;\;\;\dot{b}}$, so obtaining

$$\frac{\beta_3}{2}\sigma^{\lambda b}_{\;\;\;\dot{a}}(\partial^{\dot{a}}_{b} \sigma^{\mu\dot{c}d}\Psi_{\mu d} + \partial^{\dot{c}}_{b} \sigma^{\mu\dot{a}d}\Psi_{\mu d}) + \frac{M}{2}\sigma^{\lambda b}_{\;\;\;\dot{a}}(\sigma^{\mu\dot{a}}_{\;\;\;b}\Psi^{\dot{c}}_{\mu} + \sigma^{\mu\dot{c}}_{\;\;\;b}\Psi^{\dot{a}}_{\mu}) = 0 \; ,$$

$$\frac{\beta_3}{2}\sigma^{\lambda a}_{\;\;\;\dot{b}}(\partial^{\dot{b}}_{a} \sigma^{\mu}_{\;c\dot{d}}\Psi^{\dot{d}}_{\mu} + \partial^{\dot{b}}_{c} \sigma^{\mu}_{\;a\dot{d}}\Psi^{\dot{d}}_{\mu} + \frac{M}{2}\sigma^{\lambda a}_{\;\;\;\dot{b}}(\sigma^{\mu\dot{b}}_{a}\Psi_{\mu c} + \sigma^{\mu\dot{b}}_{c}\Psi_{\mu a}) = 0 \; .$$

These equation can be re-written as

$$\frac{\beta_3}{2}[-(\sigma^{\lambda\dot{a}b}\partial_{b\dot{a}})\sigma^{\mu\dot{c}d}\Psi_{\mu d} - \partial^{\dot{c}b}\sigma^{\lambda}_{b\dot{a}}\sigma^{\mu\dot{a}d}\Psi_{\mu d}] + \frac{M}{2}[-\sigma^{\lambda\dot{a}b}\sigma^{\mu}_{b\dot{a}}\Psi^{\dot{c}}_{\mu} - \sigma^{\mu\dot{c}b}\sigma^{\lambda}_{b\dot{a}}\Psi^{\dot{a}}_{\mu} = 0 \; ,$$

$$\frac{\beta_3}{2}[-(\sigma^{\lambda}_{a\dot{b}}\partial^{\dot{b}a})\sigma^{\mu}_{c\dot{d}}\Psi^{\dot{d}}_{\mu} - \partial_{c\dot{b}}\sigma^{\lambda\dot{b}a}\sigma^{\mu}_{a\dot{d}}\Psi^{\dot{d}}_{\mu}] + \frac{M}{2}[-(\sigma^{\lambda}_{a\dot{b}}\sigma^{\mu\dot{b}a})\Psi_{\mu c} - \sigma^{\mu}_{c\dot{b}}\sigma^{\lambda\dot{b}a}\Psi_{\mu a}] = 0 \; ,$$

which after simple transformation may be presented in vector-bispinor form as

$$\beta_3\left[\partial_\lambda(\gamma_\mu\Psi_\mu) - \frac{1}{4}\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu)\right] + M\left[\Psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu)\right] = 0 \; . \qquad (39)$$

Thus, we have arrived the the following set of equation, describing particle with one value of spin and two mass states (the first equation is multiplied by $\frac{1}{4}\gamma_\lambda$)

$$\frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu\psi_\mu) + \frac{M}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu) = 0 \; , \qquad (40)$$

$$\beta_3\left[\partial_\lambda(\gamma_\mu\Psi_\mu) - \frac{1}{4}\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu)\right] + M\left[\Psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu)\right] = 0 \; . \qquad (41)$$

Summing equations in (40)–(41), we get

$$M\Psi_\lambda + \beta_3\partial_\lambda(\gamma_\mu\Psi_\mu) + \frac{1}{4}\left(c_1 + \frac{\beta_2}{2} - \beta_3\right)\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu\Psi_\mu) = 0 \qquad (42)$$

We notice that multiplying this equation by $\frac{1}{4}\gamma_\lambda$, we get

$$\frac{1}{4}M\gamma_\mu\Psi_\mu + \frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}(\partial_\mu\Psi_\mu) = 0 \; ,$$

which is equivalent to the above equation (40):

$$\frac{1}{4}\left(c_1 + \frac{\beta_2}{2}\right)\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu) - \frac{\beta_2}{2}\gamma_\lambda(\partial_\mu\psi_\mu) + \frac{M}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu) = 0 \; /$$

This means that in fact we have only one independent equation (42), whereas eq. (40) is just its consequence. Let us re-write this main equation (42) differently

$$\beta_3\gamma_\mu\, \partial_\rho\Psi_\mu + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\nu\, \partial_\mu\Psi_\nu - \frac{\beta_2}{2}\gamma_\rho\, \partial_\mu\Psi_\mu + M\Psi_\rho = 0\ , \tag{43}$$

where the shortening notation $\beta_i$ for coefficients is used

$$\beta_1 = c_1 + \frac{\beta_2}{2} - \beta_3, \quad \beta_2 = \sqrt{\frac{2}{3}}\, c_2, \quad \beta_3 = -\delta\sqrt{\frac{2}{3}}\, c_2^*\ . \tag{44}$$

Equation (43) may be presented symbolically as $(\Gamma_\mu\partial_\mu + m)\Psi = 0$ , its more detailed form is

$$\left\{\left[\beta_3\gamma_\nu \otimes (e^{\mu,\nu})_{\rho,\sigma} + \frac{\beta_1}{4}\gamma_\lambda\gamma_\mu\gamma_\nu \otimes (e^{\lambda,\nu})_{\rho,\sigma} - \frac{\beta_2}{2}\gamma_\nu \otimes (e^{\nu,\mu})_{\rho,\sigma}\right]\frac{\partial}{\partial x^\mu} + M\delta_{\rho,\sigma}\right\}\Psi_\sigma = 0\ . \tag{45}$$

## 5. Transforming the wave equation to the Petras structure

The structure of the matrices $\Gamma_\mu$ in eq. (45)

$$\beta_3\gamma_\nu \otimes e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\lambda \otimes e^{\rho,\lambda} - \frac{\beta_2}{2}\gamma_\nu \otimes e^{\nu,\mu} \tag{46}$$

is rather complicated: it includes the triple products of the Dirac matrices. There exist special transformation which reduces the wave equation to a form without such triple products. The new basis and respective wave equation should have the structure

$$\Psi' = R\Psi, \quad \Gamma'_\mu = R\Gamma_\mu R^{-1},$$

$$R = I\otimes I + a\gamma_\rho\gamma_\sigma \otimes e^{\rho,\sigma}, \quad R^{-1} = I\otimes I + b\gamma_\rho\gamma_\sigma\otimes e^{\rho,\sigma}, \quad b = -\frac{1}{1+4a}\ . \tag{47}$$

Let us find

$$R\Gamma_\mu = \left[\beta_3\gamma_\nu\otimes e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\mu\gamma_\lambda \otimes e^{\rho,\lambda} - \frac{\beta_2}{2}\gamma_\nu\otimes e^{\nu,\mu}\right] +$$

$$+a\left[\beta_3\gamma_\rho\gamma_\sigma\gamma_\nu \otimes e^{\rho,\sigma}e^{\mu,\nu} + \frac{\beta_1}{4}\gamma_\rho\gamma_\sigma\gamma_\eta\gamma_\mu\gamma_\lambda\otimes e^{\rho,\sigma}e^{\eta,\lambda} - \frac{\beta_2}{2}\gamma_\rho\gamma_\sigma\gamma_\nu\otimes e^{\rho,\sigma}e^{\nu,\mu}\right].$$

Taking into account identities $e^{\rho,\sigma}e^{\mu,\nu} = \delta_{\sigma\mu}e^{\rho,\nu}$, $\gamma_\rho\gamma_\rho = 4$ we get

$$R\Gamma_\mu = \beta_3\gamma_\nu\otimes e^{\mu,\nu} - \beta_2(\frac{1}{2}+2a)\gamma_\rho\otimes e^{\rho,\mu} + (\frac{\beta_1}{4} + a\beta_3 + a\beta_1)\gamma_\rho\gamma_\mu\gamma_\sigma\otimes e^{\rho,\sigma}\ . \tag{48}$$

Similarly, we derive the structure of the matrices $\Gamma'_\mu$:

$$R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho\otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho\otimes e^{\rho,\mu} +$$

$$+\left\{\frac{\beta_1}{4} + a(\beta_1+\beta_3) + b(\beta_1 - \frac{\beta_2}{2}) + 4ab(\beta_1 - \frac{\beta_2}{2} + \beta_3)\right\}\gamma_\rho\gamma_\mu\gamma_\sigma\otimes e^{\rho,\sigma}\ . \tag{49}$$

Due to (47) we have $4ab = -a - b$ , so relation (49) reduces to the form

$$R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} + \\ + \left\{ \frac{\beta_1}{4} + a\frac{\beta_2}{2} - b\beta_3 \right\} \gamma_\rho\gamma_\mu\gamma_\sigma \otimes e^{\rho,\sigma} \; . \tag{50}$$

Now, we demand that the coefficient at triple product of Dirac matrices vanishes, this results in

$$\Gamma'_\mu = R\Gamma_\mu R^{-1} = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} \; . \tag{51}$$

and parameters $a$ and $b$ must obey two constraints:

$$\frac{\beta_1}{4} + a\frac{\beta_2}{2} - b\beta_3 = 0 \quad \Longrightarrow \quad b = \frac{\beta_1 + 2a\beta_2}{4\beta_3}, \quad a + b + 4ab = 0 \; .$$

Excluding $b$, we get a quadratic equation for $a$:

$$a^2 + 2a\frac{\beta_2 + 2(\beta_1+\beta_3)}{8\beta_2} + \frac{\beta_1}{8\beta_2} = 0 \; ;$$

its roots are

$$a = \frac{-C \pm \sqrt{C^2 - 8\beta_1\beta_2}}{8\beta_2}, \quad C = \beta_2 + 2\beta_1 + 2\beta_3 \; . \tag{52}$$

Respective expressions for $b$ are:

$$b = \frac{1}{4\beta_3}(\beta_1 + 2\beta_2 a) = \frac{-(\beta_2 + 2\beta_3 - 2\beta_1) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{16\beta_3} \tag{53}$$

Now, turning to the formula (51),

$$\Gamma'_\mu = \beta_3(1+4b)\gamma_\rho \otimes e^{\mu,\rho} - \beta_2(\frac{1}{2}+2a)\gamma_\rho \otimes e^{\rho,\mu} + \tag{54}$$

we find

$$\beta_3(1+4b) = \frac{-(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{4} \equiv (+A+B) \; , \\ \beta_2(\frac{1}{2}+2a) = \frac{+(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{C^2 - 8\beta_1\beta_2}}{4} \equiv (-A+B) \; ; \tag{55}$$

notice that for $B$ we have two different in sign expressions (for definiteness, below use the variant with upper sign) Thus, in Petras basis, the matrices $\Gamma'_\mu$ nab be written in rather symmetric form

$$\Gamma'_\mu = (A+B)\gamma_\rho \otimes e^{\mu,\rho} + (A-B)\gamma_\rho \otimes e^{\rho,\mu} + M\Psi = 0 \; ; \tag{56}$$

correspondingly, the wave equation reads as

$$(A+B) \; \gamma_\rho \partial_\mu \Psi_\rho + (A-B) \; \gamma_\mu \; \partial_\rho \Psi_\rho + M \; \Psi_\mu = 0 \; . \tag{57}$$

## 6. On parametrization of possible mass values

Recall main notations for parameters:

$$M_1 = \frac{M}{\lambda_1} , \quad M_2 = \frac{M}{\lambda_2} , \quad M > 0 ,$$

$$\lambda_1 = \frac{c_1 + \sqrt{c_1^2 + 4\delta|c_2|^2}}{2}, \quad \lambda_2 = \frac{c_1 - \sqrt{c_1^2 + 4\delta|c_2|^2}}{2} , \tag{58}$$

$$\lambda_1\lambda_2 = -\delta|c_2|^2, \quad \lambda_1 + \lambda_2 = c_1, \quad \lambda_1 - \lambda_2 = \sqrt{c_1^2 + 4\delta|c_2|^2} ;$$

also

$$\beta_2 = \sqrt{\frac{2}{3}}\, c_2, \quad \beta_3 = -\delta\sqrt{\frac{2}{3}}\, c_2^*, \quad \beta_1 = (c_1 + \frac{\beta_2}{2} - \beta_3) ; \tag{59}$$

and parameters $A$ and $B$

$$\frac{1}{4}\left[-(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2}\,\right] \equiv +A + B ,$$
$$\frac{1}{4}\left[+(\beta_2 - 2\beta_1 - 2\beta_3) \pm \sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2}\,\right] \equiv -A + B . \tag{60}$$

First, we consider the model when

$$\delta = -1, \quad c_1 = \rho, \quad c_2 = c_2^* = \sigma, \tag{61}$$

in this case we have

$$\beta_2 = \sqrt{\frac{2}{3}}\sigma, \quad \beta_3 = \sqrt{\frac{2}{3}}\sigma , \quad \beta_1 = \rho - \frac{1}{2}\sqrt{\frac{2}{3}}\sigma ,$$
$$\lambda_1 = \frac{\rho + \sqrt{\rho^2 - 4\sigma^2}}{2} > 0, \quad \lambda_2 = \frac{\rho - \sqrt{\rho^2 - 4\sigma^2}}{2} > 0,$$
$$\lambda_1\lambda_2 = \sigma^2, \quad \lambda_1 + \lambda_2 = \rho, \quad \lambda_1 - \lambda_2 = \sqrt{\rho^2 - 4\sigma^2} \tag{62}$$

and

$$(\beta_2 - 2\beta_1 - 2\beta_3) = -2\rho , \quad (\beta_2 + 2\beta_1 + 2\beta_3) = 2(\rho + \sqrt{\frac{2}{3}}\sigma) ,$$
$$\sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2} = 2\sqrt{\rho^2 + \frac{4}{3}\sigma^2} . \tag{63}$$

So, expression for $A, B$ in terms of $\lambda_1, \lambda_2$ are

$$A = \frac{\rho}{2} = \frac{\lambda_1 + \lambda_2}{2},$$
$$B = \frac{\sqrt{\rho^2 + (4/3)\sigma^2}}{2} = \frac{\sqrt{(\lambda_1 + \lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2} . \tag{64}$$

The ration of two masses is

$$M_1 = \frac{M}{\lambda_1}, \quad M_4 = \frac{M}{\lambda_2}, \quad \frac{M_2}{M_1} = \frac{\lambda_1}{\lambda_2} = \frac{1 + \sqrt{1 - 4\sigma^2/\rho^2}}{1 - \sqrt{1 - 4\sigma^2/\rho^2}} . \tag{65}$$

It is convenient introduce an angular parametrization

$$\sin^2\gamma = \frac{4\sigma^2}{\rho^2}, \quad 4\sigma^2 \le \rho^2 , \quad \gamma \in (0, \frac{\pi}{2}) ; \tag{66}$$

then the ratio of the masses is given by

$$\frac{M_2}{M_1} = \frac{1+\cos\gamma}{1-\cos\gamma} = \frac{1}{\tan^2(\gamma/2)} \in (1, \infty) . \tag{67}$$

Now, let us examine second model, when

$$\delta = +1, \quad c_1 = \rho, \quad c_2 = c_2^* = \sigma, \tag{68}$$

for this case we have

$$\beta_2 = \sqrt{\frac{2}{3}}\sigma, \quad \beta_3 = -\sqrt{\frac{2}{3}}\sigma , \quad \beta_1 = \rho + \frac{3}{2}\sqrt{\frac{2}{3}}\sigma,$$
$$\lambda_1 = \frac{\rho + \sqrt{\rho^2 + 4\sigma^2}}{2}, \quad \lambda_2 = \frac{\rho - \sqrt{\rho^2+4\sigma^2}}{2} < 0,$$
$$\lambda_1\lambda_2 = -\sigma^2, \quad \lambda_1 + \lambda_2 = \rho, \quad \lambda_1 - \lambda_2 = \sqrt{\rho^2 + 4\sigma^2} ; \tag{69}$$

and

$$(\beta_2 - 2\beta_1 - 2\beta_3) = -2\rho , \qquad (\beta_2 + 2\beta_1 + 2\beta_3) = 2(\rho + \sqrt{\frac{2}{3}}\sigma) ,$$
$$\sqrt{(\beta_2 + 2\beta_1 + 2\beta_3)^2 - 8\beta_1\beta_2} = 2\sqrt{\rho^2 - \frac{4}{3}\sigma^2} . \tag{70}$$

So, expressions for $A, B$ in terms of $\lambda_1, \lambda_2$ are

$$A = \frac{\rho}{2} = \frac{\lambda_1 + \lambda_2}{2},$$
$$B = \frac{\sqrt{\rho^2 - (4/3)\sigma^2}}{2} = \frac{\sqrt{(\lambda_1+\lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2} . \tag{71}$$

The ration of two mass is

$$M_1 = \frac{M}{\lambda_1} > 0, \quad M_2 = \frac{M}{\lambda_2} < 0,$$

$$\frac{M_2}{M_1} = \frac{\lambda_1}{\lambda_2} = \frac{\rho + \sqrt{\rho^2+4\sigma^2}}{\rho - \sqrt{\rho^2+4\sigma^2}} = \frac{1+\sqrt{1+4\sigma^2/\rho^2}}{1-\sqrt{1+4\sigma^2/\rho^2}} < 0 . \tag{72}$$

Let us introduce the following parametrization

$$\sinh^2\Gamma = 4\sigma^2/\rho^2 , \quad \Gamma \in (0, \infty) ; \tag{73}$$

$$\frac{M_2}{M_1} = \frac{1+\cosh\Gamma}{1-\cosh\Gamma} = -\frac{1}{\tanh^2(\Gamma/2)} \in (-\infty, -1) . \tag{74}$$

Later we explain the meaning of one positive and one negative mass (for the model with $\delta = -1$) in the theory under consideration.

## 7. Separating independent components of the wave function

From this point, taking in mind further extension of the model to generally covariant case, we use metrical tensor in Minkowski space with signature $(+,-,-,-)$. Correspondingly, the wave equation (57) is written as

$$(A+B)\ \gamma^{\rho}\partial_{\mu}\Psi_{\rho} + (A-B)\ \gamma_{\mu}\ \partial^{\rho}\Psi_{\rho} + iM\ \Psi_{\mu} = 0 \tag{75}$$

where we use Dirac matrices are taken in spinor basis, and the wave function is the vector–bispinor

$$\Psi_{\rho}(x) = \begin{vmatrix} \xi_0(x) & \xi_1(x) & \xi_2(x) & \xi_3(x) \\ \eta_0(x) & \eta_1(x) & \eta_2(x) & \eta_3(x) \end{vmatrix}, \qquad \gamma^{\rho} = \begin{vmatrix} 0 & \bar{\sigma}^{\rho} \\ \sigma^{\rho} & 0 \end{vmatrix}. \tag{76}$$

First, we convolute eq. (75) with $\gamma^{\nu}$:

$$4(A-B)(\partial^{\rho}\Psi_{\rho}) + [\ (A+B)\hat{\partial} + iM\ ](\gamma^{\rho}\Psi_{\rho}) = 0\ , \quad \hat{\partial} = \gamma^{\nu}\partial_{\nu}\ , \tag{77}$$

and second, we act on eq. (75) by operator $\partial^{\nu}$:

$$(A+B)\Box(\gamma^{\rho}\Psi_{\rho}) + [\ (A-B)\hat{\partial} + iM\ ](\gamma^{\rho}\Psi_{\rho}) = 0\ , \quad \Box = \partial^{\nu}\partial_{\nu}\ . \tag{78}$$

In order to exclude the term with operator $\Box$, we should make additional calculation. Let us act on eq. (77) by operator $\hat{\partial}$; taking into account the following identity

$$\hat{\partial}\hat{\partial} = \gamma^{\alpha}\partial_{\alpha}\gamma^{\beta}\partial_{\beta} = \partial_{\alpha}\partial_{\beta}\left(\frac{\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}}{2} + \frac{\gamma^{\alpha}\gamma^{\beta}-\gamma^{\beta}\gamma^{\alpha}}{2}\right) = \Box$$

we produce

$$(A+B)\Box(\gamma_{\rho}\Psi_{\rho}) = -4(A-B)\hat{\partial}(\gamma_{\rho}\Psi_{\rho}) - iM\hat{\partial}(\gamma_{\rho}\Psi_{\rho})\ . \tag{79}$$

Now, we can substitute the relation into (78):

$$[\ 3(A-B)\hat{\partial} - iM\ ]\ (\partial^{\rho}\Psi_{\rho}) + iM\hat{\partial}(\gamma^{\rho}\Psi_{\rho}) = 0\ , \tag{80}$$

additionally let us write down eq. (77) as well

$$[\ (A+B)\hat{\partial} + iM\ ]\ (\gamma^{\rho}\Psi_{\rho}) + 4(A-B)(\partial^{\rho}\Psi_{\rho}) = 0\ . \tag{81}$$

It is convenient to introduce special notation for two bispinor functions:

$$(\gamma^{\rho}\Psi_{\rho}) = \Phi_1, \qquad (\partial^{\rho}\Psi_{\rho}) = \Phi_2\ , \tag{82}$$

then the system (80)–(81) is written as follows

$$[\ 3(A-B)\hat{\partial} - iM\ ]\Phi_2 + iM\hat{\partial}\Phi_1 = 0\ , \tag{83}$$

$$[\ (A+B)\hat{\partial} + iM\ ]\ \Phi_1 + 4(A-B)\Phi_2 = 0\ . \tag{84}$$

Let us make some transformations over these two equation(83)–(84); multiply the first one by $4(A-b)$, and the second one – by $iM$, and sum up the results (the fist equation (83) of the system remains the same)4 in this way we arrive at the new system

$$(5A-3B)\hat{\partial}\ \Phi_1 + \frac{12}{iM}(A-B)^2\hat{\partial}\ \Phi_2 + iM\ \Phi_1 = 0\ , \tag{85}$$

$$-iM\hat{\partial}\ \Phi_1 - 3(A-B)\hat{\partial}\ \Phi_2 + iM\ \Phi_2 = 0\ . \tag{86}$$

The last system may be presented in the matrix form

$$\hat{\partial} \begin{vmatrix} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{vmatrix} \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} + iM \begin{vmatrix} \Phi_1 \\ \Phi_2 \end{vmatrix} = 0 \; . \tag{87}$$

The matrix $W$ in the equation

$$W = \begin{vmatrix} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{vmatrix}, \qquad \hat{\partial}\, W\Phi + iM\Phi = 0$$

is to be reduced to a diagonal form with the help of a linear transformation in 2-dimensional space:

$$\begin{vmatrix} \Phi'_1 \\ \Phi_2 \end{vmatrix} = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} \begin{vmatrix} \Phi'_1 \\ \Phi_2 \end{vmatrix}, \qquad \Phi' = S\Phi, \quad \hat{\partial}(SWS^{-1})\Phi' + iM\Phi' = 0 \; ;$$

further we get

$$SW = W'S \; , \quad W' = \begin{vmatrix} W_1 & 0 \\ 0 & W_2 \end{vmatrix},$$

or

$$\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} \begin{vmatrix} (5A-3B) & (12/iM)(A-B)^2 \\ -iM & -3(A-B) \end{vmatrix} = \begin{vmatrix} W_1 & 0 \\ 0 & W_2 \end{vmatrix} \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}.$$

This is equivalent to linear sub-systems:

$$[(5A-3B) - R_1]s_{11} - iMs_{12} = 0 \; ,$$

$$\frac{12}{iM}(A-B)^2 s_{11} + [-3(A-B) - R_1]s_{12} = 0 \; ;$$

$$[(5A-3B) - R_2]s_{21} - iMs_{22} = 0 \; ,$$

$$\frac{12}{iM}(A-B)^2 s_{21} + [-3(A-B) - R_2]s_{22} = 0 \; .$$

From vanishing the determinant of the matrix

$$\det \begin{vmatrix} (5A-3B) - W_1 & (12/iM)(A-B)^2 \\ -iM & -3(A-B) - W_2 \end{vmatrix} = 0$$

we obtain two eigenvalues $W_1$ and $W_2$:

$$W^2 - 2AW - 3(A^2 - B^2) = 0,$$

$$W_1 = A - \sqrt{4A^2 - 3B^2} \; , \quad W_2 = A + \sqrt{4A^2 - 3B^2}.$$

To fix elements of the matrix $S$, it suffices to use only one equation from each subsystem:

$$[(5A-3B) - R_1]s_{11} - iMs_{12} = 0 \; , \quad [(5A-3B) - R_2]s_{21} - iMs_{22} = 0 \; .$$

and a solution (one from possible) is

$$s_{11} = 1, \quad s_{12} = \frac{5A - 3B - W_1}{iM} = \frac{4A - 3B + \sqrt{4A^2 - 3B^2}}{iM} \; ;$$

$$s_{21} = 1, \qquad s_{22} = \frac{5A - 3B - W_2}{iM} = \frac{4A - 3B - \sqrt{4A^2 - 3B^2}}{iM} \; .$$

Thus, we have arrived to a couple of unlinked Dirac-like equations

$$(\gamma^{\nu}\partial_{\nu} + i\frac{M}{W_1})\Phi'_1 = 0 \; , \qquad (\gamma^{\nu}\partial_{\nu} + i\frac{M}{W_2})\Phi'_2 = 0 \tag{88}$$

these are what we need, because two identities are readily verified $W_1 \equiv \lambda_1, \; W_2 = \lambda_2$ . New (primed) bispinors relate to initial ones by the formulas

$$\Phi'_1 = \Phi_1 + \frac{5A - 3B - \lambda_1}{iM}\Phi_2, \qquad \Phi'_2 = \Phi_1 + \frac{5A - 3B - \lambda_2}{iM}\Phi_2 \; , \tag{89}$$

where $\Phi_1 = \gamma^{\rho}\Psi_{\rho}, \; \Phi_2 = \partial^{\rho}\Psi_{\rho}$. Initial wave equation (75), being written in the form

$$(A + B)\partial_{\nu}\Phi_1 + (A - B)\gamma_{\nu}\Phi_2 + iM\Psi_{\nu} = 0 \; , \tag{90}$$

provides us with possibility to determine the complete 16-component vector-bispinor $\Psi$ through known bispinors $\Phi_1$ and $\Phi_2$.

## 8. Interaction with external fields

We start with equation in Minkowski space

$$(A + B)\; \gamma^{\rho}\partial_{\nu}\Psi_{\rho} + (A - B)\; \gamma_{\nu}\; \partial^{\rho}\Psi_{\rho} + iM\; \Psi_{\nu} = 0 \; . \tag{91}$$

Extension to generally covariant case (we are to use the tetrad formalism [4]) and to presence of external electromagnetic field is done as shown

$$(A + B)\; D_{\nu}\gamma^{\rho}(x)\Psi_{\rho}(x) + (A - B)\; \gamma_{\nu}(x)D^{\rho}\Psi_{\rho} + iM\; \Psi_{\nu}(x) = 0 \; ; \tag{92}$$

we use notations

$$D_{\alpha} = \nabla_{\alpha} + \Gamma_{\alpha}(x) + ieA_{\alpha}(x) \; , \qquad \hat{D} = \gamma^{\alpha}(x)D_{\alpha}, \quad D^{\alpha}D_{\alpha} = \Box \; . \tag{93}$$

Note two commutation rules

$$\gamma^{\rho}(x)D_{\nu} = D_{\nu}\gamma^{\rho}(x) \; , \qquad D_{\sigma}g_{\alpha\beta}(x) = g_{\alpha\beta}(x)D_{\sigma} \; .$$

First, we convolute eq. (92) with $\gamma^{\nu}$:

$$4(A - B)(D^{\rho}\Psi_{\rho}) + [\; (A + B)\hat{D} + iM \;]\; (\gamma^{\rho}\Psi_{\rho}) = 0 \; . \tag{94}$$

Act on eq. (92) by operator $D^{\nu}$, this results in

$$(A + B)\Box(\gamma^{\rho}\Psi_{\rho}) + [\; (A - B)\hat{D} + iM \;](D^{\rho}\Psi_{\rho}) = 0 \; . \tag{95}$$

In order to exclude the term with operator $\Box$, we should make additional calculation. Let us act on eq. (94) by operator $\hat{D}$; taking into account the following identity

$$(A + B)\hat{D}\hat{D}(\gamma^{\rho}\Psi_{\rho}) = -4(A - B)\hat{D}(D^{\rho}\Psi_{\rho}) - iM\hat{D}(\gamma^{\rho}\Psi_{\rho}) \; . \tag{96}$$

Taking into account the identity; (see [4]; $F_{\alpha\beta}(x)$ stands for the tensor of external electromagnetic field $R(x)$ is the Ricci scalar)

$$\hat{D}\hat{D} = D_\alpha D_\beta \left[ \frac{\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha}{2} + \frac{\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha}{2} \right] =$$

$$= \Box - \Sigma(x) \; , \quad \Sigma(x) = \left\{ -ieF_{\alpha\beta}\sigma^{\alpha\beta}(x) + \frac{R}{4} \right\} , \tag{97}$$

we derive the formula

$$(A+B)\Box(\gamma^\rho\Psi_\rho) = -4(A-B)\hat{D}(D^\rho\Psi_\rho) - iM\hat{D}(\gamma^\rho\Psi_\rho) + (A+B)\Sigma(x)(\gamma^\rho\Psi_\rho) \; . \tag{98}$$

Using the last relation we may exclude form (95) the term with operator $\Box$.

In this way we obtain two equations

$$(A+B)\hat{D}\Phi_1 + iM\Phi_1 + 4(A-B)\Phi_2 = 0 \; , \tag{99}$$

$$3(A-B)\hat{D}\Phi_2 - iM\Phi_2 + iM\hat{D}\Phi_1 + (A+B)\Sigma(x)\Phi_1 = 0 \; ; \tag{100}$$

where the notation are used

$$\gamma^\rho(x)\Psi_\rho(x) = \Phi_1(x), \qquad D^\rho(x)\Psi_\rho(x) = \Phi_2(x) \tag{101}$$

In (99)-(100), let us multiply the second equation by $4(A-B)$, the first equation multiply by $iM$, and sum up the results; the second equation remains the same. In this way, we derive the system

$$\hat{D} \left\{ (5A-3B)\Phi_1 + \frac{12}{iM}(A-B)^2\Phi_2 \right\} + iM\Phi_1 + \frac{4(A^2-B^2)}{iM}\Sigma(x)\Phi_1 = 0,$$

$$\hat{D}\left\{ -iM\Phi_1 - 3(A-B)\Phi_2 \right\} + iM\Phi_2 - (A+B)\Sigma(x)\Phi_1 = 0. \tag{102}$$

In matrix form the system reads

$$\hat{D}W \left| \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right| + iM \times I_2 \left| \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right| + (A+B)\Sigma(x) \left| \begin{array}{cc} V_1 & 0 \\ V_2 & 0 \end{array} \right| \left| \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right| = 0 \; , \tag{103}$$

where

$$W = \left| \begin{array}{cc} (5A-3B) & 12(A-B)^2/iM \\ -iM & -3(A-B) \end{array} \right| , \quad V = \left| \begin{array}{cc} \beta & 0 \\ -1 & 0 \end{array} \right| = \left| \begin{array}{cc} \frac{4(A-B)}{iM} & 0 \\ -1 & 0 \end{array} \right| . \tag{104}$$

Linear transformation over function $\Phi_1, \Phi_2$, which reduces the system to a diagonal form, is known (see in previous section). Symbolically, the problem under consideration is presented as

$$\hat{D}W\Phi + iM \; I_2 \; \Phi + (A+B)\Sigma(x) \; V\Phi = 0, \quad \Phi' = S\Phi,$$

$$\hat{D}W'\Phi' + iM \; I_2 \; \Phi' + (A+B)\Sigma(x) \; V'\Phi' = 0,$$

$$S = \left| \begin{array}{cc} 1 & a \\ 1 & b \end{array} \right| , \quad S^{-1} = \frac{iM}{\lambda_1 - \lambda_2} \left| \begin{array}{cc} b & -a \\ -1 & 1 \end{array} \right| , \quad a = \frac{5A-3B-\lambda_1}{iM}, \; b = \frac{5A-3B-\lambda_2}{iM} \; ,$$

$$W' = SWS^{-1} = \left| \begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array} \right| , \quad V' = SVS^1 = \frac{iM}{\lambda_1 - \lambda_2} \left| \begin{array}{cc} b(\beta - a) & -a(\beta - a) \\ b(\beta - b) & -a(\beta - b) \end{array} \right| .$$

In this way, we arrive at the simplified system

$$(\lambda_1 \hat{D} + iM)\Phi'_1 + \Sigma(x)(A+B)\frac{iM}{\lambda_1 - \lambda_2}(\beta - a)(b\Phi'_1 - a\Phi'_2) = 0 \; ,$$
$$(\lambda_2 \hat{D} + iM)\Phi'_2 + \Sigma(x)(A+B)\frac{iM}{\lambda_1 - \lambda_2}(\beta - b)(b\Phi'_1 - a\Phi'_2) = 0 \; . \qquad (105)$$

Recall that

$$\Sigma(x) = -ieF_{\alpha\beta}(x)\sigma^{\alpha\beta}(x) + \frac{R(x)}{4} \; , \quad \beta = \frac{4(A-B)}{iM} \; .$$

Taking in mind identities

$$iM(\beta - a) = iM[\frac{4(A-B)}{iM} - \frac{5A - 3B - \lambda_1}{iM}] = \lambda_1 - A - B \; ,$$

$$iM(\beta - b) = iM[\frac{4(A-B)}{iM} - \frac{5A - 3B - \lambda_2}{iM}] = \lambda_2 - A - B \; ,$$

we may re-write equations (105) differently

$$(\lambda_1 \hat{D} + iM)\Phi'_1 + \Sigma(x)(A+B)\frac{\lambda_1 - A - B}{\lambda_1 - \lambda_2}(b\Phi'_1 - a\Phi'_2) = 0 \; ,$$
$$(\lambda_2 \hat{D} + iM)\Phi'_2 + \Sigma(x)(A+B)\frac{\lambda_2 - A - B}{\lambda_1 - \lambda_2})(b\Phi'_1 - a\Phi'_2) = 0 \; . \qquad (106)$$

Let us introduce shortening notations

$$(A+B)\frac{\lambda_1 - A - B}{\lambda_1(\lambda_1 - \lambda_2)} = \Lambda_1, \qquad (A+B)\frac{\lambda_2 - A - B}{\lambda_2(\lambda_2 - \lambda_1)} = \Lambda_2 \; ; \qquad (107)$$

then the above equations read

$$(i\hat{D} - M_1)\Phi'_1 + \Sigma(x)\Lambda_1 \, (b' \, \Phi'_1 - a' \, \Phi'_2) = 0 \; ,$$
$$(i\hat{D} - M_2)\Phi'_2 + \Sigma(x)\Lambda_2 \, (b' \, \Phi'_1 - a' \, \Phi'_2) = 0 \; , \qquad (108)$$

where

$$a' = \frac{5A - 3B - \lambda_1}{M}, \quad b' = \frac{5A - 3B - \lambda_2}{M} \; ,$$

In the end, let us note that equations (108) allow for restrictions to Majorana case. Indeed, in any Majorana basis for Dirac matrices, $(i\gamma^a)^* = +(i\gamma^a)$, and real (imaginary) bispinors are determined by the formulas

$$(\Phi'_1)^* = \pm \, (\Phi'_1), \quad (\Phi'_2)^* = \pm \, (\Phi'_2) \; . \qquad (109)$$

Because such fields correspond to neutral particles, the term with $F_{\alpha\beta}$ vanishes and we have identity $\Sigma^*(x) = +\Sigma(x)$. So we conclude that equations (108) for particles with two masses preserve their Lorentz-invariant form for neutral Majorana particles as well.

## 9. On solving the generalized wave equation

Let us discuss possible way of constructing solutions for derived equation. This equation may be written as follows The system (110) may be re-written differently

$$\Big[i\hat{D} - M_1 + b\Lambda_1\Sigma(x)\Big]\,\Psi_1(x) - a\Lambda_1\Sigma(x)\,\Psi_2(x) = 0\,,$$

$$\Big[i\hat{D} - M_2 - a\Lambda_2\Sigma(x)\Big]\,\Psi_2(x) + b\Lambda_2\Sigma(x)\,\Psi_1(x) = 0\,, \tag{110}$$

where parameters are determined by the formulas

$$\Lambda_1 = (A+B)\frac{\lambda_1 - A - B}{\lambda_1(\lambda_1-\lambda_2)}, \qquad \Lambda_2 = (A+B)\frac{\lambda_1 - A - B}{\lambda_2(\lambda_2-\lambda_2)}\,,$$

$$A = \frac{\lambda_1+\lambda_2}{2}, \qquad B = \frac{\sqrt{(\lambda_1+\lambda_2)^2 + (4/3)\lambda_1\lambda_2}}{2}\,,$$

$$a = \frac{5A - 3B - \lambda_1}{M}, \qquad b = \frac{5A-3B-\lambda_2}{M}\,.$$

The form (110) indicates a possible way to solve the system. We may apply the exclusion method. First, we are to get an inverse operator $\Sigma^{-1}(x)$ for operator $\Sigma(x)$. The system may be rewritten as

$$\frac{\Sigma^{-1}(x)}{a\Lambda_1}\Big(i\hat{D} - M_1 + b\Lambda_1\Sigma(x)\Big)\,\Psi_1(x) - \Psi_2(x) = 0\,,$$
$$\frac{\Sigma^{-1}(x)}{b\Lambda_2}\Big(i\hat{D} - M_2 - a\Lambda_2\Sigma(x)\Big)\,\Psi_2(x) + \Psi_1(x) = 0\,; \tag{111}$$

whence it follows equations for separated bispinors: functions

$$\Big[\Sigma(i\hat{D} - M_2 - a\Lambda_2\Sigma)\Sigma^{-1}(i\hat{D} - M_1 + b\Lambda_1\Sigma) + ab\Lambda_1\Lambda_2\Sigma^2\Big]\,\Psi_1(x) = 0\,, \tag{112}$$

$$\Big[\Sigma(i\hat{D} - M_1 + b\Lambda_1\Sigma)\Sigma^{-1}(i\hat{D} - M_2 - a\Lambda_2\Sigma) + ab\Lambda_1\Lambda_2\Sigma^2\Big]\,\Psi_2(x) = 0\,. \tag{113}$$

It would be desirable to get explicit solutions of such generalized wave equations in presence of some external fields: magnetic, electric, or gravitational ones.

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