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On reflecting the Dirac-Weyl-Majorana fermions by an effective medium generated by Lobachevsky geometry

E.M. Ovsiyuk, O. V. Veko, Ya.A. Voynova, V.V. Kisel, and V.M. Red'kov*

I.P.Shamiakin Mozyr State Pedagogical University

Gymnasium, Kalinkovichi

Secondary school, Yelsk region, Belarus

Belarusian State University of Informatics and Radioelectronics

B. I. Stepanov Institute of Physics of NAS of Belarus

Previously it was shown that in electrodynamic context the Lobachevsky geometry can simulate an effective medium acting as an ideal mirror, oriented perpendicularly to the axes z. In the present paper, an analogue of that effect is investigated for spin 1/2 fields. Solutions of the Dirac equation are constructed which describe waves in space which are reflected from an effective potential barrier without penetrating it. The depth of penetration into the medium is determined by characteristics of the quantum states and by the curvature radius of the Lobachevsky space; for waves with $k_1 = 0, k_2 = 0$ the effective reflecting barrier vanishes. Results are valid for Majorana fermions as well, some relevant details are specified. It is shown that for Weyl fermions, the reflecting effect vanishes. So, effects of non-Euclidean geometry can substantially depend on the type of a fermion.

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1. Introduction

Non-Euclidean geometry can be seen as a base for modeling effective medias in the electrodynamic context [1]. In particular, the Lobachevsky geometry while using quasi-Cartesian coordinates, effectively simulates an electrodynamic medium with the following constitutive law

$$D^i = \epsilon_0 \epsilon^{ik} E_k \;, \qquad B_i = \mu_0 \mu^{ik} H^k \;, \quad \epsilon^{ik}(x) = \mu^{ik}(x) = \left| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2z} \end{array} \right| ; \tag{1}$$

the medium is non-homogeneous along the direction z. The Maxwell flat space equations in such a medium may be reduced [2, 3] in the end to a single differential equation of the

*E-mail: e.ovsiyuk@mail.ru, vekoolga@mail.ru, voinyuschka@mail.ru, redkov@dragon.bas-net.by

form

$$\left(\frac{d^2}{dz^2} + \epsilon - U(z)\right) \varphi(z) = 0 \tag{2}$$

that can be associated with a Schrödinger-like equation with potential $U(z) = (a^2+b^2)e^{2z}$. This potential describes effective repulsing force acting on a 'particle':

$$F_z = -2(a^2 + b^2)e^{2z}.$$

In the context of quantum mechanics, that equation describes the motion of a particle in potential field tenting to infinity by exponential law, the particle is reflected by this potential without penetrating through it.

Thus, the Lobachevsky geometry effectively acts as (spreading in space) an ideal mirror. The depth z_0 of the penetration into that medium is determined by parameters of solutions and by the curvature radius of the Lobachevsky space [2], [3]. Note that at $a = k_1 = 0,\ b = k_2 = 0$ the barrier vanishes.

That analysis was extended [4] to the case of non-relativistic scalar particle; the main reflection features are the same. Some preliminary and non-complete study was performed in [5–7] for a relativistic Dirac particle: formal solutions in Lobachevsky space were constructed in terms of confluent hypergeometric functions though the reflection effect was not explicitly described [5–7].

In the present paper, the effects of Lobachevsky geometry are investigated for three types of spin 1/2 particle: Weyl, Dirac, and Majorana's. It is proved the effect of reflection for Dirac and Majorana particles and is demonstrated the absence of such an effect for Weyl particle.

2. Majorana spinor field

Let us fix the Majorana basis by the following transformation [8] from spinor one:

$$\Psi_M = A\ \Psi\ ,\ \Gamma^a_M = A\gamma^a A^{-1}\ , \quad A = \frac{1-\gamma^2}{\sqrt{2}}\ ,\ A^{-1} = \frac{1+\gamma^2}{\sqrt{2}};$$
$$\gamma^0_M = \gamma^0\gamma^2\ , \quad \gamma^1_M = \gamma^1\gamma^2\ , \quad \gamma^2_M = \gamma^2, \quad \gamma^3_M = \gamma^3\gamma^2\ . \tag{3}$$

Explicitly, the matrices are given by

$$\gamma^0_M = \left|\begin{array}{cccc} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{array}\right|,\ \gamma^1_M = \left|\begin{array}{cccc} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{array}\right|,\ \gamma^2_M = \left|\begin{array}{cccc} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{array}\right|,\ \gamma^3_M = \left|\begin{array}{cccc} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{array}\right|; \tag{4}$$

these matrices are purely imaginary. Therefore, in this representation, the Dirac wave operator becomes explicitly real

$$\left(i\gamma^a \frac{\partial}{\partial x^a} - m\right) \Psi_M = 0;$$

in other words there exist independent equations for real and imaginary parts of the Dirac wave function:

$$\Psi_M = \mathrm{Re}\,\Psi + i\ \mathrm{Im}\,\Psi = \Psi_+ + \Psi_-,$$

$$\left(i\gamma^a \frac{\partial}{\partial x^a} - m \right) \Psi_+ = 0 \, , \quad \left(i\gamma^a \frac{\partial}{\partial x^a} - m \right) \Psi_- = 0 \, ; \tag{5}$$

they describe so-called Majorana fermions, Ψ_+ and Ψ_-, with charge parity respectively $+1$ and -1.

For six generators

$$\sigma^{ab} = \frac{1}{4}(\gamma^a\gamma^b - \gamma^b\gamma^a) \equiv \frac{1}{2}\gamma^a\gamma^b, \quad \sigma^{ab} = -\sigma^{ba} \; ;$$

we have

$$\sigma^{01} = \frac{1}{2}\left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array} \right| , \; \sigma^{02} = \frac{1}{2}\left| \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| , \; \sigma^{03} = \frac{1}{2}\left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right| ,$$

$$\sigma^{12} = \frac{1}{2}\left| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| , \; \sigma^{13} = \frac{1}{2}\left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right| , \; \sigma^{23} = \frac{1}{2}\left| \begin{array}{cccc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right| . \tag{6}$$

As known, bispinor Lorentz transformations in an arbitrary basis are determined by the formula (for more details see in [8])

$$S(k,\; k^*) = \frac{1}{2}(k_0 + k_0^*) - \frac{1}{2}(k_0 - k_0^*)\gamma^5 \; + k_1(\sigma^{01} + i\sigma^{23}) + k_1^*(\sigma^{01} - i\sigma^{23}) +$$

$$+ k_2(\sigma^{02} + i\sigma^{31}) + k_2^*(\sigma^{02} - i\sigma^{31}) \; + k_3(\sigma^{03} + i\sigma^{12}) + k_3^*(\sigma^{03} - i\sigma^{12});$$

where complex 4-vector parameter k_a is used. With the notation $k_a = m_a - in_a$, the previous formula reads

$$S(m_a, n_a) = (m_0 + n_0 i\gamma^5) + (m_1\sigma^{01} + m_2\sigma^{02} + m_3\sigma^{03}) + (n_1\sigma^{23} + n_2\sigma^{31} + n_3\sigma^{12}) \; . \tag{7}$$

We see that in Majorana basis this bispinor transformations are real, so the Majorana particle are Lorentz invariant objects.

To describe interaction of the Majorana particles with gravitational fields it suffices to restrict the Dirac covariant equation to Majorana components in any chosen Majorana basis:

$$\{ \, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x)) - m \} \; \Psi(x) = 0 \, ,$$

$$\gamma^{\alpha}(x) = \gamma^a \, e^{\alpha}_{(a)}(x) \, , \quad \Gamma_{\alpha}(x) = \frac{1}{2} \, \sigma^{ab} \, e^{\beta}_{(a)} \, \nabla_{\alpha} \, (\, e^{\alpha}_{(b);\beta} \,) \tag{8}$$

Due to the properties

$$(i \, \gamma^a_M)^* = +\gamma^a_M \, , \quad (\sigma^{ab}_M)^* = +\sigma^{ab}_M \, , \quad (i \, \gamma^5_M)^* = +\gamma^5_M \, .$$

the covariant Dirac wave operator is real

$$[\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]^* = [\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,] \; ; \tag{9}$$

this means existence independent equations for both Majorana components. Ψ_+ and Ψ_-:

$$[\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]\Psi_+ = 0 \, , \tag{10}$$

$$[\, i\gamma^{\alpha}(x) \, (\, \partial_{\alpha} \, + \, \Gamma_{\alpha}(x) \,) \, - \, m \,]\Psi_- = 0 \, . \tag{11}$$

3. Separating the variables

Let us consider the spin 1/2 particle on the background of Lobachevsky geometry in quasi-Cartesian coordinates (t, x, y, z are dimensionless; it is convenient to start with the Dirac case):

$$dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 \ , \quad \sqrt{-g} = e^{-2z} \ , \quad z \in (-\infty, +\infty). \tag{12}$$

The most simple form of the covariant Dirac equation in orthogonal coordinates is (see in [8])

$$\left[i\gamma^a \, \left(e^{\alpha}_{(a)} \frac{\partial}{\partial x^\alpha} + \frac{1}{2} (\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} e^{\alpha}_{(a)}) \right) - m \right] \Psi(x) = 0 \ ; \tag{13}$$

in the diagonal tetrad $e^{\beta}_{(a)} = \text{diag}\,(1, e^z, e^z, 1)$, eq. (13) takes the form

$$\left[i\gamma^0 \frac{\partial}{\partial t} + i\gamma^1 e^z \frac{\partial}{\partial x} + i\gamma^2 e^z \frac{\partial}{\partial y} + i\gamma^3 \left(\frac{\partial}{\partial z} - 1 \right) - m \right] \Psi = 0 \ . \tag{14}$$

There exist three operators: $i\partial_t, \ -i\partial_x, \ -i\partial_y$ commuting with the wave operator in (14); so solutions are searched in the form

$$\Psi^{\epsilon, k_1, k_2} = e^{-i\epsilon t} \ e^{ik_1 x} \ e^{ik_2 y} \ \left| \begin{array}{c} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{array} \right| . \tag{15}$$

With the use of the spinor representation for Dirac matrices, we get 4 equations for $f_i(z), \ i = 1, 2, 3, 4$ (we simplify notation: $k_1 = a, k_2 = b$; also it is convenient to separate the simple factor: $f_i = e^z F_i$):

$$\begin{aligned} & - i\epsilon F_3 - iae^z F_4 - be^z F_4 - \frac{d}{dz} F_3 + im \ F_1 = 0 \ , \\ & -i\epsilon F_4 - iae^z F_3 + be^z F_3 + \frac{d}{dz} F_4 + im \ F_2 = 0 \ , \\ & -i\epsilon F_1 + iae^z F_2 + be^z F_2 + \frac{d}{dz} F_1 + imF_3 = 0 \ , \\ & -i\epsilon F_2 + iae^z F_1 - be^z F_1 - \frac{d}{dz} F_2 + imF_4 = 0 \ . \end{aligned} \tag{16}$$

There exists yet other commuting (generalized helicity) operator

$$\Sigma = \frac{1}{2} \ (e^z \gamma^2 \gamma^3 \frac{\partial}{\partial x} + e^z \gamma^3 \gamma^1 \frac{\partial}{\partial y} + \gamma^1 \gamma^2 \frac{\partial}{\partial z}) \ . \tag{17}$$

From eigenvalue equation $\Sigma \ \Psi = p \ \Psi$ we get

$$\begin{aligned} & ae^z F_2 - ibe^z F_2 - i\frac{d}{dz} F_1 = pF_1 \ , \quad ae^z F_1 + ibe^z F_1 + i\frac{d}{dz} F_2 = pF_2 \ , \\ & ae^z F_4 - ibe^z F_4 - i\frac{d}{dz} F_3 = pF_3 \ , \quad ae^z F_3 + ibe^z F_3 + i\frac{d}{dz} F_4 = pF_4 \ . \end{aligned} \tag{18}$$

Considering two set of equations on F_i jointly, we derive algebraic equations for F_i:

$$- i\epsilon F_3 - ipF_3 + imF_1 = 0 \,, \quad -i\epsilon F_4 - ipF_4 + imF_2 = 0 \,,$$
$$-i\epsilon F_1 + ipF_1 + imF_3 = 0 \,, \quad -i\epsilon F_2 + ipF_2 + imF_4 = 0 \,. \qquad (19)$$

Further we get two values for p and respective linear restrictions on F_i:

$$p = \pm\sqrt{\epsilon^2 - m^2} \,, \qquad F_3 = \frac{\epsilon - p}{m} F_1 \,, \qquad F_4 = \frac{\epsilon - p}{m} F_2 \,. \qquad (20)$$

Taking into account (20), instead of four equations (16) we get only two ones:

$$(\frac{d}{dz} - ip)F_1 + ie^z(a - ib)F_2 = 0 \,, \qquad (\frac{d}{dz} + ip\;)F_2 - ie^z(a + ib)F_1 = 0 \,; \qquad (21)$$

the corresponding wave functions are

$$\Psi^{\epsilon,a,b,\lambda} = e^{-i\epsilon t}\; e^{iax}\; e^{iby} e^z \left| \begin{array}{c} F_1(z) \\ F_2(z) \\ \lambda F_1(z) \\ \lambda F_2(z) \end{array} \right| , \quad \lambda = \frac{\epsilon - p}{m},\; p = \pm\sqrt{\epsilon^2 - m^2}. \qquad (22)$$

Let us detail transition to Weyl fermions. In accordance with spinor structure of the Dirac wave function

$$\psi(x) = \left| \begin{array}{c} \xi(x) \\ \eta(x) \end{array} \right| , \quad \xi(x) = \left| \begin{array}{c} \xi^1(x) \\ \xi^2(x) \end{array} \right| , \quad \eta(x) = \left| \begin{array}{c} \eta_{\dot 1}(x) \\ \eta_{\dot 2}(x) \end{array} \right| , \qquad (23)$$

we obtain substitutions Weyl 2-spinor ξ (anti-neutrino) and η (neutrino):

$$\xi = e^{-i\epsilon t}\; e^{iax}\; e^{iby} e^z \left| \begin{array}{c} F_1(z) \\ F_2(z) \end{array} \right| , \quad \eta = e^{-i\epsilon t}\; e^{iax}\; e^{iby} e^z \left| \begin{array}{c} F_3(z) \\ F_4(z) \end{array} \right| . \qquad (24)$$

Correspondingly, we have two independent subsystems:

$$- i\epsilon F_3 - iae^z F_4 - be^z F_4 - \frac{d}{dz} F_3 = 0, \quad -i\epsilon F_4 - iae^z F_3 + be^z F_3 + \frac{d}{dz} F_4 = 0 \,, \quad (25)$$

and

$$- i\epsilon F_1 + iae^z F_2 + be^z F_2 + \frac{d}{dz} F_1 = 0, \quad -i\epsilon F_2 + iae^z F_1 - be^z F_1 - \frac{d}{dz} F_2 = 0 \,. \quad (26)$$

Helicity operator is diagonalised on these (Weyl's) subsystems as follows:

$$- i\epsilon F_3 - ipF_3 = 0 \,, \;\; -i\epsilon F_4 - ipF_4 F_2 = 0 \quad \Longrightarrow \quad p = -1 \,; \qquad (27)$$

$$- i\epsilon F_1 + ipF_1 = 0 \,, \;\; -i\epsilon F_2 + ipF_2 = 0 \quad \Longrightarrow \quad p = +1 \,; \qquad (28)$$

neutrino and anti-neutrino are eigenfunctions of the helicity operator with opposite eigenvalues.

Now, let us consider transition to Majorana particles. Decomposition of quasi-flat Dirac waves into the sum of Majorana's waves $\Psi = \Psi_+ + \Psi_-$ in spinor basis is given by the formulas

$$\Psi^c = \gamma^2 \Psi^* = \left| \begin{array}{cc} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{array} \right| \left| \begin{array}{c} \xi^* \\ \eta^* \end{array} \right| = \left| \begin{array}{c} -\sigma^2 \eta^* \\ \sigma^2 \xi^* \end{array} \right| , \quad \Psi = \Psi_+ + \Psi_- = \frac{\Psi + \Psi^c}{2} + \frac{\Psi - \Psi^c}{2},$$

$$\Psi_{+} = \begin{vmatrix} \xi_{+} = (\xi - \sigma_2 \eta^*)/2 \\ \eta_{+} = (\eta + \sigma_2 \xi^*)/2 \end{vmatrix} , \qquad \Psi_{-} = \begin{vmatrix} \xi_{-} = (\xi + \sigma_2 \eta^*)/2 \\ \eta_{-} = (\eta - \sigma_2 \xi^*)/2 \end{vmatrix} . \tag{29}$$

With shortening notation

$$\xi = \varphi \begin{vmatrix} F_1(z) \\ F_2(z) \end{vmatrix}, \quad \eta = \varphi \begin{vmatrix} \lambda F_1(z) \\ \lambda F_2(z) \end{vmatrix}, \quad \xi^* = \varphi^* \begin{vmatrix} F_1^*(z) \\ F_2^*(z) \end{vmatrix}, \quad \eta^* = \varphi^* \begin{vmatrix} \lambda F_1^*(z) \\ \lambda F_2^*(z) \end{vmatrix},$$

where

$$\varphi = e^{-i\epsilon t}\, e^{iax}\, e^{iby} e^{z} \ , \quad \varphi^* = (e^{-i\epsilon t}\, e^{iax}\, e^{iby} e^{z})^*,$$

we present needed decompositions in the form

$$\Psi_{+} = \begin{vmatrix} \varphi F_1 + i\varphi^* \lambda F_2^* \\ \varphi F_2 - i\varphi^* \lambda F_1^* \\ \varphi \lambda F_1 - i\varphi^* F_2^* \\ \varphi \lambda F_2 + i\varphi F_1^* \end{vmatrix}, \quad \Psi_{-} = \begin{vmatrix} \varphi F_1 - i\varphi^* \lambda F_2^* \\ \varphi F_2 + i\varphi^* \lambda F_1^* \\ \varphi \lambda F_1 + i\varphi^* F_2^* \\ \varphi \lambda F_2 - i\varphi^* F_1^* \end{vmatrix}.$$

These formulas for Majorana components are referred to spinor basis. The Majorana nature of these solutions becomes the most evident after translating the formulas to Majorana basis by the following rule

$$\Psi^{M}_{\pm} = \frac{1 - \gamma^2}{\sqrt{2}} \Psi_{\pm} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{vmatrix} \Psi_{\pm} \ . \tag{30}$$

In this way, we get

$$\Psi_{+} = \begin{vmatrix} (\varphi F_1 + i\varphi^* \lambda F_2^*) - i(\varphi \lambda F_2 + i\varphi F_1^*) \\ (\varphi F_2 - i\varphi^* \lambda F_1^*) + i(\varphi \lambda F_1 - i\varphi^* F_2^*) \\ i(\varphi F_2 - i\varphi^* \lambda F_1^*) + (\varphi \lambda F_1 - i\varphi^* F_2^*) \\ -i(\varphi F_1 + i\varphi^* \lambda F_2^*) + (\varphi \lambda F_2 + i\varphi F_1^*) \end{vmatrix} = \begin{vmatrix} \text{Re}\, \varphi F_1 + \lambda\, \text{Im}\, \varphi F_2 \\ \text{Re}\, \varphi F_2 - \lambda\, \text{Im}\, \varphi F_1 \\ \lambda\, \text{Re}\, \varphi F_1 - \text{Im}\, \varphi F_2 \\ \lambda\, \text{Re}\, \varphi F_2 + \text{Im}\, \varphi F_1, \end{vmatrix},$$

$$\Psi_{-} = \begin{vmatrix} (\varphi F_1 - i\varphi^* \lambda F_2^*) - i(\varphi \lambda F_2 - i\varphi^* F_1^*) \\ (\varphi F_2 + i\varphi^* \lambda F_1^*) + i(\varphi \lambda F_1 + i\varphi^* F_2^*) \\ i(\varphi F_2 + i\varphi^* \lambda F_1^*) + (\varphi \lambda F_1 + i\varphi^* F_2^*) \\ -i(\varphi F_1 - i\varphi^* \lambda F_2^*) + (\varphi \lambda F_2 - i\varphi^* F_1^*) \end{vmatrix} = i \begin{vmatrix} \text{Im}\, \varphi F_1 - \lambda\, \text{Re}\, \varphi F_2 \\ \text{Im}\, \varphi F_2 + \lambda\, \text{Re}\, \varphi F_1 \\ \lambda\, \text{Im}\, \varphi F_1 + \text{Re}\, \varphi F_2 \\ \lambda\, \text{Im}\, \varphi F_2 - \text{Re}\, \varphi F_1 \end{vmatrix}. \tag{31}$$

So, it suffices to find solutions of the Dirac equation, and then to restrict ourselves to Majorana particles.

4. Constructing and analyzing the Dirac solutions

Let us turn back to eqs. (21) and transform them to the variable $Z = e^z,\ Z \in (0, +\infty)$:

$$\left(\frac{d}{dZ} - \frac{ip}{Z} \right) F_1 + i(a - ib) F_2 = 0 \ , \ \left(\frac{d}{dZ} + \frac{ip}{Z} \right) F_2 - i(a + ib) F_1 = 0 \ . \tag{32}$$

These give two 2-nd order equations

$$\left(\frac{d^2}{dZ^2} + \frac{p^2 + ip}{Z^2} - a^2 - b^2 \right) F_1 = 0 \ , \quad \left(\frac{d^2}{dZ^2} + \frac{p^2 - ip}{Z^2} - a^2 - b^2 \right) F_2 = 0 \ . \tag{33}$$

We note the symmetry with respect to complex conjugation; together with any solution of (32), its conjugate will be a solution as well:

$$\left| \begin{array}{c} F_1 \\ F_2 \end{array} \right| , \qquad \left| \begin{array}{c} F_2^* \\ F_1^* \end{array} \right| . \tag{34}$$

Eqs. (33) both have one regular point $Z = 0$ and one irregular of the rank 2 in $Z = \infty$ (at $a^2 + b^2 \neq 0$); this means that we deal with the confluent hypergeometric equation. Because equations in (33) relate to each other by complex conjugation, it suffices to detail only one of them.

For $F_1(Z)$ we use the following substitution $F_1(Z) = Z^A e^{BZ} \bar{F}_1(Z)$, this results in

$$\bar{F}_1'' + \left(\frac{2A}{Z} + 2B \right) \bar{F}_1' + \frac{2AB}{Z} \bar{F}_1 + \left(\frac{A(A-1)}{Z^2} + \frac{p^2 + ip}{Z^2} \right) \bar{F}_1 + [B^2 - (a^2 + b^2)] \bar{F}_1 = 0 \; .$$

Let us fix parameters A and B as

$$A = +ip, \; 1 - ip, \quad B = \pm \sqrt{a^2 + b^2} \; ,$$

then we get

$$Z \bar{F}_1'' + (2A + 2BZ) \, \bar{F}_1' + 2AB \bar{F}_1 = 0 \; .$$

Without loss of generality, take the values $A = +ip, \; B = -\sqrt{a^2 + b^2}$. Translating the last equation to y:

$$2BZ = -y, \; y = +2\sqrt{a^2 + b^2} e^z \; ,$$

we obtain

$$y \frac{d^2}{dy^2} \bar{F}_1 + (2A - y) \frac{d}{dy} \bar{F}_1 - A \bar{F}_1 = 0 \; ,$$

it is the confluent hypergeometric equation

$$\Phi'' + (\gamma - y)\Phi' - \alpha \Phi = 0, \quad \alpha = A = ip, \quad \gamma = 2A = 2ip \; .$$

We may take the following two linearly independent solutions [9]:

$$\bar{F}_1^{(1)}(y) = \Phi(\alpha, \gamma; y) = \Phi(ip, 2ip; y) \; ,$$
$$\bar{F}_1^{(2)}(y) = y^{1-\gamma} \Phi(\alpha - \gamma + 1, 2 - \gamma; y) = y^{1-2ip} \Phi(1 - ip, 2 - 2ip; y) \; , \tag{35}$$

they provide us with two respective complete functions $F_1(Z) = Z^A e^{BZ} \bar{F}_1$:

$$F_1^{(1)} = y^{ip} e^{-y/2} \; \Phi(ip, 2ip; y) \; , \quad F_1^{(2)} = y^{1-ip} e^{-y/2} \; \Phi(1 - ip, 2 - 2ip; y) \; . \tag{36}$$

Now, applying the above mentioned symmetry, we obtain similar results for F_2:

$$F_2^{(1)} = y^{1+ip} e^{-y/2} \; \Phi(1 + ip, 2 + 2ip; y) \; , \qquad F_2^{(2)} = y^{-ip} e^{-y/2} \; \Phi(-ip, -2ip; y) \; . \tag{37}$$

It should be stressed that the question on relating four functions

$$\left\{ F_1^{(1)}, F_1^{(2)}; \;\; F_2^{(1)}, F_2^{(2)} \right\}$$

in pairs can be solved only with the use of the first order equations (32). We state the answer and after that prove them.

Below we will need to follow two possibilities depending on the sign of A: $+A, -A$):

$$I^+. \qquad F_1^{+(1)} = e^{-y/2} y^A \, \Phi(A, 2A, y) = f \; , \qquad F_2^{+(1)} = L e^{-y/2} y^{1+A} \, \Phi(1+A, 2+2A, y) = g \; ,$$

$$II^+. \qquad F_1^{+(2)} = L^* e^{-y/2} y^{1-A} \Phi(1-A, 2-2A, y) = g^* \; , \qquad F_2^{+(2)} = e^{-y/2} y^{-A} \Phi(-A, -2A, y) = f^* \; ; \tag{38}$$

$A \Longrightarrow -A$

$$I^-. \qquad F_1^{-(1)} = e^{-y/2} y^{-A} \, \Phi(-A, -2A, y) = f^* \; , \qquad F_2^{-(1)} = L^* e^{-y/2} y^{1-A} \, \Phi(1-A, 2-2A, y) = g^* \; ,$$

$$II^-. \qquad F_1^{-(2)} = L e^{-y/2} y^{1+A} \Phi(1+A, 2+2A, y) = g \; , \qquad F_2^{-(2)} = e^{-y/2} y^A \Phi(A, 2A, y) = f \; . \tag{39}$$

Let us find numerical relative coefficient L. The functions F_1, F_2 obey the first order equations

$$\left(\frac{d}{dy} - \frac{A}{y} \right) F_1 + \frac{e^{i\alpha}}{2} F_2 = 0 \; , \qquad \left(\frac{d}{dy} + \frac{A}{y} \right) F_2 + \frac{e^{-i\alpha}}{2} F_1 = 0 \; ;$$

where

$$e^{i\alpha} = i \frac{a - ib}{\sqrt{a^2 + b^2}}, \qquad e^{-i\alpha} = -i \frac{a + ib}{\sqrt{a^2 + b^2}} \; .$$

We are to prove that needed pairs are the following ones:

$$F_1^{+(1)} = e^{-y/2} y^A \, \Phi(A, 2A, y) = f \; , \quad F_2^{+(1)} = L e^{-y/2} y^{1+A} \, \Phi(1+A, 2+2A, y) = g \; ,$$

and

$$F_2^{+(2)} = e^{-y/2} y^{-A} \Phi(-A, -2A, y) = f^*, \quad F_1^{+(2)} = L^* e^{-y/2} y^{1-A} \Phi(1-A, 2-2A, y) = g^* \; .$$

In fact, it suffices to consider the first pair only. Substitution these two functions into the first equation

$$\left(\frac{d}{dy} - \frac{A}{y} \right) e^{-y/2} y^A \, \Phi(A, 2A, y) + \frac{e^{i\alpha}}{2} L e^{-y/2} y^{1+A} \, \Phi(1+A, 2+2A, y) = 0 \, ,$$

we get

$$-\frac{1}{2} e^{-y/2} y^A \Phi(A, 2A) + A e^{-y/2} y^{A-1} \Phi(A, 2A) +$$
$$+ e^{-y/2} y^A \frac{A}{2A} \Phi(A+1, 2A+1) - A e^{-y/2} y^{A-1} \, \Phi(A, 2A) +$$
$$+ \frac{e^{i\alpha}}{2} L e^{-y/2} y^{1+A} \, \Phi(1+A, 2+2A) = 0$$

or

$$-\Phi(A, 2A) + \Phi(A+1, 2A+1) + e^{i\alpha} L y \Phi(1+A, 2+2A) = 0 \; .$$

It is readily checked (by studying several first terms of the series) the identity

$$-\Phi(A, 2A) + \Phi(A+1, 2A+1) = x \frac{1}{2(2A+1)} \; \Phi(A+1, 2A+2) \; ;$$

so we arrive at a needed relationship

$$\frac{1}{2(2A+1)} + e^{i\alpha} L = 0 \quad \Longrightarrow \quad L = -\frac{e^{-i\alpha}}{2(2A+1)} = \frac{i}{2} \frac{1}{2A+1} \frac{a+ib}{\sqrt{a^2+b^2}} \; . \tag{40}$$

Now, let us describe asymptotic behavior of the functions $f(z)$ abd $g(z)$ at $z \to -\infty (y \to 0)$:

$$f \sim y^A = \left(2\sqrt{a^2+b^2} \right)^{ip} e^{ipz} \; ,$$
$$f^* \sim y^{-A} = \left(2\sqrt{a^2+b^2} \right)^{-ip} e^{-ipz} \; ; \tag{41}$$

$$g \sim L y^{1+A} = L \left(2\sqrt{a^2+b^2} \right)^{1+ip} e^{(1+ip)z} \to 0 \; ,$$
$$g^* \sim y^{-A} = L^* \left(2\sqrt{a^2+b^2} \right)^{1-ip} e^{(1-ip)z} \to 0 \; ; \tag{42}$$

In turn, with the use of the known asymptotic formula

$$\Phi(\alpha, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \, e^x (x)^{\alpha-\gamma} \; , \quad \text{Re}\, x \to +\infty$$

we obtain behavior of these functions at $z \to +\infty$ $(y =\to +\infty)$:

$$f \sim e^{-y/2} y^{ip} \, \frac{\Gamma(2ip)}{\Gamma(ip)} \, e^y y^{-ip} \to \infty \; , \quad f^* \to \infty \; ;$$

$$g \sim L e^{-y/2} y^{1-ip} \, \frac{\Gamma(2ip+2)}{\Gamma(ip+1)} \, e^y y^{1-ip} \to \infty \; , \quad g^* \to \infty \; .$$

The last two relations mean that constructed solutions f, g do not have needed behavior in the region $z \to +\infty$, so we cannot interpret them as referred to the reflection effect.

It should be noted that in the above listed solutions (38)-(39) we see evident symmetry (34). In fact, solutions of the type $(-)$ are conjugate to these of the type $(+)$. Difference between the types $(+)$ and $(-)$ is associated with two different states of polarizations for Dirac particle.

We will consider the function F_1 as **main** one; we will construct needed solutions for this main function, and then will find their counterparts F_2.

Above we have used quite definite pair of linearly independent solutions of the confluent hypergeometric equation (with tow possibilities for A: $+A, -A$)

$$Y^{+(1)} = \Phi(A, 2A, y) \; , \quad Y^{+(2)} = y^{1-2A} \Phi(1-A, 2-2A, y) \; ;$$
$$Y^{-(1)} = \Phi(-A, -2A, y) \; , \quad Y^{-(2)} = y^{1+2A} \Phi(1+A, 2+2A, y) \; . \tag{43}$$

To construct solutions with needed asymptotic, we need yet other two linearly independent solutions [9] (again with different A: $+A, -A$):

$$Y^{+(5)} = \Psi(A, 2A, y) \; , \quad Y^{+(7)} = e^y \Psi(A, 2A, -y) \; ;$$
$$Y^{-(5)} = \Psi(-A, -2A, y) \; , \quad Y^{-(7)} = e^y \Psi(-A, -2A, -y) \; . \tag{44}$$

These sets of solutions (43) - (44) relate to each other by the Kummer formulas [9]:

$$Y^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, Y^{+(1)} + \frac{\Gamma(2A-1)}{\Gamma(A)}\, Y^{+(2)} \;,$$

$$Y^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, Y^{+(1)} - \frac{\Gamma(2A-1)}{\Gamma(A)}\, Y^{+(2)} \;;$$

$$Y^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, Y^{-(1)} + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\, Y^{-(2)} \;,$$

$$Y^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, Y^{-(1)} - \frac{\Gamma(-2A-1)}{\Gamma(-A)}\, Y^{-(2)} \;.$$

After multiplying them by $y^A e^{-y/2}$ (an respectively by $y^{-A} e^{-y/2}$) we derive

$$F_1^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} f + \frac{\Gamma(2A-1)}{\Gamma(A)} \frac{1}{L^*} g^*,$$
$$F_1^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} f - \frac{\Gamma(2A-1)}{\Gamma(A)} \frac{1}{L^*} g^*; \qquad (45)$$

$$F_1^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g,$$
$$F_1^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} f^* - \frac{\Gamma(-2A-1)}{\Gamma(-A)} \frac{1}{L} g. \qquad (46)$$

The functions $F_1^{+(5)}$ and $F_1^{+(7)}$ at $z \longrightarrow -\infty$ behave as

$$F_1^{+(5)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} \left(2\sqrt{a^2+b^2}\right)^{+ip} e^{+ipz},$$
$$F_1^{+(7)} = \frac{\Gamma(1-2A)}{\Gamma(1-A)} \left(2\sqrt{a^2+b^2}\right)^{+ip} e^{+ipz} \;. \qquad (47)$$

The functions $F_1^{-(5)}$ and $F_1^{-(7)}$ at $z \longrightarrow -\infty$ behave as

$$F_1^{-(5)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz},$$
$$F_1^{-(7)} = \frac{\Gamma(1+2A)}{\Gamma(1+A)} \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz}. \qquad (48)$$

Now let us find asymptotic behavior of $F_1^{\pm(5)} z$ at $z \to +\infty$. Applying the known formulas [9]

$$Y_5 = \Psi(A, 2A, y) \sim y^{-A},$$

we get ($y \to +\infty$ ($z \to +\infty$))

$$F_1^{+(5)} = y^A e^{-y/2}\, y^{-A} \sim e^{-y/2} \sim \exp\left(-\sqrt{a^2+b^2} e^z\right) \longrightarrow 0 \;,$$
$$F_1^{-(5)} = y^{-A} e^{-y/2}\, y^{+A} \sim e^{-y/2} \sim \exp\left(-\sqrt{a^2+b^2} e^z\right) \longrightarrow 0 \;. \qquad (49)$$

Analogously, applying the formula

$$Y_7 = e^y \Psi(A, 2A, -y) \sim e^y y^{-A},$$

we find at $y \to +\infty$ $(z \to +\infty)$:

$$F_1^{+(7)} \sim y^A e^{-y/2}\, e^y y^{-A} \sim e^{+y/2} \sim \exp\left(+\sqrt{a^2+b^2}e^z\right) \longrightarrow \infty \;,$$
$$F_1^{-(7)} \sim y^{-A} e^{-y/2}\, e^y y^{+A} \sim e^{+y/2} \sim \exp\left(+\sqrt{a^2+b^2}e^z\right) \longrightarrow \infty \;. \tag{50}$$

The most interesting are solution of the type $\pm(5)$, because they tend to zero at $z \to$ whereas the behave as flat wave at $z \to -\infty$.

Let us define new combination of the solutions $F_1^{+(5)}$ and $F_1^{-(5)}$:

$$H_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} + \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} \;, \quad H^* = H \;;$$
$$G_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} - \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} \;, \quad G^* = -G \;. \tag{51}$$

They behave as follows

$$H_1(z\to-\infty) \sim \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, e^{+ipz} + \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, e^{-ipz} \;, \quad H_1(z\to+\infty) \sim 0 \;;$$
$$G_1(z\to-\infty) \sim \frac{\Gamma(1-2A)}{\Gamma(1-A)}\, e^{+ipz} - \frac{\Gamma(1+2A)}{\Gamma(1+A)}\, e^{-ipz} \;, \quad G_1(z\to+\infty) \sim 0 \;. \tag{52}$$

For such solutions we can define the concept of reflection coefficient

$$\psi \sim M_- e^{-ipz} \pm M_+ e^{+ipz}, \quad R = \left|\frac{M_-}{M_+}\right|^2 = \left|\frac{\Gamma(1+2A)}{\Gamma(1-2A)}\frac{\Gamma(1-A)}{\Gamma(1+A)}\right|^2 = 1; \tag{53}$$

remind that $A = +ip,\ A^* = -a$. With notation

$$\frac{\Gamma(1-2A)}{\Gamma(1-A)} = \rho + i\sigma, \qquad \frac{\Gamma(1+2A)}{\Gamma(1+A)} = \rho - i\sigma, \tag{54}$$

these two types of standing waves are described as follows:

$$H_1(z\to-\infty) = 2(\rho\cos pz - \sigma\sin pz) \;,$$
$$G_1(z\to-\infty) = 2i(\sigma\cos pz + \rho\sin pz) \;. \tag{55}$$

The first is real, the second is imaginary. The choice of complex (and conjugate) coefficients when constructing the functions F, G in (51) influences only the total amplitude od standing waves and their phase shifts.

Note that direct interpretation of the effect in terms of ' barrier - reflection' finds difficulty because in eqs. (33) we see the complex potentials:

$$\left(\frac{d^2}{dZ^2} + \frac{p^2+ip}{Z^2} - (a^2+b^2)\right) F_1 = 0 \;, \quad \left(\frac{d^2}{dZ^2} + \frac{p^2-ip}{Z^2} - (a^2+b^2)\right) F_2 = 0 \;. \tag{56}$$

The structure of these equations assumes relationship $F_2 = F_1^*$. With this fact in mind, we can derive for functions

$$H = cF_1 + c^* F_1^*, \quad G = cF_1 - c^* F_1^* \;.$$

other equations which contain real potentials. Indeed, we get

$$\left(\frac{d^2}{dZ^2} + \frac{p^2}{Z^2} - (a^2+b^2)\right) H + \frac{ip}{Z^2} G = 0 \, ,$$
$$\left(\frac{d^2}{dZ^2} + \frac{p^2}{Z^2} - (a^2+b^2)\right) G + \frac{ip}{Z^2} H = 0 \, . \tag{57}$$

In the coordinate $z = \ln Z$ they read (to eliminate term with the first derivative let us separate a special factor, $H \Longrightarrow e^{z/2} H$, $G \Longrightarrow e^{z/2} G$):

$$\left(\frac{d^2}{dz^2} + p^2 - \frac{1}{4} - (a^2+b^2) e^{2z}\right) H + ipG = 0 \, ,$$

$$\left(\frac{d^2}{dz^2} + p^2 - \frac{1}{4} - (a^2+b^2) e^{2z}\right) G + ipH = 0 \, .$$

We easily find the critical point z_0, on the right of with the function should fall dawn to zero:

$$p^2 - 1/4 = (k_1^2 + k_2^2)\, e^{2z_0} \qquad \Longrightarrow \qquad z_0 = \ln \sqrt{\frac{p^2 - 1/4}{a^2+b^2}} \; ; \tag{58}$$

In vicinity of this point z_0, equations become simpler:

$$\frac{d^2}{dz^2} H + ipG = 0 \, , \quad \frac{d^2}{dz^2} G + ipH = 0 \, . \tag{59}$$

Their solutions may have only exponential form:

$$H = M e^{\nu(z-z_0)}, \quad G = N e^{\nu(z-z_0)} \; ;$$

at this we obtain algebraic equations with four solutions

$$\left| \begin{array}{cc} \nu^2 & ip \\ ip & \nu^2 \end{array} \right| \left| \begin{array}{c} M \\ N \end{array} \right| = 0 \qquad \Longrightarrow \qquad \nu = -\frac{1 \pm i}{\sqrt{2}} \sqrt{p}, \; +\frac{1 \pm i}{\sqrt{2}} \sqrt{p} \, . \tag{60}$$

Two first with negative real parts are what we need

Note that in usual units the critical point z_0 is giveb=n by

$$z_0 = \rho \; \ln \sqrt{\frac{(E^2 - M^2c^4)/c^2\hbar^2 - 1/4\rho^2}{(K_1^2 + K_2^2)}} \; ; \tag{61}$$

where K_1, K_2 are the wave numbers; ρ stands for the curvature radius of the Lobachevsky space Note that when K_1, K_2 tends to zero, the depth of penetrating z_0 tends to infinity.

Having constructed the needed main functions – solutions of equations for F_1, let us find the form of relevant functions – solutions of equation for F_2. To this end, let us turn back to solutions:

$$H_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} + \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} = C F_1^{+(5)} + C^* F_1^{-(5)} \, ,$$
$$G_1 = \left(2\sqrt{a^2+b^2}\right)^{-ip} F_1^{+(5)} - \left(2\sqrt{a^2+b^2}\right)^{+ip} F_1^{-(5)} = C F_1^{+(5)} - C^* F_1^{-(5)} \, . \tag{62}$$

Substitute here expressions for $F_1^{\pm(5)}$:

$$H_1 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}f + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}g^*\right) + C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}g\right)$$

$$G_1 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}f + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}g^*\right) - C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}f^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}g\right) \tag{63}$$

Now, by these functions H_1, G_1 , we are to find relevant H_2, G_2. Because we have linear task – see (21):

$$H_1 \Longrightarrow H_2, \qquad \left(\frac{d}{dz} - ip\right)H_1 + ie^z(a-ib)H_2 = 0\ ;$$

$$G_1 \Longrightarrow G_2, \qquad \left(\frac{d}{dz} - ip\right)G_1 + ie^z(a-ib)G_2 = 0\ ,$$

we may use yet known results – see (38)-(39):

$$\begin{array}{lll} F_1^{+(1)} = f & \Longrightarrow & F_2^{+(1)} = g\ , \\ F_1^{+(2)} = g^* & \Longrightarrow & F_2^{+(2)} = f^*\ , \\ F_1^{-(1)} = f^* & \Longrightarrow & F_2^{-(1)} = g^*\ , \\ F_1^{-(2)} = g & \Longrightarrow & F_2^{-(2)} = f\ . \end{array}$$

In this way, we get

$$H_2 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right) + C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right),$$

$$G_2 = C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g + \frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right) - C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^* + \frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right). \tag{64}$$

In fact, transition from (63) to (64)reduced to the change $f \Longleftrightarrow g$.

Allowing for asymptotic behavior of f, g at $z \to -\infty$:

$$f \sim y^A = \left(2\sqrt{a^2+b^2}\right)^{ip} e^{ipz}\ ,$$

$$f^* \sim y^{-A} = \left(2\sqrt{a^2+b^2}\right)^{-ip} e^{-ipz}\ ; \tag{65}$$

$$g \sim Ly^{1+A} = L\left(2\sqrt{a^2+b^2}\right)^{1+ip} e^{(1+ip)z} \to 0\ ,$$

$$g^* \sim y^{-A} = L^*\left(2\sqrt{a^2+b^2}\right)^{1-ip} e^{(1-ip)z} \to 0\ ; \tag{66}$$

we establish behavior of H_2, G_2:

$$H_2(G_2) = \frac{C^2}{L^*}\frac{\Gamma(2A-1)}{\Gamma(A)}\ e^{-ipz} + (-)\frac{C^{*2}}{L}\frac{\Gamma(-2A-1)}{\Gamma(-A)}\ e^{+ipz}\ .$$

Here, we see standing waves of two types:

$$H_2(z \to -\infty) = 2(\rho' \cos pz - \sigma' \sin pz)\ ,$$

$$G_2(z \to -\infty) = 2i(\sigma' \cos pz + \rho' \sin pz)\ . \tag{67}$$

5. Studying the Majorana case

To get results for Majorana particles, we start with the formulas (31)

$$\Psi_{+}=\left|\begin{array}{c}\text{Re }\varphi F_{1}+\lambda\text{ Im }\varphi F_{2}\\ \text{Re }\varphi F_{2}-\lambda\text{ Im }\varphi F_{1}\\ \lambda\text{ Re }\varphi F_{1}-\text{Im }\varphi F_{2}\\ \lambda\text{ Re }\varphi F_{2}+\text{Im }\varphi F_{1},\end{array}\right|,\quad \Psi_{-}=i\left|\begin{array}{c}\text{Im }\varphi F_{1}-\lambda\text{ Re }\varphi F_{2}\\ \text{Im }\varphi F_{2}+\lambda\text{ Re }\varphi F_{1}\\ \lambda\text{ Im }\varphi F_{1}+\text{Re }\varphi F_{2}\\ \lambda\text{ Im }\varphi F_{2}-\text{Re }\varphi F_{1}\end{array}\right|, \tag{68}$$

in which we should follow two different solutions (H_1,H_2) and G_1,G_2):

$$\begin{gathered}F_1=H_1=C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}f+\frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}g^*\right)+C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}f^*+\frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}g\right),\\ F_2=H_2=C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g+\frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right)+C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^*+\frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right);\\ F_1=G_1=C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}f+\frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}g^*\right)-C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}f^*+\frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}g\right),\\ F_2=G_2=C\left(\frac{\Gamma(1-2A)}{\Gamma(1-A)}g+\frac{\Gamma(2A-1)}{\Gamma(A)}\frac{1}{L^*}f^*\right)-C^*\left(\frac{\Gamma(1+2A)}{\Gamma(1+A)}g^*+\frac{\Gamma(-2A-1)}{\Gamma(-A)}\frac{1}{L}f\right).\end{gathered} \tag{69}$$

We will present these two solutions in short form:

$$\begin{gathered}F_1=H_1=(\alpha f+\alpha^*f^*)+(\gamma^*g^*+\gamma g)=2\text{Re}(\alpha f+\gamma g)\ ,\quad F_1^*=F_1\ ,\\ F_2=H_2=(\alpha g+\alpha^*g^*)+(\gamma^*f^*+\gamma f)=2\text{Re}(\alpha g+\gamma f)\ ,\quad F_2^*=F_2\ ;\end{gathered} \tag{70}$$

$$\begin{gathered}F_1=G_1=(\alpha f-\alpha^*f^*)+(\gamma^*g^*-\gamma g)=2i\ \text{Im}(\alpha f-\gamma g)\ ,\quad F_1^*=-F_1\ ,\\ F_2=G_2=(\alpha g-\alpha^*g^*)+(\gamma^*f^*-\gamma f)=2i\ \text{Im}(\alpha g-\gamma f)\ ,\quad F_2^*=-F_2\ .\end{gathered} \tag{71}$$

Allowing for (70), we find expressions for Majorana solutions $\Psi_\pm$ related to (H_1,H_2):

$$\Psi_{+}=\left|\begin{array}{c}\text{Re }\varphi\times\text{Re}(\alpha f+\gamma g)+\lambda\text{ Im }\varphi\times\text{Re}(\alpha g+\gamma f)\\ \text{Re }\varphi\times\text{Re}(\alpha g+\gamma f)-\lambda\text{ Im }\varphi\times\text{Re}(\alpha f+\gamma g)\\ \lambda\text{ Re }\varphi\times\text{Re}(\alpha f+\gamma g)-\text{Im }\varphi\times\text{Re}(\alpha g+\gamma f)\\ \lambda\text{ Re }\varphi\times\text{Re}(\alpha g+\gamma f)+\text{Im }\varphi\times\text{Re }\ \varphi(\alpha f+\gamma g)\end{array}\right|,$$

$$\Psi_{-}=i\left|\begin{array}{c}\text{Im }\varphi\times\text{Re}(\alpha f+\gamma g)-\lambda\text{ Re }\varphi\times\text{Re}(\alpha g+\gamma f)\\ \text{Im }\varphi\times\text{Re}(\alpha g+\gamma f)+\lambda\text{ Re}\varphi\times\text{Re}(\alpha f+\gamma g)\\ \lambda\text{ Im }\varphi\times\text{Re}(\alpha f+\gamma g)+\text{Re }\varphi\times\text{Re}(\alpha g+\gamma f)\\ \lambda\text{ Im }\varphi\times\text{Re}(\alpha g+\gamma f)-\text{Re }\varphi\times\text{Re}(\alpha f+\gamma g)\end{array}\right|. \tag{72}$$

In the same manner, with the use of (71) we find expressions for Majorana solutions $\Psi_\pm$ related to (G_1,G_2):

$$\Psi_{+}=\left|\begin{array}{c}\text{Re }i\varphi\times\text{Im}(\alpha f-\gamma g)+\lambda\text{ Im }i\varphi\times\text{Im}(\alpha g-\gamma f)\\ \text{Re }i\varphi\times\text{Im}(\alpha g-\gamma f)-\lambda\text{ Im }i\varphi\times\text{Im}(\alpha f-\gamma g)\\ \lambda\text{ Re }i\varphi\times\text{Im}(\alpha f-\gamma g)-\text{Im }i\varphi\times\text{Im}(\alpha g-\gamma f)\\ \lambda\text{ Re }i\varphi\times\text{Im}(\alpha g-\gamma f)+\text{Im }i\varphi\times\text{Im}(\alpha f-\gamma g),\end{array}\right|,$$

$$\Psi_{-}=i\left|\begin{array}{c}\text{Im }i\varphi\times\text{Im}(\alpha f-\gamma g)-\lambda\text{ Re }i\varphi\times\text{Im}(\alpha g-\gamma f)\\ \text{Im }i\varphi\times\text{Im}(\alpha g-\gamma f)+\lambda\text{ Re }i\varphi\times\text{Im}(\alpha f-\gamma g)\\ \lambda\text{ Im }i\varphi\times\text{Im}(\alpha f-\gamma g)+\text{Re }i\varphi\times\text{Im}(\alpha g-\gamma f)\\ \lambda\text{ Im }i\varphi\times\text{Im}(\alpha g-\gamma f)-\text{Re }i\varphi\times\text{Im}(\alpha f-\gamma g)\end{array}\right|. \tag{73}$$

To get the asymptotic at $z \to -\infty$, we shoul take into consideratuin that non-is only from function $f(z)$:

$$g(z), g^*(z) - -- > 0 \,, \quad f(z) - -- > e^{-ipz} \,.$$

So, asymptotics for Majorana solutions related to (H_1, H_2) are

$$\Psi_+ = \left| \begin{array}{c} \text{Re}\, \varphi \times \text{Re}\, \alpha e^{-ipz} + \lambda \,\text{Im}\, \varphi \times \text{Re}\, \gamma e^{-ipz} \\ \text{Re}\, \varphi \times \text{Re}\, \gamma e^{-ipz} - \lambda \,\text{Im}\, \varphi \times \text{Re}\alpha e^{-ipz} \\ \lambda \,\text{Re}\, \varphi \times \text{Re}\, \alpha e^{-ipz} - \text{Im}\, \varphi \times \text{Re}\, \gamma e^{-ipz} \\ \lambda \,\text{Re}\, \times \text{Re}\, \gamma e^{-ipz} + \text{Im}\, \varphi \times \text{Re}\, \alpha e^{-ipz}, \end{array} \right| ,$$

$$\Psi_- = i \left| \begin{array}{c} \text{Im}\, \varphi \times \text{Re}\, \alpha e^{-ipz} - \lambda \,\text{Re}\, \varphi \times \text{Re}\gamma e^{-ipz} \\ \text{Im}\, \varphi \times \text{Re}\, \gamma e^{-ipz} + \lambda \,\text{Re}\varphi \times \text{Re}\, \alpha e^{-ipz} \\ \lambda \,\text{Im}\, \varphi \times \text{Re}\, \alpha e^{-ipz} + \text{Re}\, \varphi \times \text{Re}\, \gamma e^{-ipz} \\ \lambda \,\text{Im}\, \varphi \times \text{Re}\, \gamma e^{-ipz} - \text{Re}\, \varphi \times \text{Re}\, \alpha e^{-ipz} \end{array} \right| ; \qquad (74)$$

asymptotics for Majorana solutions related to (G_1, G_2) are

$$\Psi_+ = \left| \begin{array}{c} \text{Re}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} - \lambda \,\text{Im}\, i\varphi \times \text{Im}\gamma e^{-ipz} \\ -\text{Re}\, i\varphi \times \text{Im}\gamma e^{-ipz} - \lambda \,\text{Im}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} \\ \lambda \,\text{Re}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} + \text{Im}\, i\varphi \times \text{Im}\gamma e^{-ipz} \\ -\lambda \,\text{Re}\, i\varphi \times \text{Im}\gamma e^{-ipz} + \text{Im}\, i\varphi \times \text{Im}\, \alpha e^{-ipz}, \end{array} \right| ,$$

$$\Psi_- = i \left| \begin{array}{c} \text{Im}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} + \lambda \,\text{Re}\, i\varphi \times \text{Im}\gamma e^{-ipz} \\ -\text{Im}\, i\varphi \times \text{Im}\gamma e^{-ipz} + \lambda \,\text{Re}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} \\ \lambda \,\text{Im}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} - \text{Re}\, i\varphi \times \text{Im}\gamma e^{-ipz} \\ -\lambda \,\text{Im}\, i\varphi \times \text{Im}\gamma e^{-ipz} - \text{Re}\, i\varphi \times \text{Im}\, \alpha e^{-ipz} \end{array} \right| . \qquad (75)$$

Let us write down general structure of elementary blocks which enter the formulas (74)–(75):

$$\varphi = e^{i(-\epsilon t + ax + by)} = e^{i\Delta} \,, \quad \text{Re}\, \varphi = \cos\Delta \,, \quad \text{Im}\;\; \varphi = \sin\Delta \,,$$

$$i\varphi, \quad \text{Re}\, i\varphi = \cos(\Delta + \pi/2) \,, \quad \text{Im}\;\; i\varphi = \sin(\Delta + \pi/2) \,,$$

$$\text{Re}\, \alpha e^{-ipz} \sim ... \cos(pz + ...) \,, \quad \text{Im}\, \alpha e^{-ipz} \sim ... \sin(pz + ...) \,,$$

$$\text{Re}\, \gamma e^{-ipz} \sim ... \cos(pz + ...) \,, \quad \text{Im}\, \gamma e^{-ipz} \sim ... \sin(pz + ...) \,.$$

All terms have similar general structure:

$$2\sin(\Delta + ..) \cos(pz + ...) = \sin(\Delta + pz + ...) + \sin(\Delta - pz + ...) \,,$$

$$2\cos(\Delta + ..) \cos(pz + ...) = \cos(\Delta + pz + ...) + \cos(\Delta - pz + ...) \,,$$

$$2\sin(\Delta + ..) \sin(pz + ...) = \cos(\Delta - pz + ...) - \cos(\Delta + pz + ...) \,,$$

$$2\cos(\Delta + ..) \sin(pz + ...) = \sin(\Delta + pz + ...) - \sin(\Delta - pz + ...) \,.$$

Thus, for Majorana particle, at fixed (x, y) in $\Delta(t, x, y)$ we have standing waves (superpositions of two running waves in the variable z). In other words, for Majorana particle we get the effect of complete reflection on effective barrier generated be Lobachevsky geometry. Also, it should be noted that Majorana components for solutions H_1, H_1 and G_1, G_2 represent standing wave (being real or imaginary) in the whole region of the variable z (not only at $z \to -\infty$).

6. Weyl particles

Finally, we are to discuss the effects of Lobachevsky geometry on Weyl 2-component fields. Recall equations for Weyl anti-neutrino

$$(\frac{d}{dz} - i\epsilon)F_1 + ie^z(a - ib)F_2 = 0 \; , \quad (\frac{d}{dz} + i\epsilon)F_2 - ie^z(a + ib)F_1 = 0 \; ; \tag{76}$$

and neutrino

$$(\frac{d}{dz} + i\epsilon)F_3 + ie^z(a - ib)F_4 = 0 \; , \quad (\frac{d}{dz} - i\epsilon)F_4 - ie^z(a + ib)F_3 = 0 \; . \tag{77}$$

Both subsystems are symmetrical under complex conjugation: they have as solutions the pairs

$$\begin{vmatrix} F_1 \\ F_2 \end{vmatrix} , \quad \begin{vmatrix} F_2^* \\ F_1^* \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} F_3 \\ F_4 \end{vmatrix} , \quad \begin{vmatrix} F_4^* \\ F_3^* \end{vmatrix} \tag{78}$$

respectively. We may have used the known results for the system (21)

$$\begin{aligned} &(\frac{d}{dz} - ip)F_1 + ie^z(a - ib)F_2 = 0 \; , \\ &(\frac{d}{dz} + ip)F_2 - ie^z(a + ib)F_1 = 0 \; . \end{aligned} \tag{79}$$

Recall that $p = \pm\sqrt{\epsilon^2 - m^2}$, so at $m = 0$ we have $p = -\epsilon, \; +\epsilon$. Therefore, eqs. (79) when $p = -\epsilon$ coincide with (77) and refer to neutrino; at $p = +\epsilon$ eqs. (79) give eq. (76) and refer to anti-neutrino. However, the point of prime significance is that for in Weyl case it is forbidden to combine solutions of eqs. (77) and (76). Therefore, only part of results obtained for Dirac particle can be retained in Weyl case.

For definiteness, below let us follows the anti-neutrino (that is $p = +\epsilon$; to get neutrino case it suffices to make formal change $p \Longrightarrow -p = -\epsilon$). We have two 2-nd order equations

$$\left(\frac{d^2}{dZ^2} + \frac{p^2 + ip}{Z^2} - a^2 - b^2 \right) F_1 = 0 \; , \quad \left(\frac{d^2}{dZ^2} + \frac{p^2 - ip}{Z^2} - a^2 - b^2 \right) F_2 = 0 \; ; \tag{80}$$

they have the following asymptotic behavior:

$$z \to -\infty, \quad F_1 \sim e^{ipz}, \; e^{(1-ip)z} \to 0, \quad F_2 \sim e^{-ipz}, \; e^{(1+ip)z} \to 0, \quad ;$$

$$z \to +\infty, \quad F_1 \sim e^{\pm\sqrt{a^2+b^2}e^z}, \quad F_2 \sim e^{\pm\sqrt{a^2+b^2}e^z} \; .$$

We see that if we cannot use solutions with opposite values of p, it is impossible to construct solutions referred to reflecting process (it concerns both F_1 and F_2)

In other words, the Weyl fields cardinally differ from Maxwell, Dirac or Majorana cases: for Weyl fermions, the reflecting effect vanishes. General conclusion may be drawn that effects of non-Euclidean geometry can substantially depend on the type of the fermion.

7. Conclusion

Previously it was shown that in electrodynamic context the Lobachevsky geometry can simulate an effective medium acting as an ideal mirror, oriented perpendicularly to the axes z. In the present paper, an analogue of that effect is investigated for spin 1/2 field. Solutions of the Dirac equation are constructed explicitly, they describe waves in space which are reflected from effective potential barrier without penetrating it. The depth of penetration into the medium is determined by characteristics of the quantum states and by the curvature radius of the Lobachevsky space; for waves with $k_1 = 0, k_2 = 0$ the effective reflecting barrier vanishes. Results are valid for Majorana fermions as well, some relevant details are specified. It is shown that for Weyl fermions, the reflecting effect vanishes. General conclusion may be drawn that effects of non-Euclidean geometry can substantially depend on the type of the fermion.

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