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Scalar–Tensor Gravitation

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In the frameworks of scalar-tensor gravitational theory the Lagrangian of scalar field whose source is a trace of the total energy-momentum tensor of matter and scalar field is founded. At appointed choice of scalar potential, parameters of such a scalar field can simulate a dark matter in accordance with observational data.

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1. Introduction

As is known today, we live in the universe where there is an abundance of dark matter and dark energy and the acceleration of cosmological expansion is observed [1], [2]. One of the possible approach to theoretical description of these observations is the modification of General Relativity. In this connection the interest in Brans-Dicke scalar theories is resumed [3]. In such approach gravitational field is depended on the metric and scalar potential.

Scalar fields are commonly used as candidates for the dark energy [4, 5]. Nevertheless, there is no unambiguous criterion for the choice of the field Lagrangian in scalar field theories. Moreover, as it was shown in [6], any scalar field in the slow-roll regime can model the cosmological constant and hence leads to an appropriate cosmological scenario.

There is an alternative approach to the problem of scalar field Lagrangian determination, which was first realized in the Relativistic Theory of Gravity (RTG) [7]. As it was shown, the Einstein equations of the gravitational field can be derived if one considers a tensor field with the source being the total stress-energy tensor of both field and matter. This theory can be regarded as a gauge theory of the group of Lie variations for dynamical variables. The related transformations are variations of the form of the function for generally covariant transformations. The requirement on action to be invariant for this group under the transformations of the dynamic variables alone requires replacing the "nondynamic" Minkowski metric γ^{ik} with expression g^{ik}: $\tilde{g}^{ik} = \sqrt{-g}g^{ik} = \sqrt{-\gamma}(\gamma^{ik} + \sqrt{k}\psi^{ik})$, where $\gamma = det\gamma_{ik}, g = detg_{ik}$, k is the Einstein constant, and thus introducing the gauge gravitational potential ψ^{ik}. The expression g^{ik} is interpreted here as the metric of the effective space-time from which the connection, the Cristoffel bracket, can be uniquely constructed. The RTG field equations in its massless variant are the Einstein ones for this effective metric, added the conditions, restricting the spin states of the field ψ^{ik}: $D_i\tilde{g}^{ik} = 0$, where D_i is the covariant derivative in Minkowsky space. This conditions play a significant role in RTG, removing gauge arbitrariness of Einstein equations and they coincide with the Fock harmonical conditions in Galilean coordinates.

Although RTG field equations coincide with General Relativity ones locally, its global solutions, generally speaking, will be different, since this solutions are defined on the various manifolds. RTG, founding on the simple space-time topology, allows to introduce the global Galilean coordinate system, that distinguishes RTG from the bimetric theories, in which a flat space plays an auxiliary role and its topology does not define the character of the physical processes. This distinction may take place at interpretation of the field solutions, since the coordinate system in RTG is defined by the Minkowsky metric, but it is fixed by noncovariant coordinate conditions in GR. In the present work we consider a scalar field with a similar property: we require the scalar field source to be the trace of the stress-energy tensor of both matter and the field itself, at that both the scalar field lagrangian and the cosmological scenario can be obtained in general relativity as well.

2. Scalar field equations

Consider a scalar field φ with the source being the total trace of the stress-energy tensor of the scalar field and matter fields. This condition implies that the scalar field equation has the following form:

$$(\Box - m^2)\varphi = k(T^\varphi + T^M), \tag{1}$$

where the d'Alembertian is defined as $\Box = -\nabla_i\nabla^i$, the constants m and k have the physical meaning of the mass and the interaction constant of the scalar field respectively (here Latin indices run from 0 to 3, the metric signature $(+,-,-,-)$ is used, the speed of light is assumed to be $c = 1$).

Equation (1) allows to determine the Lagrangian of the field. For this purpose, one writes the Lagrangian of matter and the scalar field in the most general form:

$$L = F(\varphi, u)\sqrt{-g} + L^M(\varphi, g^{ik}, \Psi^M), u = \frac{1}{2}\varphi_{,i}\varphi^{,i}, \tag{2}$$

where F is some function, Ψ^M denotes the set of matter fields. The traces of the stress-energy tensors of matter and the scalar field are given by the following expressions:

$$T^\varphi = 2u\frac{\partial F}{\partial u} - 4F, \quad T^M = \frac{2}{\sqrt{-g}}g^{ik}\frac{\delta L^M}{\delta g^{ik}}. \tag{3}$$

With the help of expressions (3) field equation (1) takes the form

$$(\Box - m^2)\varphi = k(2u\frac{\partial F}{\partial u} - 4F + \frac{2}{\sqrt{-g}}g^{ik}\frac{\delta L^M}{\delta g^{ik}}). \tag{4}$$

The equation is obtained by variation of field Lagrangian (2) can be written as follows:

$$\begin{aligned}&\frac{\partial F}{\partial \varphi} - 2u\frac{\partial^2 F}{\partial u \partial \varphi} - \frac{\partial^2 F}{\partial u^2}\varphi^{;i}\varphi^{;k}\varphi_{;i;k}\\ &- \frac{\partial F}{\partial u}\varphi^{;i}_{;i} + \frac{1}{\sqrt{-g}}\frac{\delta L^M}{\delta \varphi} = 0.\end{aligned} \tag{5}$$

The condition that equations (4) and (5) are coincided leads to restrictions both on the scalar field Lagrangian and the nature of interaction between the scalar field and matter. Specifically, $F(\varphi, u)$ must be linear in u:

$$F(\varphi, u) = A(\varphi)u + B(\varphi), \tag{6}$$

where A and B satisfy the system of equations

$$A' = -2kA^2, B' = -m^2\varphi A + 4kAB. \tag{7}$$

In solving system (7) we require that $A(0) = 1$. This condition guaranteers the ordinary dynamic part of the field lagrangian $1/2\varphi_{,i}\varphi^{,i}$ for $\varphi \to 0$.

The solution of the system (7) has the form

$$A = \frac{1}{1+2k\varphi}, B = -\frac{1}{2}m^2\varphi^2 - \frac{1}{2}C(1+2k\varphi)^2, \tag{8}$$

where C is a constant.

The interaction between scalar field and matter must satisfy the following condition:

$$\frac{\delta L^M}{\delta \varphi} = -\frac{2k}{1+2k\varphi}g^{ik}\frac{\delta L^M}{\delta g^{ik}}. \tag{9}$$

For this condition hold for an arbitrary type of matter it is sufficient to require that the metric and the scalar field occur in matter Lagrangian only in combination

$$f_{ik} = (1 + 2k\varphi)g_{ik}. \qquad (10)$$

With the help of the above mentioned conditions, lagrangian reduces to

$$L = \frac{1}{2}\left(\frac{\varphi_{,i}\varphi^{,i}}{1+2k\varphi} - m^2\varphi^2 - C(1+2k\varphi)^2\right)\sqrt{-g} + L^M\left(\frac{1}{1+2k\varphi}g^{ik}, \Psi^M\right). \qquad (11)$$

We write the field equations (1) explicitly:

$$\varphi^{;i}_{;i} + m^2\varphi = -k(-\frac{\varphi_{,i}\varphi^{,i}}{1+2k\varphi} + 2m^2\varphi^2 + 2C(1+2k\varphi)^2 + T^M). \qquad (12)$$

We note that Lagrangian (11) coincides (for $C = 0$) with the one obtained in [8] for the scalar field with the source being the total trace of its stress-energy tensor. The approach of the present work differs form [8] in including the interaction between the scalar field and matter. In addition, the results of the present work are more general since in [8] the linear in u Lagrangian of the form (6) is initially assumed and the condition $B(0) = 0$ is used, which leads to $C = 0$.

Consider the scalar field in the state with the minimum energy. The minimum of the energy is achieved at the points where the potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2 + \frac{C}{2}(1+2k\varphi)^2 \qquad (13)$$

has minimum. Here φ is assumed to vary from $-1/2k$ to $+\infty$, which ensures the positivity of the denominator in Lagrangian (11). Under the condition $C > -m^2/4k^2$, the potential has a minimum at the point

$$\varphi_0 = -\frac{2kC}{m^2+4k^2C}, \qquad V(\varphi_0) = \frac{m^2C}{2(m^2+4k^2C)}. \qquad (14)$$

For $C > 0$ the minimum value of the potential is positive: $V(\varphi_0) > 0$, therefore it can be identified with the cosmological constant in the Einstein equations. For $0 > C > -m^2/4k^2$ the minimum value of the potential is negative: $V(\varphi_0) < 0$. In this case $|V(\varphi_0)|$ can be interpreted as the squared graviton mass μ^2, since the Einstein equations in the linear approximation can be written as [7]:

$$(\Box + V(\varphi_0))\,\psi_{ik} = V(\varphi_0)\gamma_{ik}. \qquad (15)$$

Here, in contrast to the free massive Fierz-Pauli equation in the Minkowski space, the term $V(\varphi_0)\gamma_{ik}$ is the stress-energy tensor of the scalar field in the ground state. Assuming that $|C|$ is small as compared with m^2/k^2 we get the estimate $\mu^2 \approx |C|/2$.

We write the gravitational field Lagrangian in the same form as in general relativity. The total lagrangian of the theory takes the form:

$$L = -\frac{1}{16\pi G}R\sqrt{-g} + \frac{1}{2}(\frac{\varphi_{,i}\varphi^{,i}}{1+2k\varphi} - m^2\varphi^2 - C(1+2k\varphi)^2)\sqrt{-g} + L^M. \qquad (16)$$

For such a choice of the Lagrangian, the postulate about the source holds for the gravitational field [7]: the source of the gravitational field is the total stress-energy tensor of all fields of matter including the scalar field and the gravitational field.

Lagrangian (16) can be reduced to the Lagrangian of the scalar-tensor theory of gravity [9]. We use f_{ik} (10) as a new metric and the quantity ψ:

$$\psi = \frac{1}{1+2k\varphi} \qquad (17)$$

as a new scalar field. The scalar curvature R is expressed in terms of the new variables in the following way:

$$R = \frac{1}{\psi}\tilde{R} - \frac{3}{\psi^2}f^{ik}\psi_{;i;k} + \frac{3}{2\psi^3}f^{ik}\psi_{,i}\psi_{,k}, \qquad (18)$$

where $\tilde{R}$ is the scalar curvature that is defined using the metric g_{ik}, the covariant derivatives are also defined using the metric f_{ik}. With the help of equation (18) the Lagrangian of the theory in the new variables takes the form:

$$L = -\frac{1}{16\pi G}\psi\tilde{R}\sqrt{-f} + \frac{1}{8k^2\psi^2}$$
$$(1 - \frac{3k^2\psi}{4\pi G}f^{ik}\psi_{,i}\psi_{,k})\sqrt{-f} -$$
$$\frac{1}{2}(C + \frac{m^2}{4k^2}(\psi-1)^2)\sqrt{-f} + L^M, \qquad (19)$$

where $f = \det(f_{ik})$.

Note that the constant $C/2$ enters into lagrangian (19) in the same way the cosmological constant enters into the standard Lagrangian of general relativity.

3. Cosmological scenario

Consider the scenario of the Universe evolution in the given scalar-tensor theory. The metric of the homogenous and isotropic Universe is the Robertson-Walker metric with a flat 3-dimensional space:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \qquad (20)$$

The reason for choosing the flat model is mostly the observational evidence. In addition, if the problem is solved in the framework of RTG, flat model is the only acceptable one [7].

We are interested in the evolution of the Universe at the epoch following the annihilation of electron-positron pairs. At the given epoch we can treat matter as consisting of the cold matter that includes the barionic and dark matter and the radiation that includes photons and three types of neutrino and antineutrino. The equation of state is $p^{CM} = 0$ for the cold matter and $p^r = \epsilon^r/3$ for the radiation.

The equations of matter evolution follow from the gravitational field equations and have the form

$$(T^{(CM)k}_{\ i} + T^{(r)k}_{\ i} + T^{\varphi k}_{\ i})_{;k} = 0, \qquad (21)$$

where the indices CM, r, φ denote the quantities related to the cold matter, the radiation and the scalar field respectively.

The scalar field and the metric appear in matter lagrangian only in the combination f_{ik} (10). Therefore equations (21) can be written in the following form:

$$\tilde{\nabla}_k\left(\tilde{T}_i^{(CM)k} + \tilde{T}_i^{(r)k}\right) = 0, \qquad (22)$$

where the quantities $\tilde{\nabla}_i$ and $\tilde{T}_{ik}$ are defined using the metric f_{ik} (10). Specifically

$$\tilde{T}_{ik} = \frac{2}{\sqrt{-f}}\frac{\delta L^M}{\delta f^{ik}}. \qquad (23)$$

Equations (22) coincide with the equations of matter evolution in general relativity, hence the energy densities $\tilde{\epsilon} = \tilde{T}_0^0$ of matter and radiation satisfy the equations:

$$\tilde{\epsilon}^{CM}\tilde{a}^3 = const, \qquad \tilde{\epsilon}^r\tilde{a}^4 = const, \qquad (24)$$

where $\tilde{a} = \sqrt{1+2k\varphi}a$ is the scale factor defined using the metric f_{ik}. Taking into account the relation $\epsilon = (1+2k\varphi)^2\tilde{\epsilon}$ we reduce equations (24) to the form

$$\frac{\epsilon^{CM}a^3}{\sqrt{1+2k\varphi}} = const, \qquad \epsilon^r a^4 = const. \qquad (25)$$

Constants in equations (25) are found from initial conditions. With the assumption that at present time the influence of the scalar field on observations is negligible (the exact criterion will be given below) the following equalities hold at present time [10]:

$$\epsilon^r(0) = \left(1 + \frac{21}{8}\left(\frac{4}{11}\right)^{4/3}\right)\frac{\pi^2 T_0^4}{15(\hbar c)^3},$$
$$\epsilon^{CM}(0) = (1-\Omega_{DE})\epsilon_c - \epsilon^r(0), \qquad (26)$$

where $\epsilon_c = 3H_0^2/8\pi G$ is the critical density, H_0 is the Hubble constant at present time, Ω_{DE} is the dark energy density in the units of critical density, $T_0 = 2.7$ K$= 2.3\cdot 10^{-4}$ erg is the temperature of the cosmic microwave background.

Scalar field lagrangian (11) can be transformed into a standard form by the variable change:

$$\Phi = \frac{1}{k}\sqrt{1+2k\varphi}, \qquad \varphi = \frac{k^2\Phi^2-1}{2k}. \tag{27}$$

Then, the equation (11) takes the following form:

$$L = \frac{1}{2}(\Phi_{,i}\Phi^{,i} - V(\Phi)\sqrt{-g} + L^M(\frac{1}{k^2\Phi^2}g^{ik}, \Psi^M), \tag{28}$$

where the potential $V(\Phi)$ is defined by expression (13). We write it as a function of the variable Φ:

$$V(\Phi) = \frac{m^2}{8k^2}\left(k^2\Phi^2 - 1\right)^2 + \frac{1}{2}Ck^4\Phi^4. \tag{29}$$

Field equations obtained by varying (28) have the form

$$\Phi^{;i}_{;i} + V'(\Phi) = -\frac{1}{\Phi}T^M. \tag{30}$$

The evolution of the Universe is determined by the Einstein equations and scalar field equation (30). Three equations are independent among them:

$$\dot{H} = -4\pi G\left(\epsilon^{CM} + \frac{4}{3}\epsilon^r + \epsilon^\varphi + p^\varphi\right), \tag{31}$$

$$\ddot{\Phi} + 3H\dot{\Phi} + V'(\Phi) = -\frac{\epsilon^{CM}}{\Phi}, \tag{32}$$

$$\frac{3H^2}{8\pi G} = \epsilon^{CM} + \epsilon^r + \epsilon^\varphi. \tag{33}$$

Here, dot denotes the derivative with respect to time t, the Hubble constant is defined by $H = \dot{a}/a$, the energy density and pressure of the scalar field are given by

$$\epsilon^\varphi = T^{\varphi}{}^0_0 = \frac{1}{2}\dot{\Phi}^2 + V(\Phi),$$
$$p^\varphi = -T^{\varphi}{}^1_1 = \frac{1}{2}\dot{\Phi}^2 - V(\Phi). \tag{34}$$

The energy densities ϵ^{CM} and ϵ^r are determined by equations (25). The analysis of the numerical solution of system (31)with the initial conditions:

$$a(0) = 1, \quad H(0) = H_0, \quad \Phi(0) = \Phi_0, \quad \dot{\Phi}(0) = \dot{\Phi}_0.$$

where $t = 0$ denotes the present time,shows that for a certain restriction on parameters the so-called slow-roll regime, in which the scalar field well models the dark energy, is possible [12].

4. Conclusion

In the present work the nonlinear scalar field interacting with the gravitational field and matter is introduced in the way analogical to the introduction of nonlinear tensor field describing gravitation in RTG. The requirement that the source of the field is the trace of its own stress-energy tensor leads to the lagrangian containing three arbitrary parameters. These parameters are connected with the scalar field mass, the cosmological constant and for a certain restriction on one of the parameters with the graviton mass. The analysis of the cosmological solution for the homogenous and isotropic Universe shows that the scalar field may model the dark energy in agreement with modern observational data.

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