Structures on Three-dimensional Pseudo-Riemannian Spaces

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The problem of establishing links between the curvature and the topological structure of a manifold is one of the important problems of the geometry. In general, the purpose of the research of manifolds of various types is rather complicated. Therefore, it is natural to consider this problem in a narrower class of pseudo-Riemannian manifolds, for example, in the class of homogeneous pseudo-Riemannian manifolds. In this paper, for all three-dimensional pseudo-Riemannian homogeneous spaces, it is determined under what conditions the space is Ricci-flat, Einstein, Ricci-parallel, locally-symmetric or conformally-flat.

Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. We can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A *pseudo-Riemannian homogeneous space* is a triple $(\overline{G}, M, \mathbf{g})$, where $\mathbf{g}$ is an invariant pseudo-Riemannian metric on M. The $\mathfrak{g}$-module $\mathfrak{m} = \bar{\mathfrak{g}}/\mathfrak{g}$ corresponding to the isotropy action of G on $\bar{\mathfrak{g}}/\mathfrak{g}$ has the form: $x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g},\ x, y \in \bar{\mathfrak{g}}$, and the bilinear form B is also an invariant bilinear form on the $\mathfrak{g}$ -module $\bar{\mathfrak{g}}/\mathfrak{g}$, i.e. $B(x.v_1, v_2) + B(v_1, x.v_2) = 0$ for all $x \in \mathfrak{g}$, $v_1, v_2 \in \bar{\mathfrak{g}}/\mathfrak{g}$. Moreover, if G is connected, then the converse is also true. The local classification of pseudo-Riemannian homogeneous spaces is equivalent to the description of effective pairs of Lie algebras supplied with an invariant nondegenerate symmetric bilinear form on the isotropy module.

All three-dimensional pseudo-Riemannian homogeneous spaces where classified in[1]. Based on the classification, we study geometry of each class. It is well known that geometry of different spaces is related to their Riemann curvature tensor, the $(1, 3)$-tensor field which is defined by the identity $R(x, y)z = [\Lambda(x), \Lambda(y)]z - \Lambda([x, y])z$, where x, y, z are arbitrary tangent vector fields on the base manifold and Λ is the Levi-Civita connection deduced by the well known Koszul formula,

$$2\mathbf{g}(\Lambda(x)y, z) = \mathbf{g}(x, [z, y]) + \mathbf{g}(y, [z, x]) + \mathbf{g}(z, [x, y]).$$

The Ricci tensor is defined by the following contraction on the curvature tensor's indices, $\mathrm{Rich}(x, y) = \mathrm{tr}\{z \to R(z, x)y\}$. Several geometric properties are related to the Ricci tensor. A manifold $(M, \mathbf{g})$ is called *Ricci flat* if it's Ricci tensor vanishes identically. A more general condition appears in *Einstein property*, that is $\mathrm{Rich} = \lambda\mathbf{g}$ for a real constant λ. Ricci parallel condition is also means that the covariant derivative of the Ricci tensor will be zero. If the covariant derivative of the curvature tensor vanishes, i.e., $\Lambda(R) = 0$, the manifold is called *locally symmetric*. The Cotton tensor on a pseudo-Riemannian manifold is a third-order tensor concomitant of the metric:

$$C(x,y,z) = \nabla_z \operatorname{Rich}(x,y) - \nabla_y \operatorname{Rich}(x,z) + \frac{1}{2(n-1)} \left(\nabla_y R\mathbf{g}(x,z) - \nabla_z R\mathbf{g}(x,y)\right),$$

where $x, y, z \in \mathfrak{m}$. The vanishing of the Cotton tensor for $n = 3$ is necessary and sufficient condition for the manifold to be *conformally flat*.

For all three-dimensional pseudo-Riemannian homogeneous spaces, it is determined under what conditions the space is Ricci-flat, Einstein, Ricci-parallel, locally-symmetric or conformally-flat. In addition, for all these spaces, Levi-Cevita connections, curvature and torsion tensors, holonomy algebras, scalar curvatures, Ricci tensors are written out in explicit form. The results can find applications in mathematics and physics, since many fundamental problems in these fields are reduced to the study of invariant objects on homogeneous spaces.

References

[1] Mozhey N. P. Affine connections on three-dimensional pseudo-Riemannian homogeneous spaces. I *Izvestiya vysshikh uchebnykh zavedeniy. Matematika.* **12** (2013) 51-69.