Journal of Physics: Conference Series

PAPER • **OPEN ACCESS**

On modeling neutrinos oscillations by geometry methods in the frames of the theory for a fermion with three mass parameters

Journal of Physics: Conference Series **1416** (2019) 012040 doi:10.1088/1742-6596/1416/1/012040

On modeling neutrinos oscillations by geometry methods in the frames of the theory for a fermion with three mass parameters

Ya.A. Voynova[1], E A Ovsiyuk[2], V V Kisel [3], V A Pletyukhov[4], V V Gilewsky [5], V M Red'kov[1]

[1] B.I. Stepanov Institute of Physics, NAS of Belarus, Independence ave. 68-2, 220072 Minsk
[2] Mozyr State Pedagogical University, Studencheskaya st. 28, Mozyr 247760, Belarus
[3] Belarus State University of Informatics and Radio-electronics, P. Brovki st 6, 220013 Minsk
[4] Brest State University, Kosmonavtov st 21, 224016 Brest, Belarus
[5] JINP-Sosny, NAS of Belarus, Minsk

E-mail: e.ovsiyuk@mail.ru , voinovayanina@mail.ru , vasiliy-bspu@mail.ru , terphys@brsu.brest.by , valentin.gilewsky@gmail.com , v.redkov@ifanbel.bas-net.by

Abstract. Starting from the general Gel'fand-Yaglom approach, we develop the theory of a new wave equation for a spin 1/2 fermion, which is characterized by three mass parameters. On the base of a 20-component wave function, three auxiliary bispinors are introduced, in absence of external fields these bispinors obey three separate Dirac-like equations with different masses. In presence of external electromagnetic fields or gravitational non-Euclidean background with a non-vanishing Ricci scalar curvature, the main equation is not split into separated three equations, instead a quite definite mixing of three Dirac-like equations arises. It is shown that for neutral Majorana particle, a generalized equation with three mass parameters exists as well. Such a generalized Majorana equation is not split into three separated equations in the curved space-time background, if the Ricci scalar of that space-time does not vanish. We have studied in detail the Majorana case, assuming approximation when an external cosmological background is taken into account by a constant Ricci parameter, $R =$ const, and the Cartesian coordinates are used. With the help of a special linear transformation, the system of three linked Majorana equations transforms into three separate ones, with modified mass parameters, the last are solvable in the usual way. The spectrum of arising mass parameters is studied analytically and numerically.

1. Fermion with 3 mass parameter, theory in absence of external fields

Existence of different and more general wave equations than commonly used ones is well known [?, ?, ?, ?]. In the context of existence of the similar neutrinos of different masses, we examine a possibility within the theory of relativistic wave equations to describe a spin 1/2 particle with three mass parameters. Such a generalized equation for fermion with 3 mass parameters is based on the use of 20-component wave function (bispinor Ψ_0 and vector bispinor Ψ_μ). Omitting many details of this theory in the frames of Gel'fand–Yaglom approach [?], we start with a system of equations in spin-tensor form (first, consider the model in absence of external fields)

$$c_1\hat{\partial}(\gamma_\mu\Psi_\mu) + \frac{c_3}{\sqrt{6}}[\hat{\partial}(\gamma_\mu\Psi_\mu) - 4(\partial_\mu\Psi_\mu)] + M(\gamma_\mu\Psi_\mu) = 0, \qquad (1)$$

$$c_2\hat{\partial}\Psi_0 - i\frac{4c_4}{\sqrt{6}}[-\frac{1}{4}\hat{\partial}(\gamma_\mu\Psi_\mu) + (\partial_\mu\Psi_\mu)] + M\Psi_0 = 0, \tag{2}$$

$$-\frac{2fc_3^*}{\sqrt{6}}[\partial_\lambda(\gamma_\mu\Psi_\mu) - \frac{1}{4}\gamma_\lambda\hat{\partial}(\gamma_\mu\Psi_\mu)] + i\frac{2gc_4^*}{\sqrt{6}}(\partial_\lambda\Psi_0 - \frac{1}{4}\gamma_\lambda\hat{\partial}\Psi_0) + M[\Psi_\lambda - \frac{1}{4}\gamma_\lambda(\gamma_\mu\Psi_\mu)] = 0. \tag{3}$$

Numerics c_1, c_2 are real, and c_3, c_4 are complex; $f, g \in \{\pm 1\}$; we use notation $\hat{\partial} = \gamma_\mu\partial_\mu$. Physical sense of the numerical parameters will be clarified later. The system can be transformed to a the form of equations with respect to three bispinors $\gamma_\mu\Psi_\mu,\ \Psi_0,\ \partial_\mu\Psi_\mu$:

$$\frac{1}{c_2(c_1 + c_3/\sqrt{6})}\left\{c_1c_2(c_1 + c_3/\sqrt{6}) + fc_2|c_3|^2 - \frac{g}{\sqrt{6}}c_3|c_4|^2\right\}\hat{\partial}(\gamma_\mu\Psi_\mu)$$
$$-ig\frac{c_3c_4^*}{c_2}\hat{\partial}\Psi_0 - \frac{4c_3}{M\sqrt{6}}\frac{fc_2|c_3|^2 + gc_1|c_4|^2}{c_2(c_1 + c_3/\sqrt{6})}\hat{\partial}(\partial_\mu\Psi_\mu) + M(\gamma_\mu\Psi_\mu) = 0,$$
$$\frac{i}{\sqrt{6}}\frac{c_4}{c_2(c_1 + c_3/\sqrt{6})}(\sqrt{6}fc_2c_3^* - g|c_4|^2)\hat{\partial}(\gamma_\mu\Psi_\mu) + \frac{c_2^2 + g|c_4|^2}{c_2}\hat{\partial}\Psi_0$$
$$-i\frac{4c_4}{M\sqrt{6}}\frac{fc_2|c_3|^2 + gc_1|c_4|^2}{c_2(c_1 + c_3/\sqrt{6})}\hat{\partial}(\partial_\mu\Psi_\mu) + M\Psi_0 = 0,$$
$$\frac{M}{4}\frac{1}{c_2(c_1 + c_3/\sqrt{6})}\left\{\sqrt{6}fc_2c_3^* - g|c_4|^2 - c_2(c_1 + c_3/\sqrt{6})\right\}\hat{\partial}(\gamma_\mu\Psi_\mu)$$
$$-igM\frac{\sqrt{6}}{4}\frac{c_4^*}{c_2}\hat{\partial}\Psi_0 - \frac{fc_2|c_3|^2 + gc_1|c_4|^2}{c_2(c_1 + c_3/\sqrt{6})}\hat{\partial}(\partial_\mu\Psi_\mu) + M(\partial_\mu\Psi_\mu) = 0.$$

In brief, it is presented in a matrix form as follows

$$K\hat{\partial}\begin{vmatrix} \gamma_\mu\Psi_\mu \\ \Psi_0 \\ \partial_\mu\Psi_\mu \end{vmatrix} = M\begin{vmatrix} \gamma_\mu\Psi_\mu \\ \Psi_0 \\ \partial_\mu\Psi_\mu \end{vmatrix}, \quad K = \begin{vmatrix} A_1 & B_1 & R_1 \\ A_2 & B_2 & R_2 \\ A_3 & B_3 & R_3 \end{vmatrix}. \tag{4}$$

The numerical matrix K can be diagonalized, there arises a cubic equation for diagonal elements λ_i:

$$\Psi' = S\Psi, \quad SKS^{-1} = K' = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}, \quad S = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ r_1 & r_2 & r_3 \end{vmatrix},$$
$$\lambda^3 - \lambda^2(c_1 + c_2) + \lambda(c_1c_2 - f|c_3|^2 - g|c_4|^2) + fc_2|c_3|^2 + gc_1|c_4|^2 = 0, \tag{5}$$

The rows of the matrix S satisfy to corresponding subsystems:

$$a_1A_1 + a_2A_2 + a_3A_3 = \lambda_1 a_1\,, \quad a_1B_1 + a_2B_2 + a_3B_3 = \lambda_1 a_2\,, \quad a_1R_1 + a_2R_2 + a_3R_3 = \lambda_1 a_3\,;$$
$$b_1A_1 + b_2A_2 + b_3A_3 = \lambda_2 b_1\,, \quad b_1B_1 + b_2B_2 + b_3B_3 = \lambda_2 b_2\,, \quad b_1R_1 + b_2R_2 + b_3R_3 = \lambda_2 b_3\,;$$
$$r_1A_1 + r_2A_2 + r_3A_3 = \lambda_3 r_1\,, r_1B_1 + r_2B_2 + r_3B_3 = \lambda_3 r_2\,, r_1R_1 + r_2R_2 + r_3R_3 = \lambda_3 r_3\,.$$

Their solutions have similar structure:

$$a_2 = a_1\frac{-igc_3c_4^*(c_1 + \frac{c_3}{\sqrt{6}})\lambda_1}{\lambda_1c_2(\lambda_1 - c_2) + (\lambda_1 - c_2)[fc_2|c_3|^2 + gc_1|c_4|^2] - \lambda_1(c_1 + \frac{c_3}{\sqrt{6}})g|c_4|^2},$$

$$a_3 = -a_1\frac{4a_1}{\sqrt{6}M}\frac{c_3(\lambda_1 - c_2)[fc_2|c_3|^2 + gc_1|c_4|^2]}{\lambda_1c_2(\lambda_1 - c_2) + (\lambda_1 - c_2)[fc_2|c_3|^2 + gc_1|c_4|^2] - \lambda_1(c_1 + \frac{c_3}{\sqrt{6}})g|c_4|^2},$$

and results for b_j, r_j are similar: $b_1, b_2, b_3 \Longrightarrow \lambda_2,\ r_1, r_2, r_3 \Longrightarrow \lambda_3$.

In this way we transform the system into three unlinked Dirac-like equations with different masses: $M_1 = M/\lambda_1, M_2 = M/\lambda_2, M_3 = M/\lambda_3$. We are to examine possible values of the roots $\lambda_1, \lambda_2, \lambda_3$ of characteristic cubic equations (5). It is readily proved that real positive values are possible if $c_1 > 0,\ c_2 > 0,\ f = -1\ ,\ g = -1$. Using simplifying notations

$$|c_4|^2 = a^2, \quad |c_3|^2 = b^2, \quad \Gamma = \frac{c_1 a^2 + c_2 b^2}{(c_1 + c_2)^3}, \tag{6}$$

we can introduce simple parametrization for the roots:

$$M_3 = \frac{M}{\lambda_3} = \frac{\mu}{\cos\alpha}, \quad \mu = \frac{M}{(c_1 + c_2)}, \quad \alpha \in (0, \frac{\pi}{2}), \tag{7}$$

$$M_{1,2} = \frac{M}{\lambda_2} = \frac{\mu}{\sin^2(\alpha/2) \pm \sqrt{\sin^4(\alpha/2) - \Gamma/\cos\alpha}}\,. \tag{8}$$

Thus, in absence of external electromagnetic fields, the initial system is reducible to the form of three separated Dirac-like equations for bispinors:

$$\begin{aligned}
\Phi_1 &= a_1(\gamma_\mu \Psi_\mu) + a_2\Psi_0 + a_3(\partial_\mu \Psi_\mu), \quad (\hat{\partial} + M_1)\Phi_1 = 0\ , \quad M_1 = M/\lambda_1\,;\\
\Phi_2 &= b_1(\gamma_\mu \Psi_\mu) + b_2\Psi_0 + b_3(\partial_\mu \Psi_\mu), \quad (\hat{\partial} + M_2)\Phi_2 = 0, \quad M_2 = M/\lambda_2\,;\\
\Phi_3 &= r_1(\gamma_\mu \Psi_\mu) + r_2\Psi_0 + r_3(\partial_\mu \Psi_\mu), \quad (\hat{\partial} + M_3)\Phi_3 = 0\ , \quad M_3 = M/\lambda_3\,.
\end{aligned} \tag{9}$$

2. The presence of electromagnetic field

Now, let us take unto account external electromagnetic fields. To this end, we are to turn back to the system (1)—(3) and modify the derivative operator, $D_\mu = \partial_\mu - ieA_\mu(x)$:

$$c_2\hat{D}\Psi_0 - \frac{4ic_4}{\sqrt{6}}\left[(D_\mu\Psi_\mu) - \frac{1}{4}\hat{D}(\gamma_\mu\Psi_\mu)\right] + M\Psi_0 = 0, \tag{10}$$

$$(c_1 + \frac{c_3}{\sqrt{6}})\hat{D}(\gamma_\mu\Psi_\mu) - \frac{4c_3}{\sqrt{6}}(D_\mu\Psi_\mu) + M(\gamma_\mu\Psi_\mu) = 0, \tag{11}$$

$$\begin{aligned}
&-\frac{2fc_3^*}{\sqrt{6}}\left[\frac{3}{4}D^2 - \frac{1}{4}(-ieF_{\lambda\rho})\,\sigma_{\lambda\rho}\right](\gamma_\mu\Psi_\mu)\\
&+\frac{2igc_4^*}{\sqrt{6}}\left[\frac{3}{4}D^2 - \frac{1}{4}(-ieF_{\lambda\rho})\sigma_{\lambda\rho}\right]\Psi_0 - \frac{M}{4}\hat{D}(\gamma_\mu\Psi_\mu) + M\ (D_\lambda\Psi_\lambda) = 0.
\end{aligned} \tag{12}$$

Equation in (12) contains a 2-nd order operator $D^2 = D_\nu D_\nu$. With the help of two first equations, we may exclude such an operator from the third equation. Further, we can reduce the system to a different (but equivalent) form

$$\begin{aligned}
&A_1\hat{D}\,\bar{\Phi}_1 + B_1\hat{D}\,\bar{\Phi}_2 + R_1\hat{D}\,\bar{\Phi}_3 + M\,\bar{\Phi}_1 + \frac{4}{3}\frac{f|c_3|^2}{M}\Sigma\,\bar{\Phi}_1 - \frac{4i}{3}\frac{gc_3c_4^*}{M}\Sigma\,\bar{\Phi}_2 = 0,\\
&A_2\hat{D}\,\bar{\Phi}_1 + B_2\hat{D}\,\bar{\Phi}_2 + R_2\hat{D}\,\bar{\Phi}_3 + M\,\bar{\Phi}_2 + \frac{4i}{3}\frac{fc_3^*c_4}{M}\Sigma\,\bar{\Phi}_1 + \frac{4}{3}\frac{g|c_4|^2}{M}\Sigma\,\bar{\Phi}_2 = 0,\\
&A_3\hat{D}\,\bar{\Phi}_1 + B_3\hat{D}\,\bar{\Phi}_2 + R_3\hat{D}\,\bar{\Phi}_3 + M\,\bar{\Phi}_3 + \frac{2}{\sqrt{6}}fc_3^*\,\Sigma\,\bar{\Phi}_1 - i\frac{2}{\sqrt{6}}\,gc_4^*\Sigma\,\bar{\Phi}_2 = 0,
\end{aligned} \tag{13}$$

where $\bar{\Phi}_1 = \gamma_\mu\Psi_\mu,\ \bar{\Phi}_2 = \Psi_0, \quad \bar{\Phi}_3 = D_\mu\Psi_\mu\ ,\ \Sigma = -ieF_{\lambda\rho}\sigma_{\lambda\rho}$. The system (13) differs from similar one (4) in free case: it contains additional mixing terms depending on the tensor $F_{\lambda\rho}$ of the external electromagnetic field.

From equations (13), for three new bispinors

$$\Phi_1 = a_1\bar{\Phi}_1 + a_2\bar{\Phi}_2 + a_3\bar{\Phi}_3,\ \Phi_2 = b_1\bar{\Phi}_1 + b_2\bar{\Phi}_2 + b_3\bar{\Phi}_3,\ \Phi_3 = r_1\,\bar{\Phi}_1 + r_2\,\bar{\Phi}_2 + r_3\,\bar{\Phi}_3, \qquad (14)$$

we derive a system with a more symmetric structure

$$\begin{aligned}
(\lambda_1\hat{D} + M)\Phi_1 + \frac{4c_2c_3}{3M}\,\lambda_1(\lambda_1 - c_2)\,\Sigma(x)\,[fc_3^*\,\bar{\Phi}_1 - igc_4^*\bar{\Phi}_2)] = 0,\\
(\lambda_2\hat{D} + M)\Phi_2 + \frac{4c_2c_3}{3M}\,\lambda_2(\lambda_2 - c_2)\,\Sigma(x)\,[fc_3^*\,\bar{\Phi}_1 - igc_4^*\bar{\Phi}_2] = 0,\\
(\lambda_3\hat{D} + M)\Phi_3 + \frac{4c_2c_3}{3M}\,\lambda_3(\lambda_3 - c_2)\,\Sigma(x)\,[fc_3^*\,\bar{\Phi}_1 - igc_4^*\bar{\Phi}_2] = 0.
\end{aligned} \qquad (15)$$

Expressing $\bar{\Phi}_j$ through Φ_j, we get

$$\begin{aligned}
\hat{D}\Phi_1(x) + M_1\Phi_1(x) + Y_1\,\Sigma(x)\Phi(x) = 0\,,\\
\hat{D}\Phi_2(x) + M_2\Phi_2(x) + Y_2\,\Sigma(x)\Phi(x) = 0\,,\\
\hat{D}\Phi_3(x) + M_3\Phi_3(x) + Y_3\,\Sigma(x)\Phi(x) = 0\,.
\end{aligned} \qquad (16)$$

In the system (16) the notations are used

$$\begin{aligned}
\Phi(x) = L_1\Phi_1(x) + L_2\Phi_2(x) + L_3\Phi_3(x)\,,\\
L_1 = \frac{-L|c_4|^2 - L|c_3|^2 + c_2{}^2 - c_2(\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{Lc_2\,c_3(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}\,,\\
L_2 = \frac{-L|c_4|^2 - L|c_3|^2 + c_2{}^2 - c_2(\lambda_3 + \lambda_1) + \lambda_3\,\lambda_1}{Lc_2\,c_3(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}\,,\\
L_3 = \frac{-L|c_4|^2 - L|c_3|^2 + c_2{}^2 - c_2(\lambda_1 + \lambda_2) + \lambda_1\,\lambda_2}{Lc_2\,c_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\,.\\
Y_i = \frac{4c_3}{3M}c_2(\lambda_i - c_2), \quad i = 1, 2, 3\,, \quad L = c_1 + \frac{c_3}{\sqrt{6}}\,,
\end{aligned} \qquad (17)$$

Equations (16) can be presented in a matrix form

$$\begin{vmatrix} \hat{D} + M_1 & 0 & 0 \\ 0 & \hat{D} + M_2 & 0 \\ 0 & 0 & \hat{D} + M_3 \end{vmatrix} \begin{vmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix} + \Sigma(x) \begin{vmatrix} Y_1L_1 & Y_1L_2 & Y_1L_3 \\ Y_2L_1 & Y_2L_2 & Y_2L_3 \\ Y_3L_1 & Y_3L_2 & Y_3L_3 \end{vmatrix} \begin{vmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix} = 0\,. \qquad (18)$$

Let it be $|c_4| = a, |c_3| = b$; taking in mind that according the relations (8) parameter $(c_1 + c_2)$ may be taken up by arbitrary M, we set $c_1 = 1, c_2 = 1$. Then the the elements of the mixing matrix read

$$\begin{aligned}
Y_1L_1 = \frac{4}{3M}(\lambda_1 - 1)\frac{-L(a^2 + b^2) + 1 - (\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{L(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},\\
Y_1L_2 = \frac{4}{3M}(\lambda_1 - 1)\frac{-L(a^2 + b^2) + 1 - (\lambda_3 + \lambda_1) + \lambda_3\,\lambda_1}{L(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)},\\
Y_1L_3 = \frac{4}{3M}(\lambda_1 - 1)\frac{-L(a^2 + b^2) + 1 - (\lambda_1 + \lambda_2) + \lambda_1\,\lambda_2}{L(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},\\
Y_2L_1 = \frac{4}{3M}(\lambda_2 - 1)\frac{-L(a^2 + b^2) + 1 - (\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{L(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)},
\end{aligned}$$

$$
\begin{aligned}
Y_2 L_2 &= \frac{4}{3M}(\lambda_2 - 1)\frac{-L(a^2+b^2)+1-(\lambda_3+\lambda_1)+\lambda_3\,\lambda_1}{L(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},\\
Y_2 L_3 &= \frac{4}{3M}(\lambda_2 - 1)\frac{-L(a^2+b^2)+1-(\lambda_1+\lambda_2)+\lambda_1\,\lambda_2}{L(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)},\\
Y_3 L_1 &= \frac{4}{3M}(\lambda_3 - 1)\frac{-L(a^2+b^2)+1-(\lambda_2+\lambda_3)+\lambda_2\lambda_3}{L(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},\\
Y_3 L_2 &= \frac{4}{3M}(\lambda_3 - 1)\frac{-L(a^2+b^2)+1-(\lambda_3+\lambda_1)+\lambda_3\,\lambda_1}{L(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},\\
Y_3 L_3 &= \frac{4}{3M}(\lambda_3 - 1)\frac{-L(a^2+b^2)+1-(\lambda_1+\lambda_2)+\lambda_1\,\lambda_2}{L(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.
\end{aligned}
$$

3. Riemannian space-time geometry

Let us extend the model to a curved space-time. Instead of ict-metric, now we use the metric tensor $g_{\alpha\beta}(x)$ and slightly other Dirac matrices, also we apply more complicated derivative operations

$$
\gamma^0 = \left| \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right|, \quad \gamma^i = \left| \begin{array}{cc} 0 & -\sigma^i \\ \sigma^i & 0 \end{array} \right|, D_\alpha(x) = \nabla_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x), \hat{D} = \gamma^\alpha(x) D_\alpha(x), \tag{19}
$$

where $\Gamma_\alpha(x)$ is the conventional bispinor connection, and tetrad based local Dirac matrices $\gamma^\alpha(x) = \gamma^a e^\alpha_{(a)}(x)$. The final system of equations for a fermion with three mass parameters in Riemannian space-time has the form

$$
\begin{aligned}
&i\gamma^\alpha(x)\left[\partial_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x)\right]\Phi_1(x) - M_1\Phi_1(x) + Y_1\,\Sigma(x)\Phi(x) = 0\,,\\
&i\gamma^\alpha(x)\left[\partial_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x)\right]\Phi_2(x) - M_2\Phi_2(x) + Y_2\,\Sigma(x)\Phi(x) = 0\,,\\
&i\gamma^\alpha(x)\left[\partial_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x)\right]\Phi_3(x) - M_3\Phi_3(x) + Y_3\,\Sigma(x)\Phi(x) = 0\,,
\end{aligned} \tag{20}
$$

where $\Sigma(x) = -ieF_{\alpha\beta}\sigma^{\alpha\beta}(x) + \frac{1}{4}R(x)$, $R(x)$ is a Ricci scalar. It should be noticed that for geometrical models with non-vanising Ricci scalar, $R(x) \neq 0$, even in absence of the electromagnetic interaction, equations (20) link three bispinors in the unified system. And what is more, because in any Majorana the following properties of Dirac matrices and bispinor connection hold

$$
[i\gamma^\alpha(x)]^* = i\gamma^\alpha(x)\,, \quad \Gamma^*_\alpha(x) = \Gamma_\alpha(x)\,,
$$

the above system (20) describes Majorana-type neutral fermions with three mass parameters as well/ The system remains the same only with change: $\partial_\alpha + \Gamma_\alpha(x) + ieA_\alpha(x) \implies \partial_\alpha + \Gamma_\alpha(x)$.

Therefore, this theory permits us to take into account the effects of cosmological geometry background on such a complicated fermion. The most simple physically interpretable example is as follows: we may employ an approximation when locally the use of Cartesian coordinates is justified, and an external cosmological background is described by a non-vanishing constant Ricci scalar.

4. Model example

For simplicity, let us follow a simple 1-dimensional case $(t, x, y = 0, z = 0)$. So we start with the system

$$
\begin{aligned}
&(i\gamma^0\partial_t + i\gamma^1\partial_1 - M_1)\Phi_1 + d_1(L_1\Phi_1 + L_2\Phi_2 + L_3\Phi_3) = 0\,,\\
&(i\gamma^0\partial_t + i\gamma^1\partial_1 - M_2)\Phi_2 + d_2(L_1\Phi_1 + L_2\Phi_2 + L_3\Phi_3) = 0\,,\\
&(i\gamma^0\partial_t + i\gamma^1\partial_1 - M_3)\Phi_3 + d_3(L_1\Phi_1 + L_2\Phi_2 + L_3\Phi_3) = 0\,;
\end{aligned} \tag{21}
$$

where $d_i = Y_i \, \frac{R}{4}, \; i = 1, 2, 3$. The system (21) is transformed to the matrix form

$$(i\gamma^0\partial_t + i\gamma^1\partial_1 - M) \begin{vmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix} = - \begin{vmatrix} M - M_1 + d_1 L_1 & d_1 L_2 & d_1 L_3 \\ d_2 L_1 & M - M_2 + d_2 L_2 & d_2 L_3 \\ d_3 L_1 & d_3 L_2 & M - M_3 + d_3 L_3 \end{vmatrix} \begin{vmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix},$$

or in brief $\Delta \; \Phi = T \; \Phi \,, \; \Delta = -(i\gamma^0\partial_t + i\gamma^1\partial_1 - M)$. The 3-column Φ is subject to a linear transformation to diagonalize the mixing matrix T:

$$\bar{\Phi} = S\Phi \,, \quad STS^{-1} = T_0 = \begin{vmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{vmatrix} . \tag{22}$$

After that we will have three separate Dirac-like equations with new mass parameters $\bar{M}_1$:

$$\begin{aligned} (\, i\gamma^0\partial_t + i\gamma^1\partial_1 - \bar{M}_1 \,) \, \bar{\Phi}_1 = 0 \,, \quad \bar{M}_1 = M + \mu_1 \,; \\ (\, i\gamma^0\partial_t + i\gamma^1\partial_1 - \bar{M}_2 \,) \, \bar{\Phi}_2 = 0 \,, \quad \bar{M}_2 = M + \mu_2 \,; \\ (\, i\gamma^0\partial_t + i\gamma^1\partial_1 - \bar{M}_3 \,) \, \bar{\Phi}_3 = 0 \,, \quad \bar{M}_3 = M + \mu_3 \,, \end{aligned} \tag{23}$$

by physical reason we assume the real-valuedness of μ_i, and positiveness of $M + \mu_i$. To find the transformation S, we should solve the equation $ST = T_0 S$. It leads to three linear subsystems. For diagonal elements in T_0 we get a cubic algebraic equation. To get a more simple form of this equation we are to make several steps. First, we apply the substitution $M_i = \frac{M}{\lambda_i}$, where M is arbitrary. We may simplify the task without loss of generality by setting $c_1 = c_2 = 1$, then the cubic equation for λ_i give simple roots

$$\lambda^3 - 2\lambda^2 + (1+k)\lambda - k = 0 \,, \; \lambda_3 = 1 \,, \lambda_{1,2} = \frac{1}{2} \mp \frac{1}{2}\sqrt{1-4k} \,, \; k = (a^2 + b^2) \in (0, \frac{1}{4}) \,. \tag{24}$$

Correspondingly, the masses M_i equal to

$$M_3 = M \,, \quad M_{1,2} = \frac{2M}{1 \mp \sqrt{1-4k}} \,, \quad k \in (0, \frac{1}{4}) \,. \tag{25}$$

It is convenient to introduce dimensionless parameter r: $R = 6r \; M^2 \implies d_i = Mrb(\lambda_i - \frac{1}{2}) = MD_i$, and the notation $L = 1 + \frac{b}{\sqrt{6}}, 0 < 2b < 1$. Besides, the roots may be done dimensionless as well $\mu_i = M \; \Delta_i$.

In this way, we arrive at the following cubic equation for Δ_i:

$$\Delta^3 + \frac{2k-1}{k}\Delta^2 + \left[1 + \left(1 - \frac{\sqrt{6}}{2(\sqrt{6}+b)} \frac{1-4k}{2k} \right) r \right] \Delta + \left(1 + \frac{\sqrt{6}}{2(\sqrt{6}+b)} \frac{1-4k}{2k} \right) r = 0 \,, \tag{26}$$

where $0 < k < \frac{1}{4}$, $0 < b < 2$. Taking in mind relationship $R = 6rM^2$, we expect dimensionless parameter r is small, because effect of geometry in the model under consideration should be small. Besides, there exist two physically different possibilities: $r > 0$ at positive curvature, and $r < 0$ at negative curvature. We have followed several case of weak and strong gravitation of different curvature sings:

$$\begin{aligned} r = +10^{-30} \,, r = +10^{-5}, r = +10^{-3}, r = +10^{-2}, r = +1 \; ; \\ r = -10^{-30} \,, r = -10^{-5}, r = -10^{-3}, r = -10^{-2}, r = -1 \; . \end{aligned} \tag{27}$$

The cases $r = \pm 10^{-2}, \pm 1$ correspond to very strong curvature of space. Numerical study showed that dependence of the roots Δ_i upon parameter $b \in (0, 2)$ is very small, by this reason below we take the value $b = 0$.

Numerical study shows that at the values of curvature in the region $r = \pm 10^{-2}, \pm 1$ the model becomes non-interpretable, because there appear complex-valued roots.

Acknowledgments
The present work was developed within the framework of the project BRFFR A18-RA-015, set within the framework of the agreement between the National Academy of Sciences of Belarus and the Romanian Academy, and of the project BRFFR F18Y-009.

Special thanks to EU Project MOST, Grant R-f26C-50842, for financial support of attending the 26-th Internatinal Conference on Integrable Systems and Quantum Symmetries, July 8–12, 2019, Prague.

Table 1. The roots at$b = 0,\ r = -10^{-5}$.

k	Δ_1	Δ_2	Δ_3
0.24	0.0000104	0.667	1.500
0.20	0.0000125	0.382	2.618
0.16	0.0000156	0.250	4.000
0.12	0.0000208	0.162	6.171
0.08	0.0000313	0.096,	10.404
0.04	0.0000626	0.043	22.956

Table 2. The roots at$b = 0,\ r = +10^{-5}$.

k	Δ_1	Δ_2	Δ_3
0.24	0.667	1.500	-0.00001
0.20	0.382	2.618	-0.00001
0.16	0.250	3.999	-0.00002
0.12	0.162	6.171	-0.00002
0.08	0.096	10.404,	-0.00003
0.04	0.044	22.956	-0.00006

Table 3. The roots at$b = 0,\ r = -10^{-2}$.

k	Δ_1	Δ_2	Δ_3
0.24	0.011	0.637	1.519
0.20	0.013	0.363	2.624
0.16	0.017	0.231	4.002
0.12	0.025	0.137	6.172
0.08	10.403	0.485+0.256 i	0.485-0.256 i
0.04	22.955	0.226+0.470 i	0.226-0.470 i

Table 4. The roots at$b = 0,\ r = +10^{-2}$.

k	Δ_1	Δ_2	Δ_3
0.24	0.697	1.480	-0.010
0.20	0.399	2.613	-0.012
0.16	0.267	3.998	-0.015
0.12	0.181	6.171	-0.019
0.08	0.120	10.405	-0.025
0.04	0.077	22.958	=-0.035

Table 5. The roots at$b = 0,\ r = +1$.

k	Δ_1	Δ_2	Δ_3
0.24	1.265+1.13 i	-0.362	1.265-1.13 i
0.18	1.173	2.804	-0.422
0.12	0.695	6.128	-0.489
0.06	0.503,	=14.727	-0.563

Table 6. The roots at$b = 0,\ r = -1$.

k	Δ_1	Δ_2	Δ_3
0.24	2.339	-0.86+0.66 i	-0.86-0.66 i
0.18	=3.556	-0.247+0.63 i	-0.24-0.63 i
0.12	6.213	0.60+0.58 i	0.60-0.58 i
0.06	14.468	0.996+0.53 i	0.996-0.53 i

References

[1] Gel'fand I M and Yaglom A M 1948 *Zh. Eksp. Teor. Fiz.* **18** 703–733
[2] Red'kov V M 2009 *Fields in Riemannian space and the Lorentz group* (Minsk: Belarusian Science)
[3] Pletjukhov V A, Red'kov V M and Strazhev V I 2015 *Relativistic wave equations and intrinsic degrees of freedom* (Minsk: Belarusian Science)
[4] Kisel V V, Ovsiyuk E M, Balan V, Veko O V and Red'kov V M 2018 *Elementary particles with internal structure in externalfield. Vol. I. General theory; Vol. II. Physical problems* (New York: Nova Science Publishers Inc.)