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Geometric Algorithms for Finding a Point in the Intersection of Balls

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Abstract—We consider a problem of finding a point in the intersection of n balls in the Euclidean space E^m. For the case $m = 2$ we suggest two algorithms of the complexity $O(n^2 \log n)$ and $O(n^3)$ operations, respectively. For the general case we suggest an exact polynomial recursive algorithm which uses the orthogonal transformation of the space E^m.

Keywords: intersection of balls, approximation a convex set by ellipsoids, polynomial algorithm, delivery applications of drones, configuration of the swarm of drones

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1. INTRODUCTION

In [1] the following problem was formulated. There is a set of n balls in the m-dimensional Euclidean space E^m. Each ball $\mathcal{B}_i$, $1 \leqslant i \leqslant n$, is determined by setting its center O_i and radius R_i. The objective is to reveal whether the intersection of n balls is nonempty, and in the case of positive answer to find a point in this intersection.

An application of this problem in the case $m = 2$ is considered in [1]. On a plane there are n transmitting stations of various capacities. A signal from station i located at the point O_i can be received at a distance not exceeding R_i units. The objective is to determine whether it is possible to find a place for building a receiving station (taking into account only characteristics of the range of propagation of signals from transmitting stations) so that this receiving station can receive signals from all transmitting stations. This technical interpretation can be generalized to the case of three-dimensional space ($m = 3$), when the model takes into account the height of transmitting and receiving stations. Besides, in the three-dimensional case one can give different interpretations of the problem in the context of the actual application associated with the use of unmanned aerial vehicles (drones). For example, residents of small villages located in remote mountainous regions can use drones for delivering medicines, various small goods and postal packages. In village i the launch pad for the drone is located at the point O_i. Taking into account its technical characteristics, the drone from village i can travel a distance not exceeding $2R_i$ units (there and back) without recharging. It is necessary to determine coordinates of the hovering point of an air vehicle (helicopter, airship) used as a warehouse of goods, so that this warehouse could be reached for drones from all villages. A similar model can be used for planning the delivery process of medicines, medical devices, supplies and so on by drones to several rescue teams in areas of large-scale disasters. In such situations, constantly changing circumstances may require the multiple solving the problem of determining coordinates of the hovering point for an air vehicle (a warehouse) so that it could be reached by drones of all rescue teams.

In [1], two approaches to the problem under consideration in m-dimensional case are suggested. The first approach is related to the problem of minimizing a linear function under quadratic constraints. The second one is based on the well-known ellipsoid method (see, for example, [3]). Notice that the problem under consideration can be attributed to a wide class of problems of approximating a convex set by ellipsoids in the space E^m [4]. This class of problems is divided into two subclasses of internal and external approximation problems. A solution to the considered problem can be obtained by solving the following internal approximation problem: find an ellipsoid of the largest volume contained in the intersection of n given ellipsoids, if this intersection is non-empty (of course, to solve the problem we are interested in, we should consider a special case in which ellipsoids are replaced by balls). In [4] it was demonstrated that the last problem can be reduced to a convex program problem. A similar external approximation problem can be formulated as follows: find an ellipsoid of the smallest volume containing an intersection of given ellipsoids if this intersection is non-empty. Unlike the internal approximation problem mentioned above, the external approximation problem is NP-hard [4]. In [5], its special case is considered: find the ball of the smallest radius containing the intersection of given n balls. In [5] it is shown that if the intersection of n balls is non-empty and $n \leqslant m-1$, the problem of finding the ball of the smallest radius can be solved by minimizing the convex quadratic function.

Notice that all previous approaches to the problems under consideration are polynomial from the theoretical point of view. However, their implementation in practice may be not effective, especially for large values of n (see, for example, the comment in [6] concerning the ellipsoid algorithm). Besides, the implementing may be difficult due to the rather abstract nature of the used mathematical constructions (for example, see in [4] the reduction of the problem of finding the largest volume ellipsoid contained in the intersection of given ellipsoids to the convex program). In this paper we suggest an alternative geometric approach to the problem of finding a point in the intersection of n balls from the space E^m, which uses the well-known apparatus of linear algebra and analytical geometry. As a result of this alternative approach, we design exact polynomial algorithms for solving the problem.

The paper is organized as follows. In Section 2 two algorithms for solving the problem in the case $m = 2$ are presented. The more specific algorithm $BALLS1$ can be implemented in $O(n^2 \log n)$ operations. Algorithm $BALLS2$ has the more simple structure, but the higher computational complexity $O(n^3)$ operations. In Section 3 we consider the general m-dimensional case for which the recursive algorithm $BALLS3(m,n)$ is developed. During operating, algorithm $BALLS3(m,n)$ uses either algorithm $BALLS1$ or algorithm $BALLS2$. This fact determines the computational complexity of algorithm $BALLS3(m,n)$, which is either $O(n^{2m-4}(nm^2+m^3+n^2\log n))$ or $O(n^{2m-4}(nm^2+m^3+n^3))$ operations, respectively. The concluding remarks are presented in Section 4.

2. ALGORITHMS FOR FINDING A POINT IN THE INTERSECTION OF CIRCLES

In this section we describe algorithms for finding a point in the intersection of circles on a plane. However, to construct the first algorithm, we should treat points on a plane as points in three-dimensional space with zero third coordinate. Thus, we introduce a Cartesian rectangular coordinate system $(O; \vec{i}, \vec{j}, \vec{k})$.

Consider n circles on the plane Oxy. Each circle B_i, $1 \leqslant i \leqslant n$, is determined by its center $O_i(x_i, y_i, 0)$ and radius r_i. The boundary of the circle B_i is the circumference C_i defined by the equation $(x - x_i)^2 + (y - y_i)^2 = r_i^2$. Let us number the circles B_i, $1 \leqslant i \leqslant n$, in the non-increasing order of their radii: $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n$.

2.1. Pre-Processing

First, we check whether the given circles intersect in pairs. For each pair of circles B_i and B_j, $1 \leqslant i \leqslant n-1$, $i < j \leqslant n$, calculate the distance d_{ij} between their centers.

If $d_{ij} > r_i + r_j$, then the circles B_i and B_j do not intersect, so the problem has no solution.

If $d_{ij} \leqslant r_i - r_j$, then the circle B_j is inside the circle B_i. Therefore, we can exclude the circle B_i from further consideration and solve the problem for $n-1$ circles.

If $d_{ij} = r_i + r_j$, then the circles B_i and B_j have a unique common point M, where their borders (circumferences) C_i and C_j touch. We find the vector $\overrightarrow{O_iO_j}$ and normalize it. Let $\overrightarrow{e_{ij}}$ be a unit vector with the same direction as the vector $\overrightarrow{O_iO_j}$. Lay aside the vector $r_i\overrightarrow{e_{ij}}$ from the point O_i and get the point M where the circles B_i and B_j touch. It remains to check whether the found point M belongs to all other circles. If it belongs, then the point M is the solution to the problem. Otherwise, the problem has no solution.

2.2. The Main Part of Algorithm BALLS1

In the following consideration, we assume w.l.o.g. that for any pair of circles B_i and B_j, $1 \leqslant i \leqslant n-1$, $i < j \leqslant n$, the inequality $r_i - r_j < d_{ij} < r_i + r_j$ holds. This means that the circumferences C_i and C_j intersect at two points.

Fix some circumference C_i and find its intersection points with the other circumference C_j, $i \neq j$. To do this, solve a system of equations:

$$\begin{cases} (x-x_i)^2 + (y-y_i)^2 = r_i^2 \\ (x-x_j)^2 + (y-y_j)^2 = r_j^2. \end{cases} \tag{1}$$

Subtracting the first equation of system (1) from the second one, we get an equation of the form $fx + gy + h = 0$, which determines a line that passes through the intersection points of the circumferences C_i and C_j. Expressing y from the equation $fx + gy + h = 0$ and substituting it into the first equation of system (1), we get a quadratic equation of the form $ax^2 + bx + c = 0$, which has two different real roots $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$. From the equation $fx + gy + h = 0$ we get the corresponding values $y_{ij}^{(1)}$ and $y_{ij}^{(2)}$. Thus, we define the points $M_{ij}^{(1)}(x_{ij}^{(1)}, y_{ij}^{(1)}, 0)$ and $M_{ij}^{(2)}(x_{ij}^{(2)}, y_{ij}^{(2)}, 0)$ of intersection of the circumferences C_i and C_j. Note that the circles B_i and B_j are guaranteed to have a common segment $[M_{ij}^{(1)}, M_{ij}^{(2)}]$ (see Fig. 1).

The points $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$ divide the circumference C_i into two arcs. In the following, only the arc that is inside the circle B_j will be of interest. Let us agree that the traversing the chosen

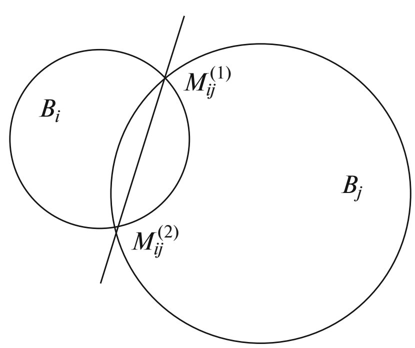

Fig. 1. The intersection of the balls B_i and B_j.

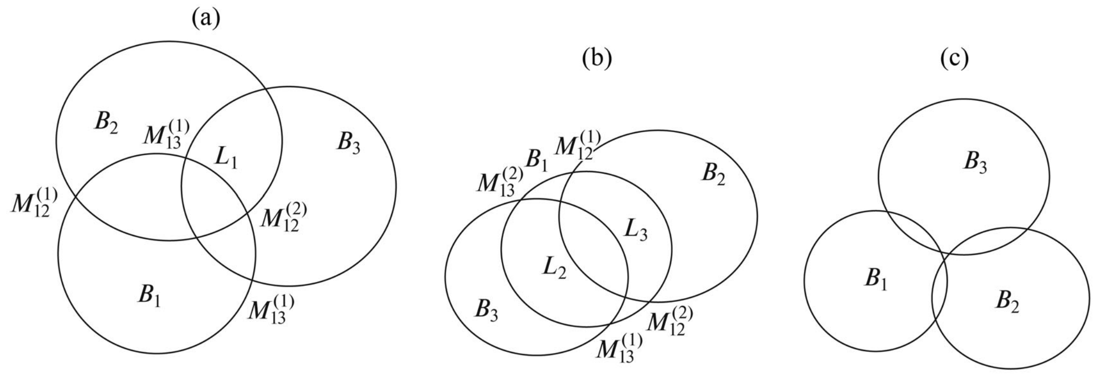

Fig. 2. The cases of mutual arrangement of three balls.

arc will be counter-clockwise. Thus, one of the points $M_{ij}^{(1)}$, $M_{ij}^{(2)}$ will be chosen as the starting point of the path while the other one will be chosen as the end point. Consider three non-coplanar vectors $\overrightarrow{O_iO_j}$, $\overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}$, $\vec{k}$ and find their mixed product. If $(\overrightarrow{O_iO_j}, \overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}, \vec{k}) > 0$, i.e., the triple of vectors $\overrightarrow{O_iO_j}$, $\overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}$, $\vec{k}$ is right, then the point $M_{ij}^{(1)}$ will be the starting point of the path (paint it white) and the point $M_{ij}^{(2)}$ will be the end point of the path (paint it black). If $(\overrightarrow{O_iO_j}, \overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}, \vec{k}) < 0$, i.e., the triple of vectors $\overrightarrow{O_iO_j}$, $\overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}$, $\vec{k}$ is left, then the point $M_{ij}^{(2)}$ will be the starting point of the path (paint it white) and the point $M_{ij}^{(1)}$ will be the end point of the path (paint it black). The start (white) point and end (black) point of the path along with agreeing on the counter-clockwise direction completely define the arc of the circumference C_i located inside the circle B_j. Denote this arc by L_{ij} and call it a proper arc of the circumference C_i for the circle B_j.

Consider all circles B_j, $1 \leqslant j \leqslant n$, $j \neq i$, and find the set of all proper arcs L_{ij} of the circumference C_i. If the intersection L_i of all proper arcs is non-empty, i.e., $L_i = \bigcap_{1\leqslant j\leqslant n, j\neq i} L_{ij} \neq \emptyset$, then the arc L_i of the circumference C_i will be inside each circle B_j, $1 \leqslant j \leqslant n$, $j \neq i$. In addition, the chord connecting the ends of the arc L_i will be inside the intersection of all circles B_j, $1 \leqslant j \leqslant n$. Note that the arc L_i is a part of the border of the intersection area of all circles. If the intersection of all proper arcs is empty, i.e., $L_i = \bigcap_{1\leqslant j\leqslant n, j\neq i} L_{ij} = \emptyset$, then either no part of the border of the intersection area of all circles belongs to the circumference C_i (i.e., the intersection of all circles is inside the open circle $B_i \setminus C_i$), or the intersection of all circles is empty. If the intersection of all circles is inside the open circle $B_i \setminus C_i$, then there is another circle, e.g., the circle B_k, $k \neq i$, such that part of the border of the intersection area of all circles is an arc L_k of the circumference C_k. Considering successively all circles B_i, $1 \leqslant i \leqslant n$, we either find such a circle B_k, or conclude that the intersection of all circles is empty.

Consider all typical cases of mutual arrangement of circles on the example of three circles (see Fig. 2).

1. The arc L_1 of the circumference C_1 is the intersection of the proper arcs L_{12} and L_{13}. Notice that the arc L_1 is a part of the border of the intersection area of the circles B_1, B_2, and B_3.

2. The border of the intersection area of the circles B_1, B_2, and B_3 does not contain any arc of the circumference C_1. The border of the intersection area of all circles consists of arcs L_2 and L_3 of circumferences C_2 and C_3.

3. The intersection of the pairwise intersecting circles B_1, B_2, and B_3 is empty.

Consider the issue how effectively define the intersection $L_i = \bigcap_{1\leqslant j\leqslant n, j\neq i} L_{ij}$ of all proper arcs of the circumference C_i, $1 \leqslant i \leqslant n$.

Each proper arc L_{ij} is defined by setting points $M_{ij}^{(1)}(x_{ij}^{(1)}, y_{ij}^{(1)}, 0)$ and $M_{ij}^{(2)}(x_{ij}^{(2)}, y_{ij}^{(2)}, 0)$ of the intersection of the circumferences C_i and C_j and indicating which of them is the starting point (the end point, respectively) of the path. To each point $M_{ij}^{(l)}(x_{ij}^{(l)}, y_{ij}^{(l)}, 0)$, $1 \leqslant l \leqslant 2$, we put into accordance an element $\phi_{ij}^{(l)}(x_{ij}^{(l)}, y_{ij}^{(l)}, \alpha_{ij}^{(l)})$. The marker-variable $\alpha_{ij}^{(l)}$ indicates the color of the point $M_{ij}^{(l)}$. If $M_{ij}^{(l)}$ is colored black, then we assume $\alpha_{ij}^{(l)} = 1$, otherwise $\alpha_{ij}^{(l)} = 0$. Thus, each proper arc is defined by two elements $\phi_{ij}^{(1)}$ and $\phi_{ij}^{(2)}$.

Let the set of all elements $\phi_{ij}^{(l)}$, $1 \leqslant l \leqslant 2$, $1 \leqslant j \leqslant n$, $j \neq i$ be found for the circumference C_i. Divide the set of elements $\phi_{ij}^{(l)}$ into two subsets. If $y_{ij}^{(l)} < 0$, then assign the element $\phi_{ij}^{(l)}(x_{ij}^{(l)}, y_{ij}^{(l)}, \alpha_{ij}^{(l)})$ to the set N_1, otherwise assign $\phi_{ij}^{(l)}$ to the set N_2. Thus, the set N_1 (set N_2) of elements $\phi_{ij}^{(l)}$ determines the set of points $M_{ij}^{(l)}$ of the circumference C_i, which can be put in a one-to-one correspondence to their projections on the axis Ox (of course, coinciding points $M_{ij}^{(l)}$ are projected into identical points on the axis Ox).

Order the set N_1 (N_2) of elements $\phi_{ij}^{(l)}$ in non-decreasing order (in non-increasing order) of values $x_{ij}^{(l)}$. Denote the obtained sequences of elements $\phi_{ij}^{(l)}$ as π_1 and π_2, respectively. Form a sequence $\pi = (\pi_1, \pi_2)$, which is an enumeration of the start and end points of proper arcs L_{ij} in the process of traversing the circumference C_i in the counter-clockwise direction. Number the elements of the sequence π, denoting them by $\psi_k(x_k, y_k, \alpha_k)$, i.e., $\pi = (\psi_1, \psi_2, \ldots, \psi_{2n-2})$.

Looking through the sequence π from left to right, find in it subsequences of elements corresponding to coinciding points on the circumference C_i, if such exist. Let $\bar{\pi} = (\psi_k, \psi_{k+1}, \ldots, \psi_{k+t})$ be one of such subsequences. All elements $\psi_p(x_p, y_p, \alpha_p)$ of the subsequence $\bar{\pi}$ have the same values x_p and the same values y_p, but they may have different values α_p. Divide the elements of $\bar{\pi}$ into two subsequences: we assign all elements that have $\alpha_p = 0$ to the subsequence $\bar{\pi}'$, the remaining elements will be assigned to the subsequence $\bar{\pi}''$ (they have $\alpha_p = 1$). If the element ψ_{k-1} that precedes the subsequence $\bar{\pi}$ in the sequence π has $\alpha_{k-1} = 0$, then in the sequence π replace the subsequence $\bar{\pi}$ with $(\bar{\pi}', \bar{\pi}'')$. If $\alpha_{k-1} = 1$, then replace in π the subsequence $\bar{\pi}$ with the subsequence $(\bar{\pi}'', \bar{\pi}')$. We perform similar actions with all subsequences of elements corresponding to coinciding points of the circumference C_i. Such constructions are performed in order to minimize the number of switches of the indicator α_k in the list of the start and end points of proper arcs L_{ij} when traversing the circumference C_i.

Renumber the elements of the transformed sequence π according to their location. If the neighboring elements $\psi_k(x_k, y_k, \alpha_k)$ and $\psi_{k+1}(x_{k+1}, y_{k+1}, \alpha_{k+1})$, $1 \leqslant k \leqslant 2n-3$, in the sequence π have $\alpha_k \neq \alpha_{k+1}$, then we say that there takes place an indicator switch. Looking through the sequence π, we calculate the number ρ of indicator switches.

Lemma 1. *Let the sequence π of elements $\psi_k(x_k, y_k, \alpha_k)$, $1 \leqslant k \leqslant 2n-2$ corresponding to the start and end points of proper arcs of the circumference C_i be constructed according to the rules described above. If $1 \leqslant \rho \leqslant 2$, then the intersection L_i of all proper arcs of the circumference C_i is non-empty, i.e., $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$.*

The Appendix provides a constructive proof of Lemma 1, which shows how to find the arc $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$. Thus, the problem is solved, because any point on the arc L_i (as well as any point on the chord connecting the ends of L_i) belongs to the intersection of all circles.

The following example shows that if $\rho \geqslant 3$, then one cannot make any conclusion about the existence of the non-empty intersection $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij}$ of the proper arcs of the circumference C_i.

Example. We are given a sequence of indicators $(0, 1, 0, 1)$. For this sequence, the number of the indicator switches is $\rho = 3$. Two possible situations are shown on Figs. 2b and 2c. For each

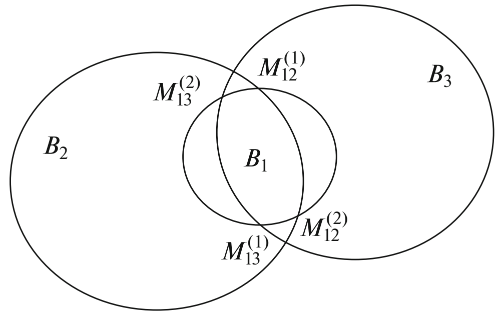

Fig. 3. The bound of the intersection area contains two arcs of the circumference C_1.

of these situations, the intersection of the proper arcs of the circumference C_1 is empty. Figure 3 shows the third possible situation, when the intersection of the proper arcs of the circumference C_1 consists of two arcs $\smile M_{12}^{(1)}M_{13}^{(2)}$ and $\smile M_{13}^{(1)}M_{12}^{(2)}$. Note that the degree measure of at least one of the arcs L_{12} and L_{13} is greater than $180°$.

Lemma 2. *The degree measure of the proper arc L_{ij} of the circumference C_i is greater than* $180°$ *only if its radius r_i is less than radius r_j of the circumference C_j, $j \neq i$.*

Corollary. *If radius r_i of the circumference C_i is greater than radii of all other circumferences C_j, $j \neq i$, and $\rho \geqslant 3$, then $L_i = \bigcap_{1\leqslant j\leqslant n, j\neq i} L_{ij} = \emptyset$.*

Recall that all circles B_i, $1 \leqslant i \leqslant n$, are numbered in the non-increasing order of their radii: $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n$. Consider the circle B_1 and determine the number ρ of the indicator switches for the circumference C_1. According to Lemma 1 and Corollary, for the circumference C_1 that has the largest radius, depending on the value of ρ, we can deduce whether the situation $L_1 \neq \emptyset$ takes place.

Suppose we checked the circumference C_1 and came to the conclusion that $L_1 = \emptyset$. If the intersection of all circles is a non-empty set, then it is located inside the open circle $B_1 \setminus C_1$. Temporarily delete the circle B_1 and consider the circle B_2, i.e., the circle with the next largest radius. We continue the process until we find a circle B_k such that for the circumference C_k we have $L_k = \bigcap_{k<j\leqslant n} L_{kj} \neq \emptyset$. Let M_{kp} and M_{kq} be the starting and ending points of the arc L_k of the circumference C_k. By construction, the point M_{kp} (and also any point of the arc L_k as well as any point of the chord connecting its ends) belongs to all circles B_k, $B_{k+1}, \ldots, B_n$. It remains to check whether the point M_{kp} (or any of the mentioned points) belongs to the circles $B_1, \ldots, B_{k-1}$. If this is so, then the checked point is the one being searched for. Otherwise, the problem has no solution.

Below we give a formal description of the algorithm.

Algorithm 1. $BALLS1$

Input: Circles $B_1, \ldots, B_n$, determined by their centers $O_i(x_i, y_i, 0)$ and radii r_i, $i = \overline{1, n}$.

Output: A common point for all circles $B_1, \ldots, B_n$ or the answer that the problem has no solution.

1. Number the circles $B_1, \ldots, B_n$ such that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_n$.
2. **FOR** $i = 1$ to n **DO**
3. $N_1 = \emptyset, N_2 = \emptyset$.
4. **FOR** $j = i + 1$ to n **DO**
5. For the circumferences C_i and C_j find their intersection points $M_{ij}^{(1)}(x_{ij}^{(1)}, y_{ij}^{(1)}, 0)$ and $M_{ij}^{(2)}(x_{ij}^{(2)}, y_{ij}^{(2)}, 0)$ by solving the system (1).

6. **IF** $(\overrightarrow{O_iO_j}, \overrightarrow{M_{ij}^{(1)}M_{ij}^{(2)}}, \vec{k}) > 0$ **THEN** set $\alpha_{ij}^{(l)} = 0$, $\alpha_{ij}^{(2)} = 1$
(for the arc L_{ij}, point $M_{ij}^{(1)}$ is the start point, and $M_{ij}^{(2)}$ is the end point)
ELSE set $\alpha_{ij}^{(l)} = 1$, $\alpha_{ij}^{(2)} = 0$ (for the arc L_{ij}, point $M_{ij}^{(2)}$ is the start point, and $M_{ij}^{(1)}$ is the end point).
7. **FOR** $l = 1$ to 2 **DO**
8. Create an element $\phi_{ij}^{(l)}(x_{ij}^{(l)}, y_{ij}^{(l)}, \alpha_{ij}^{(l)})$.
9. **IF** $y_{ij}^{(l)} < 0$ **THEN** $N_1 = N_1 \cup \{\phi_{ij}^{(l)}\}$ **ELSE** $N_2 = N_2 \cup \{\phi_{ij}^{(l)}\}$.
END FOR l
END FOR j
10. Order the set N_1 of elements $\phi_{ij}^{(l)}(x_{ij}^{(l)}, y_{ij}^{(l)}, \alpha_{ij}^{(l)})$ in non-decreasing order of $x_{ij}^{(l)}$, and order the set N_2 in non-increasing order of $x_{ij}^{(l)}$. Denote the obtained sequences by π_1 and π_2, respectively.
11. Create a sequence $\pi = (\pi_1, \pi_2)$. Find in the sequence π subsequences of elements corresponding to coinciding points and reorder them (see above the description of the reordering process).
12. Looking through the sequence π from left to right, determine the number ρ of the indicator switches.
13. **IF** $1 \leqslant \rho \leqslant 2$ **THEN** define the arc $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$ (see the proof of Lemma 1), set $k = i$ and go to step 15.
END FOR i
14. Go to step 19.
15. Determine the start and end points M_{kp} and M_{kq} of the arc L_k of the circumference C_k.
16. **FOR** $i = 1$ to $k-1$ **DO**
17. **IF** $M_{kp} \notin B_i$, **THEN** go to step 19.
END FOR i
18. Point M_{kp} is a solution of the problem. Stop.
19. The problem has no solution. Stop.

Ordering the sets N_1 and N_2 (step 10) is the most time-consuming operation that is included in the loop on i (steps 2–13). The total computational complexity of the algorithm is $O(n^2 \log n)$ operations.

2.3. The Main Part of Algorithm BALLS2

Each circle B_i, $1 \leqslant i \leqslant n$, is a convex set. It is known that the intersection of a finite number of convex sets is a convex set (see, for example, [6]). The boundary of the intersection area of n circles consists of arcs of some of the circumferences C_i, $1 \leqslant i \leqslant n$. In fact, the intersection of circles is a convex combination (see [6]) of the points of its boundary. Among all boundary points, the junction points of the arcs are of primary interest. Using convex combinations of these points, one can find the other points of the intersection area, if necessary.

The algorithm looks through all pairs of circumferences C_i and C_j, $1 \leqslant i, j \leqslant n$, $i \neq j$, determines their intersection points $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$, and checks whether at least one of them belongs to all other circles B_k, $1 \leqslant k \leqslant n$, $k \neq i, j$. In case of positive answer, the found point (e.g., $M_{ij}^{(1)}$) is the solution to the problem.

If the intersection of circles is non-empty, then the border of the intersection area contains at least two arcs (taking into account preprocessing), and thus at least two points of arc junction.

Therefore, after iterating over the intersection points of all pairs of circumferences C_i and C_j, $1 \leqslant i$, $j \leqslant n$, $i \neq j$, the algorithm will determine the required point.

A formal description of this algorithm is given below.

Algorithm 2. $BALLS2$

Input: Circles $B_1, \ldots, B_n$, determined by their centers $O_i(x_i, y_i, 0)$ and radii $r_i, i = \overline{1, n}$.

Output: A common point for all circles $B_1, \ldots, B_n$ or the answer that the problem has no solution.

1. **FOR** $i = 1$ to $n - 1$ **DO**
2. **FOR** $j = i + 1$ to n **DO**
3. For the circumferences C_i and C_j find their intersection points $M_{ij}^{(1)}$ and $M_{ij}^{(2)}$.
4. $l = 1$
5. **FOR** $k = 1$ to n, $k \neq i$, $k \neq j$ **DO**
6. **IF** $M_{ij}^{(l)} \notin B_k$, **THEN** go to step 8.
   END FOR k
7. Point $M_{ij}^{(l)}$ is a solution to the problem. Stop.
8. $l = l + 1$
9. **IF** $l \leqslant 2$, **THEN** go to step 5.
   END FOR j
   END FOR i
10. The problem has no solution. Stop.

The loop for the variable i contains a nested loop for the variable j, which in turn contains a loop for the variable k. Therefore, the total computational complexity of the algorithm is $O(n^3)$ operations.

3. FINDING A POINT IN THE INTERSECTION OF BALLS IN THE SPACE E^m

Suppose that in m-dimensional affine Euclidean space E^m we are given a Cartesian rectangular coordinate system $(O, \vec{e_1}, \ldots, \vec{e_m})$, where $\vec{e_1}, \ldots, \vec{e_m}$ is an orthonormal basis. There are n balls. Each m-dimensional ball $\mathcal{B}_i$, $i = \overline{1, n}$, is determined by its center $O_i(x_1^{(i)}, \ldots, x_m^{(i)})$ and radius R_i. The boundary of the ball $\mathcal{B}_i$ is m-dimensional sphere S_i, defined by the equation $(x_1 - x_1^{(i)})^2 + \ldots + (x_m - x_m^{(i)})^2 = R_i^2$. Pre-processing, i.e., checking whether the given balls intersect in pairs, is performed in the same way as it was done for the case $m = 2$ (see Section 2.1). Here we take into account that in the affine Euclidean space E^m, the distance between two points $O_i(x_1^{(i)}, \ldots, x_m^{(i)})$ and $O_j(x_1^{(j)}, \ldots, x_m^{(j)})$ is defined by the formula $d_{ij} = \sqrt{(x_1^{(i)} - x_1^{(j)})^2 + \ldots + (x_m^{(i)} - x_m^{(j)})^2}$. Further we shall assume w.l.o.g. that for any pair of balls $\mathcal{B}_i$ and $\mathcal{B}_j$ such that $R_i \geqslant R_j$, the inequality $R_i - R_j < d_{ij} < R_i + R_j$ holds, where d_{ij} is the distance between their centers O_i and O_j. This means that the m-dimensional spheres S_i and S_j intersect along the "circumference" (i.e., along the $(m-1)$-dimensional sphere) C_{ij}.

Fix two spheres S_i and S_j with centers $O_i(x_1^{(i)}, \ldots, x_m^{(i)})$ and $O_j(x_1^{(j)}, \ldots, x_m^{(j)})$, and consider a system of equations

$$\begin{cases} \left(x_1 - x_1^{(i)}\right)^2 + \ldots + \left(x_m - x_m^{(i)}\right)^2 = R_i^2 \\ \left(x_1 - x_1^{(j)}\right)^2 + \ldots + \left(x_m - x_m^{(j)}\right)^2 = R_j^2. \end{cases} \tag{2}$$

From the geometric point of view, its solution is a $(m-1)$-dimensional sphere (a circumference for $m - 1 = 2$) C_{ij}, along which the m-dimensional spheres S_i and S_j intersect. Subtracting the first equation of system (2) from the second one, we get an equation of the form

$$\alpha_1 x_1 + \ldots + \alpha_m x_m + \beta = 0, \tag{3}$$

which is a hyperplane equation in the affine space E^m. The hyperplane (3) cuts off from the balls $\mathcal{B}_i$ and $\mathcal{B}_j$ the parts that make up their intersection area while the $(m-1)$-dimensional sphere C_{ij} lies in this hyperplane and is the boundary of the "slices" of the balls $\mathcal{B}_i$ and $\mathcal{B}_j$. "Slices" of the balls $\mathcal{B}_i$ and $\mathcal{B}_j$, i.e., $(m-1)$-dimensional balls (circles for $m-1=2$) with the border C_{ij}, are glued along the hyperplane (3). In the space E^m $(m-1)$-dimensional ball with the boundary C_{ij} is an analogue of the chord connecting the intersection points of two circumferences (for the case $m=2$).

Using some orthogonal transformation of the Cartesian coordinate system (some rotation of the coordinate system around its origin in the case $m=3$), Eq. (3) can be reduced to the form $x_m = x^*$, where x^* is a constant. Substituting $x_m = x^*$ into the equation $(x_1 - x_1^{(i)})^2 + \ldots + (x_m - x_m^{(i)})^2 = R_i^2$ (the first equation of system (2)), we get the canonical equation of the $(m-1)$-dimensional sphere C_{ij}. Below we show how to construct matrix T of such orthogonal transformation (operator).

Looking through the coefficients $\alpha_1, \ldots, \alpha_m$ of Eq. (3), one can find the coefficient $\alpha_l \neq 0$. Set the coordinates x_i, $i = \overline{1,m}$, $i \neq l$, of the points $P_0, P_1, \ldots, P_{m-1}$ on the hyperplane (3), according to the following table:

	x_1	x_2	$\ldots$	x_l	$\ldots$	x_m
P_0	0	0	$\ldots$	$\tilde{x}_l^{(0)}$	$\ldots$	0
P_1	1	0	$\ldots$	$\tilde{x}_l^{(1)}$	$\ldots$	0
P_2	0	1	$\ldots$	$\tilde{x}_l^{(2)}$	$\ldots$	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
P_{m-1}	0	0	$\ldots$	$\tilde{x}_l^{(m-1)}$	$\ldots$	1

The x_l coordinate of each of these points is found from Eq. (3).

The system of vectors $\vec{f}_1 = \overrightarrow{P_0P_1}, \ldots, \vec{f}_{m-1} = \overrightarrow{P_0P_{m-1}}$ is linearly independent, each of these vectors being orthogonal to the vector $\vec{f}_m = (\alpha_1, \ldots, \alpha_m)$. Thus, the system of vectors $\vec{f}_1, \ldots, \vec{f}_m$ can serve as a basis in the space E^m. Let us orthogonalize the system of vectors $\vec{f}_1, \ldots \vec{f}_m$:

$$\vec{h}_1 = \frac{\vec{f}_1}{|\ \vec{f}_1\ |};\ \vec{g}_i = \vec{f}_i - (\vec{f}_i, \vec{h}_1)\vec{h}_1 - \ldots - (\vec{f}_i, \vec{h}_{i-1})\vec{h}_{i-1}, \vec{h}_i = \frac{\vec{g}_i}{|\ \vec{g}_i\ |},\ i = 2, \ldots, m-1;\ \vec{h}_m = \frac{\vec{f}_m}{|\ \vec{f}_m\ |}.$$

As a result, we have got the orthonormal basis $\vec{h}_1, \ldots, \vec{h}_m$. Make up matrix T of the transition from the original orthonormal basis $\vec{e}_1, \ldots, \vec{e}_m$ to the basis $\vec{h}_1, \ldots, \vec{h}_m$, writing the coordinates of vectors $\vec{h}_1, \ldots, \vec{h}_m$ in the basis $\vec{e}_1, \ldots, \vec{e}_m$ to the columns of matrix T. Matrix T is also a matrix of an orthogonal operator that transfers the hyperplane (3) to a hyperplane of the form $x_m = x^*$.

Recalculate the coordinates of centers O_k of the balls $\mathcal{B}_k$, $k = \overline{1,n}$, using the formula

$$\widetilde{X}^{(k)} = T^{-1} X^{(k)}, \tag{4}$$

where $X^{(k)} = (x_1^{(k)}, \ldots, x_m^{(k)})^T$ is the column vector of coordinates of the point O_k in the basis $\vec{e}_1, \ldots, \vec{e}_m$, and $\widetilde{X}^{(k)} = (\tilde{x}_1^{(k)}, \ldots, \tilde{x}_m^{(k)})^T$ is the column vector of its coordinates in the basis $\vec{h}_1, \ldots, \vec{h}_m$.

For the spheres S_i and S_j, consider the system of Eqs. (2) with the new coordinates of their centers O_i and O_j. Subtracting the first equation of system (2) from the second one, we get the equation $x_m = x^*$.

Further, for each sphere S_k, $k = \overline{1,n}$, $k \neq j$, we consider its equation

$$\left(x_1 - \tilde{x}_1^{(k)}\right)^2 + \ldots + \left(x_m - \tilde{x}_m^{(k)}\right)^2 = R_k^2 \tag{5}$$

and substitute $x_m = x^*$ in it. If $R_k^2 - (x^* - \tilde{x}_m^{(k)})^2 \geqslant 0$, we assume

$$r_k^2 = R_k^2 - \left(x^* - \tilde{x}_m^{(k)}\right)^2. \tag{6}$$

This means that the m-dimensional sphere S_k intersects with the plane $x_m = x^*$. The result of this intersection is a $(m-1)$-dimensional sphere (a circumference for $m - 1 = 2$), given by the equation

$$\left(x_1 - \tilde{x}_1^{(k)}\right)^2 + \ldots + \left(x_{m-1} - \tilde{x}_{m-1}^{(k)}\right)^2 = r_k^2. \tag{7}$$

If all spheres S_k (and therefore, all balls $\mathcal{B}_k$), $k = \overline{1,n}$, $k \neq j$, intersect with the hyperplane $x_m = x^*$, then at this stage the problem is reduced to finding a point in the intersection of $n-1$ balls in the space E^{m-1}. If such a point $M_0(\tilde{x}_1^0, \ldots, \tilde{x}_{m-1}^0, x^*)$ is found, it remains only to recalculate its coordinates according to the formula

$$X^0 = T\widetilde{X}^0, \tag{8}$$

where $\widetilde{X}^0 = (\tilde{x}_1^0, \ldots, \tilde{x}_{m-1}^0, x^*)^T$ are coordinates of the point M_0 in the basis $\vec{h}_1, \ldots, \vec{h}_m$, and $X^0 = (x_1^0, \ldots, x_m^0)^T$ are its coordinates in the original basis $\vec{e}_1, \ldots, \vec{e}_m$. Formula (8) specifies the inverse coordinate transformation with respect to the transformation defined by formula (4).

If $n-1$ balls from the space E^{m-1} defined by the hyperplane $x_m = x^*$ do not intersect, then we proceed with the next stage of the algorithm, i.e., we consider the next fixed pair of spheres S_i and S_j. We also proceed with the next stage of the algorithm when not all spheres S_k, $k = \overline{1,n}$, $k \neq j$, intersect with the hyperplane $x_m = x^*$, i.e., if for some sphere S_k we have $R_k^2 - (x^* - \tilde{x}_m^{(k)})^2 < 0$.

In general, if the intersection of balls is non-empty, then the border of the intersection area consists of parts of some bounding spheres (taking into account the preprocessing, there are at least two such spheres). Parts of spheres limited the area of intersection of balls are jointed by hyperplanes passing through the area of intersection. Enumerating all such hyperplanes (i.e., all pairs of m-dimensional spheres S_i and S_j, $1 \leqslant i$, $j \leqslant n$, $i \neq j$), we find the hyperplane containing the desired point.

Below we give a brief formal description of the key stages of the recursive algorithm $BALLS3(m,n)$, which solves the problem in the space E^m, where $m \geqslant 3$.

Algorithm 3. $BALLS3(m,n)$

Input: m-dimensional balls $\mathcal{B}_1^{(m)}, \ldots, \mathcal{B}_n^{(m)}$ given by their centers $O_i^{(m)}(x_1^{(i)}, \ldots, x_m^{(i)})$ and radii $R_i^{(m)}$, $i = \overline{1,n}$.

Output: A common point for all balls $\mathcal{B}_1^{(m)}, \ldots, \mathcal{B}_n^{(m)}$ or the answer that the problem has no solution.

1. Perform preprocessing.
2. **FOR** $i = 1$ to $n-1$ **DO**
3. $j = 2$.
4. **FOR** $j \leqslant n$ **DO**

5. From system (2), define the hyperplane $\alpha_1 x_1 + \ldots + \alpha_m x_m + \beta = 0$, containing the intersection of the spheres $S_i^{(m)}$ and $S_j^{(m)}$.
6. Construct matrix T of an orthogonal transformation that transfers the hyperplane $\alpha_1 x_1 + \ldots + \alpha_m x_m + \beta = 0$ to the hyperplane $x_m = x^*$.
7. Find matrix T^{-1} of the inverse orthogonal transformation.
8. Orthogonal transformation: for all balls $\mathcal{B}_k^{(m)}$, $k = \overline{1,n}$, $k \neq i$, recalculate the coordinates of their centers $O_k^{(m)}$ using the formula (4).
9. From system (2) of equations of spheres with new coordinates of their centers define a plane $x_m = x^*$ containing the intersection of the spheres $S_i^{(m)}$ and $S_j^{(m)}$.
10. **FOR** $k = 1$ to n, $k \neq j$ **DO**
11. **IF** the hyperplane $x_m = x^*$ does not intersect the sphere $S_k^{(m)}$
    THEN go to step 14
    ELSE for $(m-1)$-dimensional ball $\mathcal{B}_k^{(m-1)}$ determine its radius $R_k^{(m-1)} = r_k$ by formula (6) and the center $O_k^{(m-1)}$ (discarding the last coordinate of the point $O_k^{(m)}$).
    END FOR k
12. **IF** $m > 3$ **THEN** for the set of balls $\mathcal{B}_k^{(m-1)}$, $1 \leqslant k \leqslant n$, $k \neq j$, execute $BALLS3(m-1, n-1)$
    ELSE execute $BALLS1$ or $BALLS2$ for this set of balls.
13. **IF** there is found the point $M_0(\tilde{x}_1^0, \ldots, \tilde{x}_{m-1}^0, x^*)$ in the intersection of $(m-1)$-dimensional balls $\mathcal{B}_k^{(m-1)}$, $1 \leqslant k \leqslant n$, $k \neq j$
    THEN inverse orthogonal transformation: recalculate the coordinates of the point M_0 using formula (8) and stop.
14. Set $j = j + 1$ and go to step 4.
    END FOR j
  END FOR i
15. The problem has no solution. Stop.

Estimate the computational complexity of algorithm $BALLS3(m, n)$. In the space E^m, the distance d_{ij} between two points O_i and O_j can be calculated in $O(m)$ operations. Therefore, preprocessing (step 1) requires $O(n^2 m)$ operations. Steps 5 and 9 are performed in $O(m)$ operations.

The construction of a system of vectors $\vec{f}_1, \ldots, \vec{f}_m$ can be performed in $O(m^2)$ operations (see table), and the orthogonalizing of the vector system can be performed in $O(m^3)$ operations (taking into account that the scalar product of two vectors is calculated in $O(m)$ operations). Hence, matrix T, whose columns are the coordinates of the orthonormal system of vectors $\vec{h}_1, \ldots \vec{h}_m$, can be constructed in $O(m^3)$ operations (step 6). The inverse matrix T^{-1} can be constructed in $O(m^2)$ operations (step 7) using the following known method based on the Gauss method. The extended matrix $[T|I_m]$, where I_m is identity matrix of the order m, is subjected to elementary transformations over the rows that bring this matrix to the form $[I_m|C]$. The result is matrix $T^{-1} = C$.

Calculation of coordinates of a point in the new basis by formula (4) can be performed in $O(m^2)$ operations. Therefore, step 8 is performed in $O(nm^2)$ operations. The cycle for k (steps 10, 11) is performed in $O(n)$ operations. Recalculation of coordinates of the point M_0 by formula (8) (step 13) is performed in $O(m^2)$ operations.

Taking into account two main cycles for i (steps 2–14) and for j (steps 4–14), the total computational complexity $\mathcal{T}(m, n)$ of algorithm $BALLS3(m, n)$ can be expressed by the following recurrent equation:

$$\mathcal{T}(m,n) = O\left(n^3m^2 + n^2m^3\right) + n^2\mathcal{T}(m-1,n-1), \;\; m \geqslant 3, \tag{9}$$

where $\mathcal{T}(2,n) = O(n^2 \log n)$ if algorithm $BALLS1$ is used, and $\mathcal{T}(2,n) = O(n^3)$ if algorithm $BALLS2$ is used. The solution $\mathcal{T}(m,n)$ of the recurrent Eq. (9) for $m \geqslant 3$ does not exceed $O(n^{2m-4}(nm^2 + m^3 + \mathcal{T}(2,n)))$. Therefore, in the space E^m, the original problem can be solved in $O(n^{2m-4}(nm^2 + m^3 + n^2 \log n))$ or in $O(n^{2m-4}(nm^2 + m^3 + n^3))$ operations. In particular, for three-dimensional space, the complexity of the algorithm will be $O(n^4 \log n)$ or $O(n^5)$ operations.

4. CONCLUSION

In this paper we suggest the exact polynomial algorithms for finding a point in the intersection of n balls in m-dimensional Euclidean space. In the cases $m = 2$ and $m = 3$ that are valuable from the practical point of view, the presented algorithms can be used for developing software for various dynamic systems including a variety of drones. For example, for a given drone swarm configuration, it is required to determine the location of the control drone or determine whether the desired change of the configuration is possible without losing control of all drones by the control drone. The other practical interpretations were discussed in Section 1.

It should be mentioned that it is possible to perform a slight modification of the presented algorithms to find not a unique point belonging to the intersection of balls, but several such points (if this intersection is non-empty and does not consist of a unique point). This means that it is also possible to determine a linear combination of these points that belongs entirely to the intersection area. Note that such linear combination of corner points of the boundary of the intersection area does not coincide with the set of points of the ball of the largest volume contained in the intersection of balls, which should be found in the internal approximation problem [4]. Therefore, the approach proposed in this paper is an alternative one not only as the method. It also gets the different set of solutions providing additional opportunities for decision-making.

APPENDIX

Proof of Lemma 1. 1. Let $\rho = 1$, i.e., the sequence π has a single indicator switch. Note that in this case, the indicator was switched on the element with the number $n-1$, i.e., $\alpha_{n-1} \neq \alpha_n$. Two possible situations can occur.

(a) $\alpha_{n-1} = 0$, $\alpha_n = 1$. In this case, the sequence π can be split into two subsequences: $\pi = (\pi^{(1)}, \pi^{(2)})$, where $\pi^{(1)} = (\psi_1, \psi_2, \ldots, \psi_{n-1})$, $\pi^{(2)} = (\psi_n, \psi_{n+1}, \ldots, \psi_{2n-2})$. All elements of the subsequence $\pi^{(1)}$ have a zero indicator, i.e., they correspond to the white points on the circumference C_i (the starting points of the proper arcs). All elements of the subsequence $\pi^{(2)}$ have an indicator 1, i.e., they correspond to the black points on the circumference C_i (the end points of the proper arcs). Any proper arc L_{ij} of the circumference C_i starts at the white point and ends at the black point, provided that the movement along the arc goes in the counter-clockwise direction. The sequence π corresponds to the enumeration of points on the circumference C_i in the counter-clockwise direction. Therefore, the traversing any arc L_{ij} from the start to the end point corresponds to a certain subsequence $(\psi_p, \psi_{p+1} \ldots, \psi_{n-1}, \psi_n, \psi_{n+1}, \ldots, \psi_q)$ of the sequence π, where $(\psi_p, \psi_{p+1}, \ldots, \psi_{n-1})$ is a sequence of elements with indicator 0, and $(\psi_n, \psi_{n+1}, \ldots, \psi_q)$ is a sequence of elements with indicator 1. Note that whatever the numbers p and q, $1 \leqslant p \leqslant n-1$, $n \leqslant q \leqslant 2n-2$, any such subsequence must include the elements ψ_{n-1} and ψ_n. Therefore, the arc L_i connecting the points of the circumference C_i determined by the elements ψ_{n-1} (start point) and ψ_n (end point) will be contained in any proper arc L_{ij}, i.e., $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$.

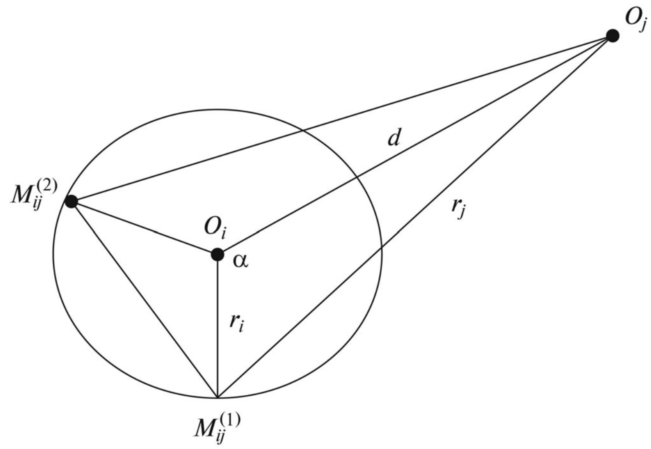

Fig. 4.

(b) $\alpha_{n-1} = 1$, $\alpha_n = 0$. In this case, all elements of the subsequence $\pi^{(1)}$ have indicator 1, and all elements of the subsequence $\pi^{(2)}$ have indicator 0. The movement along the circumference C_i in the counter-clockwise direction can be done in the order $\pi^* = (\pi^{(2)}, \pi^{(1)}) = (\psi_n, \psi_{n+1}, \ldots, \psi_{2n-2}, \psi_1, \psi_2, \ldots, \psi_{n-1})$. Then the movement along any proper arc L_{ij} from the start to the end point corresponds to a certain subsequence $(\psi_p, \psi_{p+1} \ldots, \psi_{2n-2}, \psi_1, \psi_2, \ldots, \psi_q)$ of the sequence π^*, where $(\psi_p, \psi_{p+1}, \ldots, \psi_{2n-2})$ is the subsequence of elements with indicator 0, and $(\psi_1, \psi_2, \ldots, \psi_q)$ is the subsequence of elements with indicator 1. Whatever the numbers p and q, $n \leqslant p \leqslant 2n-2$, $1 \leqslant q \leqslant n-1$, the elements ψ_{2n-2} and ψ_1 must be included in any such subsequence $(\psi_p, \psi_{p+1}, \ldots, \psi_{2n-2}, \psi_1, \psi_2, \ldots, \psi_q)$. Therefore, the arc L_i connecting the points of the circumference C_i corresponding to the elements ψ_{2n-2} and ψ_1 will be contained in any proper arc, i.e., $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$.

2. Let $\rho = 2$, i.e., the indicator switches twice: once from 0 to 1, and once from 1 to 0, and the order of these switches can be arbitrary. Therefore, there are two possible situations.

(a) Let the first indicator switch occur from 0 to 1 on the element with the number k, i.e., $\alpha_k = 0$, $\alpha_{k+1} = 1$. In this case, the sequence π can be divided into three subsequences: $\pi = (\pi^{(1)}, \pi^{(2)}, \pi^{(3)})$, where $\pi^{(1)} = (\psi_1, \psi_2, \ldots, \psi_k)$, $\pi^{(2)} = (\psi_{k+1}, \psi_{k+2}, \ldots, \psi_{k+n-1})$, $\pi^{(3)} = (\psi_{k+n}, \psi_{k+n+1}, \ldots, \psi_{2n-2})$, $1 \leqslant k \leqslant n-2$. All elements of the subsequences $\pi^{(1)}$ and $\pi^{(3)}$ have indicator 0, and all elements of the subsequence $\pi^{(2)}$ have indicator 1. The movement along the circumference C_i in the counter-clockwise direction can be done in the order $\hat{\pi} = (\pi^{(3)}, \pi^{(1)}, \pi^{(2)})$. Then the movement along any proper arc L_{ij} from the start to the end points corresponds to some subsequence $(\psi_p, \ldots, \psi_k, \psi_{k+1}, \ldots, \psi_q)$, where $(\psi_p, , \ldots, \psi_k)$ is a subsequence of elements with indicator 0, and $(\psi_{k+1}, \ldots, \psi_q)$ is a subsequence of elements with indicator 1, $k+n \leqslant p \leqslant 2n-2$, $1 \leqslant p \leqslant k$, $k+1 \leqslant q \leqslant k+n-1$, $1 \leqslant k \leqslant n-2$. The elements ψ_k and ψ_{k+1} must be included into any such subsequence $(\psi_p, \ldots, \psi_k, \psi_{k+1}, \ldots, \psi_q)$. Therefore, the arc L_i connecting the points corresponding to the elements ψ_k and ψ_{k+1} will be contained in any proper arc, i.e., $L_i = \bigcap_{1 \leqslant j \leqslant n, j \neq i} L_{ij} \neq \emptyset$.

(b) Let the first indicator switch occur from 1 to 0 on the element with the number k, i.e., $\alpha_k = 1$, $\alpha_{k+1} = 0$. As in the case 2.a, the sequence π can be split into the same three subsequences $\pi = (\pi^{(1)}, \pi^{(2)}, \pi^{(3)})$. However, in this case, all elements of the subsequences $\pi^{(1)}$ and $\pi^{(3)}$ have indicator 1, while all elements of the subsequence $\pi^{(2)}$ have indicator 0. The movement along the circumference C_i in the counter-clockwise direction can be done in the order $\tilde{\pi} = (\pi^{(2)}, \pi^{(3)}, \pi^{(1)})$. The movement along any proper arc L_{ij} from the start to the end points corresponds to a certain subsequence $(\psi_p, \ldots, \psi_{k+n-1}, \psi_{k+n}, \ldots, \psi_q)$, where $(\psi_p, , \ldots, \psi_{k+n-1})$ is the subsequence of elements with indicator 0, and $(\psi_{k+n}, \ldots, \psi_q)$ is the subsequence of elements with indicator 1,

$k+1 \leqslant p \leqslant k+n-1$, $k+n \leqslant q \leqslant 2n-2$, $1 \leqslant q \leqslant k$, $1 \leqslant k \leqslant 2n-2$. The elements ψ_{k+n-1} and ψ_{k+n} must be included into any such subsequence $(\psi_p, \ldots, \psi_{k+n-1}, \psi_{k+n}, \ldots, \psi_q)$. Therefore, the arc L_i connecting the points corresponding to the elements ψ_{k+n-1} and ψ_{k+n} will be contained in any proper arc, i.e., $L_i = \bigcap_{1\leqslant j\leqslant n, j\neq i} L_{ij} \neq \emptyset$.

Lemma 1 is proved.

Proof of Lemma 2. Let the degree measure of the proper arc L_{ij} be $2\alpha > 180°$. To be definite, we consider the point $M_{ij}^{(1)}$ as the beginning of the arc L_{ij}, and the point $M_{ij}^{(2)}$ as its end. The degree measure of the proper arc L_{ij} is equal to the sum of two identical obtuse angles $M_{ij}^{(1)}O_iO_j$ and $M_{ij}^{(2)}O_iO_j$ (see Fig. 4). Let d be the distance between centers O_i and O_j of the circumferences C_i and C_j. Then from the triangle $M_{ij}^{(1)}O_iO_j$ according to the law of cosines we have $r_j^2 = r_i^2 + d^2 - 2r_i d\cos\alpha$. Since α is an obtuse angle, we have $\cos\alpha < 0$, hence $r_j > r_i$, which was required to be proved.

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