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EXACT ANALYTIC REPRESENTATIONS FOR THE INTEGRAL CHARACTERISTICS OF A FOUR-POINT COHERENCE FUNCTION FOR LASER BEAMS IN TURBULENT MEDIA

N. N. Rogovtsov[a*] and V. Ya. Anisimov[b]

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A new integral-functional equation is derived for the four-dimensional Fourier transform of the four-point coherence function of laser beams in turbulent media and two families of exact analytical solutions of this equation are found. These solutions hold for any level of fluctuations of the refractive index of air. They are used to obtain exact analytic representations of the integral characteristics of the four-point coherence function. In particular, the truncated spectral characteristics of the spatial correlation function of the intensities are found. These representations can be used to test asymptotic, numerical, and other methods for finding this function and to describe its integral properties.

Keywords: *turbulent medium, multiple scattering, fluctuation, four-point coherence function, integral characteristic, truncated spectral characteristics, invariant embedding, bijective transformation, analytic representation, laser beam.*

Introduction. A whole series of scientific and technical problems in ranging, information transformation in open optical communications systems, geodesy, plasma theory, the optics of scattering media, radiative transfer theory, and astrophysics cannot be solved correctly without studying the effect of regular or random (discrete or continuous) spatial-temporal variations in the geometric and physical (in particular, optical) characteristics of microscopically inhomogeneous media on the scattering of radiation and the formation of radiative fields in these media. Geophysical media, in particular the earth's atmosphere in its various states and natural or other inclusions in it, are classical examples of these media. In the transparent part of the atmosphere, random fluctuations (they can be regarded as continuous random variations) in the refractive index $n(\mathbf{r}, t)$ ($\mathbf{r}$ is the radius vector of the observation point and t is time) of the air, which are caused by its turbulent motion, have a strong influence on the propagation of a laser beam. Despite the very small ($\sim 10^{-6}$–10^{-5}) amplitudes of the fluctuations in the refractive index of the air, over sufficiently long paths a laser beam passes through a very large number of optical inhomogeneities which creates an effect of accumulated distortions in the parameters of the original beam owing to multiple scattering on these inhomogeneities. This kind of scattering of the laser light takes place essentially completely in the forward direction, and backscattering can be neglected. Changes in the polarization characteristics of the laser light owing to refractive index fluctuations in the earth's atmosphere are also very insignificant. These two statements are justified in [1–3], with one based substantially on the fact that the characteristic size of the inhomogeneities in the refractive index of the earth's atmosphere exceeds the wavelength λ of the light. In many papers ([1–18] and references therein) various theoretical and experimental methods are used to establish some fundamental aspects of the propagation of electromagnetic (especially, laser) radiation in turbulent media. These theoretical studies mainly employ a scalar quasioptical (parabolic) approximation for the wave and related equations. Up to now, however, no exact and efficient methods of searching for the statistical moments (higher than second order) complex amplitudes of the wave fields in randomly inhomogeneous media have been developed. It should be emphasized specially that moments of this type are used to express such important characteristics as the relative dispersion (flicker index) and spatial intensity correlation function [13].

In this paper we derive a new integral-functional equation for the four-dimensional Fourier transform of the four-point coherence function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ (it depends on nine variables) of a laser beam. The free parameters in this equation

*To whom correspondence should be addressed.

[a]Belarusian National Technical University, Minsk, 220013, Belarus; email: rogovtsov@bntu.by; [b]Belarusian State University of Informatics and Radioelectronics, Minsk, 220013, Belarus; email: anissimov@bsuir.by. Translated from Zhurnal Prikladnoi Spektroskopii, Vol. 87, No. 2, pp. 204–211, March–April, 2020. Original article submitted December 17, 2019.

include one scalar parameter and one two-dimensional vector. A number of heuristic procedures from a reduction method for common invariance relations (GIRRM), which is one of the general and effective methods for solving multidimensional problems in radiative transfer theory and mathematical physics [19–28], are used in deriving this equation. An analysis of the properties of the derived integral-functional equation is used for the first time to find exact, explicit analytic representations for families of integral characteristics of the four-point coherence function of laser beams in turbulent atmospheres. These representations are true for any level of fluctuations in the refractive index of air. They can be used for analyzing the accuracy of asymptotic, numerical, and other methods of searching for the four-point coherence function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ and become the basis for a rigorous analysis of its properties.

Statement of the Problem. Let $[V]$ be a closed half-space on the boundary S of which lies the plane $OXYZ$ of a rectangular right-hand Cartesian coordinate system. Here we direct the Z axis into $[V]$. We assume that $[V]$ is filled randomly by a nonuniform medium whose properties are identical to those of some transparent part of the turbulent atmosphere of the earth. On an arbitrary plane z = const (const ≥ 0) we take four points M_1, M_2, M_3, and M_4, whose positions are defined by the radius vectors $\mathbf{r}_1 = (\boldsymbol{\rho}_{11}, \boldsymbol{\rho}_{12}, z)$, $\mathbf{r}_2 = (\boldsymbol{\rho}_{21}, \boldsymbol{\rho}_{22}, z)$, $\mathbf{r}_1' = (\rho_{11}', \rho_{12}', z)$, and $\mathbf{r}_2' = (\rho_{21}', \rho_{22}', z)$. We introduce the notation $\boldsymbol{\rho}_1 = (\rho_{11}, \rho_{12})$, $\boldsymbol{\rho}_2 = (\rho_{21}, \rho_{22})$, $\boldsymbol{\rho}_1' = (\rho_{11}', \rho_{12}')$, and $\boldsymbol{\rho}_2' = (\rho_{21}', \rho_{22}')$. We assume that the semi-infinite medium is irradiated by a spatially bounded monochromatic linearly polarized beam of radiation for which the projections of the electric field on the X and Y axes can be written as $e^{i(\omega t - kz)}U(\boldsymbol{\rho}; z)$, where i is the imaginary unit, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength of the radiation, ω is its circular frequency, and $U(\boldsymbol{\rho}; z)$ is the complex amplitude which is a random function and varies insignificantly over distances on the order of the wavelength ($\boldsymbol{\rho} = (\rho_1, \rho_2)$ is a two-dimensional vector parallel to the OXY plane). We assume that the power of the beam is finite. Besides these assumptions, we assume that the volume of known information on the coherence properties of the radiation beam is sufficient to specify the four-point coherence function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ on the $z = 0$ plane. This function is defined as follows [13]:

$$\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z) = \langle U(\rho_1; z)U(\rho_2; z)U^*(\rho_1'; z)U^*(\rho_2'; z)\rangle \,, \tag{1}$$

where $\langle ... \rangle$ denotes the operation of averaging over the ensemble of realizations; * denotes complex conjugation; the quantities $U(\boldsymbol{\rho}_j; z)$, where $j \in \{1, 2, 3, 4\}$, signify the complex amplitudes of the wave field on the plane specified by the applicate z and parallel OXY plane at the points M_1, M_2, M_3, and M_4, respectively. In [1–3], different methods are used to obtain an initial equation for the functions $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ in terms of the above-described assumptions and properties of a turbulent medium:

$$\left[2ik\frac{\partial}{\partial z} + \sum_{m=1}^{2}\left(\Delta_m - \Delta_m'\right) + \frac{ik^3}{4}F_{22}\left(\rho_1, \rho_2, \rho_1', \rho_2'; z\right)\right]\Gamma_{22}\left(\rho_1, \rho_2, \rho_1', \rho_2'; z\right) = 0 \,, \quad z \in (0, +\infty) \,. \tag{2}$$

Here Δ_m and Δ_m' are two-dimensional Laplace operators which are specified in the system OXY by the symbolic equations $\Delta_m = \dfrac{\partial^2}{\partial \rho_{m1}^2} + \dfrac{\partial^2}{\partial \rho_{m2}^2}$ and $\Delta_m' = \dfrac{\partial^2}{\partial \rho_{m1}'^2} + \dfrac{\partial^2}{\partial \rho_{m2}'^2}$, $m \in \{1, 2\}$. The function $F_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ in Eq. (2) determines the average influence of the fluctuations in the refractive index of the fluctuations in the refractive index of the turbulent atmosphere of the earth on the four-point coherence function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$ during propagation of the radiation in the positive Z direction. It can be written in the form [13]

$$\begin{aligned} F_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z) = 2\pi[&H(\rho_1 - \rho_1'; z) + H(\rho_2 - \rho_2'; z) + H(\rho_1 - \rho_2'; z) \\ &+ H(\rho_2 - \rho_1'; z) - H(\rho_2 - \rho_1; z) - H(\rho_2' - \rho_1'; z)] \,, \end{aligned} \tag{3}$$

where $H(\rho; z)$ is defined by the equation

$$H(\boldsymbol{\rho}; z) = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi_\varepsilon^{\mathrm{o}}(\mathbf{q}; z)\left(1 - \cos\left(\mathbf{q}\cdot\boldsymbol{\rho}\right)\right) dq_1 dq_2 \,, \quad \mathbf{q} = \left(q_1, q_2\right) ,$$

and $\overset{\circ}{\Phi}_\varepsilon(\mathbf{q};\ z)$ = const $\Phi_\varepsilon(\mathbf{q}; z)$, where $\Phi_\varepsilon(\mathbf{q}; z)$ has the significance of a spectral density of the fluctuations in the dielectric constant ε of the air, which, since $n = \sqrt{\varepsilon}$, is directly related to the density of the fluctuations in the refractive index n (const, a positive number determined by the choice of the form of the forward and inverse Fourier transforms). Symbols of the type $\boldsymbol{a}\cdot\boldsymbol{l}$ here and in the following denote the real or formal scalar product of the elements $\boldsymbol{a}$ and $\boldsymbol{l}$. The function $\overset{\circ}{\Phi}_\varepsilon(\mathbf{q};\ z)$ satisfies the equation $\overset{\circ}{\Phi}_\varepsilon(\mathbf{q};\ z) = \overset{\circ}{\Phi}_\varepsilon(-\mathbf{q};\ z)$, which is used in deriving the desired integral-functional equation. To obtain a unique solution of Eq. (2) it is necessary to specify the boundary conditions. We assume that $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2';\ z)\big|_{z=0}$ is a known function. This is the first boundary condition. Given the finite power of the radiation passing through the $z = 0$ plane and the absence of other sources within the open half-space V (i.e., $z > 0$), the function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2';\ z)$ for arbitrary $z \geq 0$ must tend to zero, if any one of $|\rho_1|$, $|\rho_2|$, $|\rho_1'|$, $|\rho_2'|$ go to $+\infty$. This behavior of the function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2';\ z)$ at infinity serves as a second boundary condition. These conditions are then used to obtain the integral-functional equation and exact analytic representations for the integral characteristics of the four-point coherence function.

Procedure for Deriving the Unknown Relationships and Equations. We transform Eq. (2) sequentially with two replacements of variables that are related in form to two standard substitutions of variables that are used in the theory of the propagation of laser radiation in a turbulent atmosphere [13]. These substitutions are bijective and can be written in vector form:

$$\boldsymbol{\omega}_s = \boldsymbol{\rho}_s - \boldsymbol{\rho}_s', \quad \boldsymbol{\tau}_s = \boldsymbol{\rho}_s + \boldsymbol{\rho}_s', \ s \in \{1, 2\}, \quad \mathbf{u} = \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2, \quad \mathbf{p} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 . \tag{4}$$

After substitutions of Eqs. (4) and including Eq. (3), Eq. (2) takes the form

$$\left[\frac{ik}{2}\frac{\partial}{\partial z} + \nabla_{\omega_1}\cdot\nabla_{\boldsymbol{u}} + \nabla_{\omega_1}\cdot\nabla_{\boldsymbol{p}} + \nabla_{\omega_2}\cdot\nabla_{\boldsymbol{p}} - \nabla_{\omega_2}\cdot\nabla_{\boldsymbol{u}} + \frac{ik^3}{16}F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{u};\ z\right)\right]$$

$$\times\, \Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{p};\ z\right) = 0\ , \tag{5}$$

where $\nabla_{\omega_1}, \nabla_{\omega_2}, \nabla_{\boldsymbol{u}}, \nabla_{\boldsymbol{p}}$ are two-dimensional Hamiltonian operators, and the function $\Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{p};\ z\right)$ is equal to $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2';\ z)$ with the following substitutions:

$$\rho_1 = 2^{-1}\left(\boldsymbol{\omega}_1 + 2^{-1}\left(\mathbf{u} + \mathbf{p}\right)\right), \quad \rho_2 = 2^{-1}\left(\boldsymbol{\omega}_2 + 2^{-1}\left(\mathbf{p} - \mathbf{u}\right)\right), \tag{6}$$

$$\rho_1' = 2^{-1}\left(2^{-1}\left(\mathbf{u} + \mathbf{p}\right) - \boldsymbol{\omega}_1\right), \quad \rho_2' = 2^{-1}\left(2^{-1}\left(\mathbf{p} - \mathbf{u}\right) - \boldsymbol{\omega}_2\right).$$

The function $F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u};\ z\right)$ in Eq. (5) has the form

$$F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u};\ z\right) = 8\pi\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\overset{\circ}{\Phi}_\varepsilon\left(\mathbf{q};\ z\right)[1 + \cos\left(2^{-1}\left(\mathbf{q}\cdot\mathbf{u}\right)\right)\cos\left(2^{-1}\left(\mathbf{q}\cdot\left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\right)\right)\right) \tag{7}$$

$$- \cos\left(2^{-1}\left(\mathbf{q}\cdot\left(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2\right)\right)\right)\left(\cos\left(2^{-1}\left(\mathbf{q}\cdot\mathbf{u}\right)\right) + \cos\left(2^{-1}\left(\mathbf{q}\cdot\left(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\right)\right)\right)\right)]\, dq_1dq_2\ .$$

Given the above boundary conditions, we first apply the two-dimensional Fourier transform with respect to the variable $\mathbf{p}$ ($\mathbf{p} = (p_1, p_2)$) to Eq. (5), and then to the resulting equation — a two-dimensional Fourier transform with respect to the variable $\mathbf{u}$ ($\mathbf{u} = (u_1, u_2)$). As a result, we have

$$\left[\frac{k}{2}\frac{\partial}{\partial z} - \left(\left(\boldsymbol{\gamma} + \boldsymbol{\zeta}\right)\cdot\frac{\partial}{\partial\boldsymbol{\omega}_1}\right) - \left(\left(\boldsymbol{\gamma} - \boldsymbol{\zeta}\right)\cdot\frac{\partial}{\partial\boldsymbol{\omega}_2}\right)\right]\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma};\ z\right) \tag{8}$$

$$+ \frac{k^3}{32\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\zeta}\cdot\boldsymbol{u})}F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u};\ z\right)\overline{\Gamma_{22}^{\times}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \boldsymbol{\gamma};\ z\right) du_1du_2 = 0\ ,$$

where

$$\overline{\Gamma_{22}^{\times}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \boldsymbol{\gamma}; z\right) = (2\pi)^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\gamma}\cdot\boldsymbol{p})} \Gamma_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{p}; z\right) dp_1 dp_2,$$

$$\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) = (2\pi)^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\zeta}\cdot\boldsymbol{u})} \overline{\Gamma_{22}^{\times}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \boldsymbol{\gamma}; z\right) du_1 du_2, \quad \text{and} \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2), \ \boldsymbol{\zeta} = (\zeta_1, \zeta_2).$$

With a series of actions representing invariant embedding, partition [19, 20, 22–27], or bijective transformation procedures, this equation can be reduced to the form of an integral-functional equation. For this purpose we break the function $F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}; z\right)$ into two terms, which contain a single arbitrary scalar parameter ξ and one arbitrary real two-dimensional vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$. The following equation holds:

$$F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}; z\right) = \varkappa_1(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2; z; \xi, \boldsymbol{\alpha}) + \varkappa_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{u}; z; \xi, \boldsymbol{\alpha}), \tag{9}$$

in which

$$\varkappa_1(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2; z; \xi, \boldsymbol{\alpha}) = 8\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{\Phi}_{\varepsilon}(\mathbf{q}; z) \chi_1\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2; \mathbf{q}; \xi, \boldsymbol{\alpha}\right) dq_1 dq_2,$$

$$\varkappa_2(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}; z; \xi, \boldsymbol{\alpha}) = 8\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{\Phi}_{\varepsilon}(\mathbf{q}; z) \chi_2\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{q}; \xi, \boldsymbol{\alpha}\right) dq_1 dq_2,$$

$$\chi_1\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{q}; \xi, \boldsymbol{\alpha}\right) = 1 + \xi \cos\left(2^{-1}\left(\mathbf{q}\cdot\boldsymbol{\alpha}\right)\right) \cos\left(2^{-1}\left(\mathbf{q}\cdot(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\right)\right) - \cos\left(2^{-1}\left(\mathbf{q}\cdot(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\right)\right)$$

$$\times \left(\xi \cos\left(2^{-1}\left(\mathbf{q}\cdot\boldsymbol{\alpha}\right)\right) + \cos\left(2^{-1}\left(\mathbf{q}\cdot(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\right)\right)\right),$$

$$\chi_2\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{q}; \xi, \boldsymbol{\alpha}\right) = \left(\cos\left(2^{-1}\left(\mathbf{q}\cdot(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\right)\right) - \cos\left(2^{-1}\left(\mathbf{q}\cdot(\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\right)\right)\right)$$

$$\times \left(\cos\left(2^{-1}\left(\mathbf{q}\cdot\mathbf{u}\right)\right) - \xi \cos\left(2^{-1}\left(\mathbf{q}\cdot\boldsymbol{\alpha}\right)\right)\right).$$

In place of the function $F_{22}^{\times}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}; z\right)$, we substitute its representation in the form of the right-hand side of Eq. (9) in Eq. (8). We transform the result using the Euler formula, the equation $\overset{\circ}{\Phi}_{\varepsilon}(\mathbf{q}; z) = \overset{\circ}{\Phi}_{\varepsilon}(-\mathbf{q}; z)$, the even character of the cosine, the definitions of the functions $\overline{\Gamma_{22}^{\times}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \boldsymbol{\gamma}; z\right)$ and $\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right)$, and the elementary bijective substitutions of the variables in the three double integrals that show up when reducing the initial Eq. (8) to a form convenient for further discussions. As a result, we obtain the integral-differential equation

$$\left[\frac{k}{2}\frac{\partial}{\partial z} - \left((\boldsymbol{\gamma} + \boldsymbol{\zeta})\cdot\frac{\partial}{\partial \boldsymbol{\omega}_1}\right) - \left((\boldsymbol{\gamma} - \boldsymbol{\zeta})\cdot\frac{\partial}{\partial \boldsymbol{\omega}_2}\right) + \frac{k^3}{16}\varkappa_1\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2; z; \xi, \boldsymbol{\alpha}\right)\right] \tag{10}$$

$$\times \overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z\right) + 2\pi k^3 g\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z; \xi, \boldsymbol{\alpha}\right) = 0,$$

$$g\left(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\zeta}, \boldsymbol{\gamma}; z; \xi, \boldsymbol{\alpha}\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{\Phi}_{\varepsilon}\left(2(\boldsymbol{\zeta} - \mathbf{q}); z\right) [\cos\left((\boldsymbol{\zeta} - \mathbf{q})\cdot(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\right)$$

$$-\cos\left(\left(\boldsymbol{\zeta}-\mathbf{q}\right)\cdot\left(\boldsymbol{\omega}_1+\boldsymbol{\omega}_2\right)\right)\Big]\left[\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\mathbf{q},\boldsymbol{\gamma};z\right)-\xi\cos\left(\left(\boldsymbol{\zeta}-\mathbf{q}\right)\cdot\boldsymbol{\alpha}\right)\overline{\overline{\Gamma_{22}^{\times}}}\left(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};z\right)\right]dq_1dq_2 \ .$$

Equation (10) can be simplified by bijective substitution of the variables

$$\tilde{z}=2k^{-1}z,\quad \tilde{\boldsymbol{\omega}}_1=\boldsymbol{\omega}_1+2k^{-1}z\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\quad \tilde{\boldsymbol{\omega}}_2=\boldsymbol{\omega}_2+2k^{-1}z\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right) \ . \tag{11}$$

Using Eq. (11), Eq. (10) reduces to the form

$$\left[\frac{\partial}{\partial\tilde{z}}+\frac{k^3}{16}\varkappa_1\left(\tilde{\boldsymbol{\omega}}_1-\tilde{z}\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\tilde{\boldsymbol{\omega}}_2-\tilde{z}\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right);\frac{k\tilde{z}}{2};\xi,\boldsymbol{\alpha}\right)\right]\overline{\overline{\Gamma_{22}^{\times}}}\left(\tilde{\boldsymbol{\omega}}_1-\tilde{z}\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\tilde{\boldsymbol{\omega}}_2-\tilde{z}\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\boldsymbol{\zeta},\boldsymbol{\gamma};\frac{k\tilde{z}}{2}\right) \tag{12}$$

$$+\ 2\pi k^3 g\left(\tilde{\boldsymbol{\omega}}_1-\tilde{z}\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\tilde{\boldsymbol{\omega}}_2-\tilde{z}\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\boldsymbol{\zeta},\boldsymbol{\gamma};\frac{k\tilde{z}}{2};\xi,\boldsymbol{\alpha}\right)=0 \ .$$

For $\tilde{z}>0$, the last term on the left of Eq. (12) is given in terms of the unknown function $\overline{\overline{\Gamma_{22}^{\times}}}(\cdots)$; this follows from Eq. (10). If we assume that the function $g(\cdots)$ in Eq. (12) is a known quantity, then its formal solution can be written as

$$G\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}\right)=\exp\left(-f\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right)\right)[\overline{\overline{\Gamma_{22}^{\times}}}\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};0\right)$$

$$-2\pi k^3\int_0^{\tilde{z}}\exp\left(f\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}';\xi,\boldsymbol{\alpha}\right)\right)g\left(\boldsymbol{\sigma},\boldsymbol{\vartheta},\boldsymbol{\zeta},\boldsymbol{\gamma};\psi;\xi,\boldsymbol{\alpha}\right)d\tilde{z}']. \tag{13}$$

Here $G\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z}\right)=\overline{\overline{\Gamma_{22}^{\times}}}\left(\tilde{\boldsymbol{\omega}}_1-\tilde{z}\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\tilde{\boldsymbol{\omega}}_2-\tilde{z}\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\boldsymbol{\zeta},\boldsymbol{\gamma};\frac{k\tilde{z}}{2}\right)$;

$$f\left(\tilde{\boldsymbol{\omega}}_1,\tilde{\boldsymbol{\omega}}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};\tilde{z};\xi,\boldsymbol{\alpha}\right)=\frac{k^3}{16}\int_0^{\tilde{z}}\varkappa_1\left(\tilde{\boldsymbol{\omega}}_1-\tilde{z}''\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\tilde{\boldsymbol{\omega}}_2-\tilde{z}''\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right);\frac{k\tilde{z}''}{2};\xi,\boldsymbol{\alpha}\right)d\tilde{z}''.$$

$$\boldsymbol{\sigma}=\tilde{\boldsymbol{\omega}}_1-\tilde{z}'\left(\boldsymbol{\gamma}+\boldsymbol{\zeta}\right),\quad \boldsymbol{\vartheta}=\tilde{\boldsymbol{\omega}}_2-\tilde{z}'\left(\boldsymbol{\gamma}-\boldsymbol{\zeta}\right),\quad \psi=\frac{k\tilde{z}'}{2}.$$

In deriving Eq. (13) we have used the fact that the function $\overline{\overline{\Gamma_{22}^{\times}}}(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\zeta},\boldsymbol{\gamma};0)$ is expressed directly in terms of the function $\Gamma_{22}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\boldsymbol{\rho}_1',\boldsymbol{\rho}_2';0)$, the values of which have been assumed known from the start. Equation (13) is the unknown integral-differential equation for the unknown function $\overline{\overline{\Gamma_{22}^{\times}}}(\cdots)$ and makes it possible to represent it in the form of a sum of two (or three, for $\xi\neq 0$) terms which can be found or estimated in some cases without solving it.

Exact Analytic Representations. Equation (10) implies that the second term in the square brackets in Eq. (13) goes to zero for arbitrary ξ and α if either of the following equations is true:

$$\left(\tilde{\boldsymbol{\omega}}_1-\tilde{\boldsymbol{\omega}}_2-2\tilde{z}'\boldsymbol{\zeta}\right)=\pm\left(\tilde{\boldsymbol{\omega}}_1+\tilde{\boldsymbol{\omega}}_2-2\tilde{z}'\boldsymbol{\gamma}\right),\quad \tilde{z}'\in[0,+\infty) \ . \tag{14}$$

If the "+" sign is chosen in Eq. (14), then the equalities

$$\boldsymbol{\zeta}=\boldsymbol{\gamma},\quad \tilde{\boldsymbol{\omega}}_2=\mathbf{0}=(0,0),\quad \check{\mathbf{u}}_1=\mathbf{b}=\left(b_1,b_2\right) \tag{15}$$

should hold, where $\mathbf{b}$ is an arbitrary vector parallel to the *OXY* plane. On the other hand, if the "–" sign is chosen, then the equalities

$$\zeta = -\boldsymbol{\gamma},\ \tilde{\boldsymbol{\omega}}_1 = \mathbf{0} = (0, 0), \qquad \tilde{\boldsymbol{\omega}}_2 = \mathbf{h} = (h_1, h_2) \tag{16}$$

should hold, where $\mathbf{h}$ is an arbitrary vector which, like $\mathbf{b}$, is parallel to the *OXY* plane. We note that Eqs. (4), (6), and (11) imply consistency of each of the conditions (15) and (16). If conditions (15) are satisfied, then the analytical solution of the integral-differential Eq. (13) has the form

$$\overline{\overline{\Gamma_{22}^{\times}}}\left(\mathbf{b}, \mathbf{0}, \boldsymbol{\gamma}, \boldsymbol{\gamma}; z\right) = \beta\left(z, \boldsymbol{\gamma}, \mathbf{b}\right) \overline{\overline{\Gamma_{22}^{\times}}}\left(\mathbf{b} + 4k^{-1} z\boldsymbol{\gamma}, \mathbf{0}, \boldsymbol{\gamma}, \boldsymbol{\gamma}; 0\right), \tag{17}$$

where $\beta\left(z, \boldsymbol{\gamma}, \mathbf{b}\right) = \exp\left(\left(-k^2 \int_0^z \varkappa_1^{\Delta}\left(\mathbf{b} + 4k^{-1}\left(z - z''\right)\boldsymbol{\gamma}; z''\right)\right) dz''\right)$,

$$\varkappa_1^{\Delta}\left(\boldsymbol{\delta}; z''\right) = \pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overset{\circ}{\Phi}_{\varepsilon}\left(\mathbf{q}; z''\right)(1 - \cos^2\left(2^{-1}\left(\boldsymbol{\delta}\cdot\mathbf{q}\right)\right) dq_1 dq_2 \ \text{, and } \ \boldsymbol{\delta} = \left(\delta_1, \delta_2\right).$$

Equations (9) and (11) were taken into account in deriving Eq. (17). If Eqs. (16) are satisfied, then the analytical solution of the integral-differential Eq. (13) can be written in the form

$$\overline{\overline{\Gamma_{22}^{\times}}}\left(\mathbf{0}, \mathbf{h}, -\boldsymbol{\gamma}, \boldsymbol{\gamma}; z\right) = \beta\left(z, \boldsymbol{\gamma}, \mathbf{h}\right) \overline{\overline{\Gamma_{22}^{\times}}}\left(\mathbf{0}, \mathbf{h} + 4k^{-1} z\boldsymbol{\gamma}, -\boldsymbol{\gamma}, \boldsymbol{\gamma}; 0\right). \tag{18}$$

The analytical solutions (17) and (18), with the definitions of the functions $\overline{\overline{\Gamma_{22}^{\times}}}(\cdots)$, the vectors $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \mathbf{u}, \mathbf{p}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2$, and Eqs. (15) and (16) taken into account and with a number of elementary bijective transformations, can be reduced to the following equalities:

$$\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\gamma}\cdot\boldsymbol{\rho}_1)}\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1 - \mathbf{b}, \boldsymbol{\rho}_2\ ; z\right) d\rho_{11}\ldots d\rho_{22} = \beta\left(z, 4^{-1}\boldsymbol{\gamma}, \mathbf{b}\right)\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\gamma}\cdot\boldsymbol{\rho}_1)}\Gamma_{22}\left(\cdots; 0\right) d\rho_{11}\ldots d\rho_{22}, \tag{19}$$

$$\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\gamma}\cdot\boldsymbol{\rho}_2)}\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 - \mathbf{h}; z\right) d\rho_{11}\ldots \mathrm{d}\rho_{22} = \beta\left(z, 4^{-1}\boldsymbol{\gamma}, \mathbf{h}\right)\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{i(\boldsymbol{\gamma}\cdot\boldsymbol{\rho}_2)}\Gamma_{22}\left(\cdots; 0\right) d\rho_{11}\ldots d\rho_{22}. \tag{20}$$

The integrals in Eqs. (19) and (20) are taken over the entire four-dimensional Euclidean space E_4, the elements of which are all row-vectors of the form (ρ_{11}, ρ_{12}, ρ_{21}, ρ_{22}). Here the symbol $\Gamma_{22}(\cdots; 0)$ in Eqs. (19) and (20) denotes the functions

$$\Gamma_{22}\left(\boldsymbol{\rho}_1 + (2k)^{-1} z\boldsymbol{\gamma}, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1 - (2k)^{-1} z\boldsymbol{\gamma} - \boldsymbol{b}, \boldsymbol{\rho}_2\ ; 0\right),$$

$$\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 + (2k)^{-1} z\boldsymbol{\gamma}, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 - (2k)^{-1} z\boldsymbol{\gamma} - \boldsymbol{h}\ ; 0\right).$$

To within a factor of $(2k)^{-1}$, the left-hand sides of Eqs. (19) and (20) coincide with two-dimensional Fourier transforms of the double integrals

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1 - \mathbf{b}, \boldsymbol{\rho}_2\ ; z\right) d\rho_{21} d\rho_{22},$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{22}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2 - \mathbf{h}; z\right) d\rho_{11} d\rho_{12}.$$

These double integrals and the truncated Fourier transforms (19) and (20) are integral characteristics of the four-point coherence function $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)$. We note that for $\mathbf{b} = 0$ and $h = 0$ the left-hand sides of Eqs. (19) and (20) have the significance of truncated spectral characteristics of the spatial correlation function of the intensities. With a two-dimensional inverse Fourier transform, from Eqs. (19) and (20) we obtain analytic representations for the double integrals described above:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{22}\left(\rho_1, \rho_2, \rho_1 - \mathbf{b}, \rho_2; z\right) d\rho_{21} d\rho_{22} = \left(4\pi^2\right)^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-i\left(\gamma\cdot\left(\rho_1 - \rho_1''\right)\right)\right)\beta\left(z, 4^{-1}\gamma, \mathbf{b}\right) \tag{21}$$

$$\times\ \Gamma_{22}\left(\rho_1'' + (2k)^{-1} z\gamma, \rho_2, \rho_1'' - (2k)^{-1} z\gamma - \mathbf{b}, \rho_2; 0\right) d\gamma_1 d\gamma_2 d\rho_{11}'' d\rho_{12}'' d\rho_{21} d\rho_{22}\ ,$$

and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{22}\left(\rho_1, \rho_2, \rho_1, \rho_2 - \mathbf{h}; z\right) d\rho_{11} d\rho_{12} = \left(4\pi^2\right)^{-1} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp\left(-i\left(\gamma\cdot\left(\rho_2 - \rho_2''\right)\right)\right)\beta\left(z, 4^{-1}\gamma, \mathbf{h}\right) \tag{22}$$

$$\times\ \Gamma_{22}\left(\rho_1, \rho_2'' + (2k)^{-1} z\gamma, \rho_1, \rho_2'' - (2k)^{-1} z\gamma - \mathbf{h}; 0\right) d\gamma_1 d\gamma_2 d\rho_{11} d\rho_{12} d\rho_{21}'' d\rho_{22}''\ ,$$

where $\rho_1'' = \left(\rho_{11}'', \rho_{12}''\right)$, $\rho_2'' = \left(\rho_{21}'', \rho_{22}''\right)$.

We emphasize that the right-hand sides of all the exact analytic representations (19)–(22) are expressed in terms of boundary values of the four-point coherence function on the $z = 0$ plane and the function $\varkappa_1^{\Delta}\left(\delta; z''\right)$, which were initially assumed to be known. Equations (19) and (20) have their simplest form for $\gamma = 0$. In that case they can be written as

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \Gamma_{22}\left(\rho_1, \rho_2, \rho_1 - \mathbf{b}, \rho_2\ ; z\right) d\rho_{11} \dots d\rho_{22} = \beta\left(z, \mathbf{0}, \mathbf{b}\right) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \Gamma_{22}\left(\rho_1, \rho_2, \rho_1 - \mathbf{b}, \rho_2\ ; 0\right) d\rho_{11} \dots d\rho_{22}\ , \tag{23}$$

and

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \Gamma_{22}\left(\rho_1, \rho_2, \rho_1, \rho_2 - \mathbf{h}; z\right) d\rho_{11} \dots d\rho_{22} = \beta\left(z, \mathbf{0}, \mathbf{h}\right) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \Gamma_{22}\left(\rho_1, \rho_2, \rho_1, \rho_2 - \mathbf{h}\ ; 0\right) d\rho_{11} \dots d\rho_{22}\ . \tag{24}$$

If $\mathbf{b} = \mathbf{h} = 0$, then $\beta(z, 0, 0) = 1$ for arbitrary $z \geq 0$ and the right-hand sides of Eqs. (23) and (24) are constants that are fully determined by the boundary values of the functions $\Gamma_{22}(\rho_1, \rho_2, \rho_1', \rho_2'; z)\big|_{z=0}$. Thus, the left-hand sides of Eqs. (23) and (24) for this case are invariant with respect to variations in $z \in [0, +\infty)$. For this situation, Eqs. (23) and (24) are easily obtained from the previously derived Eqs. (46.12)–(46.14) of [18]. However, if $\mathbf{b} \neq 0$ and $\mathbf{h} \neq 0$, then the right-hand sides of Eqs. (23) and (24) are not constant, and equations themselves generalize the invariant (46.15) from [18]. Here the function $\beta(z, 0, \boldsymbol{\delta})$, which fully describes the dependences of the right-hand sides of Eqs. (23) and (24) on the variable z, takes the form

$$\beta\left(z, \mathbf{0}, \boldsymbol{\delta}\right) = \exp\left[-\pi k^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left(1 - \cos^2\left(2^{-1}\left(\mathbf{q}\cdot\boldsymbol{\delta}\right)\right)\right) dq_1 dq_2 \int_0^z \overset{\circ}{\Phi}_{\varepsilon}\left(\mathbf{q}; z''\right) dz''\right]\ , \tag{25}$$

where $\boldsymbol{\delta} = \mathbf{b}$ or $\boldsymbol{\delta} = \mathbf{h}$ ($|\boldsymbol{\delta}| \neq 0$). For a Karman fluctuation spectrum [13] (the equality $\overset{\circ}{\Phi}_{\varepsilon}\left(\boldsymbol{\delta}; z''\right) \equiv \overset{\circ}{\Phi}_{\varepsilon}\left(|\boldsymbol{\delta}|; z''\right)$ holds) the function (25) approaches zero for $z \to +\infty$. Thus, the integral characteristics of the four-point coherence function determined by the left-hand sides of Eqs. (23) and (24) also approach zero as $z \to +\infty$. In addition, for this case the function $\beta(z, 0, \boldsymbol{\delta})$ can also be written in the form

$$\beta\left(z, \mathbf{0}, \boldsymbol{\delta}\right) = \exp\left[-(\pi k)^2 \int_0^{+\infty} \left(1 - J_0\left(|\boldsymbol{\delta}|\, w\right)\right)\left(\int_0^z \overset{\circ}{\Phi}_{\varepsilon}\left(w; z''\right) dz''\right) w dw\right]\ , \tag{26}$$

where $J_0(\dots)$ is the zeroth-order Bessel function of the first kind.

Conclusions. The explicit exact analytic representations for the integral (in particular, spectral) characteristics of the four-point coherence function of laser light obtained here illustrate the possibility in principle of a rigorous and efficient solution of the boundary value problems for Eq. (2) for arbitrary levels of fluctuations in the refractive index in turbulent media. Equations (19)–(24) illustrate the significant effect of the initial data for a laser beam, as well as of the type of the dependence of the integral $\int_0^z \overset{\circ}{\Phi}_\varepsilon\left(\boldsymbol{q};\, z''\right) dz''$ on the variable z and vector δ on variations in the integral characteristics of the four-point coherence function for extended path lengths z.

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