MULTIDIMENSIONAL-MATRIX REPRESENTATION OF TENSOR

Mukha V. S.
Department of Information Technologies of Automated Systems, Belarusian State University of Informatics and Radioelectronics
Minsk, Republic of Belarus
E-mail: mukha@bsuir.by

The multidimensional-matrix representation of the arbitrary order tensor is presented which is the generalization of the known matrix representation of the second order tensor. It is showed that the elements of the multidimensional matrix of the initial moments of the random vector are the components of the tensor of the respective order. The theorem on the orthogonality of the transformation matrix of the arbitrary order tensor is given.

Introduction

The multidimensional-matrix mathematical approach is used in many applications one of them is control theory [1]. This approach is based on the theory of the multidimensional matrices that has very good foundations [2,3]. On the other hand, there are attempts to use the tensor as the multidimensional matrix. So, in [4] we find that "tensors are multidimensional generalizations of matrices". The illegality of such a replacement is noted shortly in [3]. In this article, we want emphasize that the generalization of the matrix to the multidimensional case should be performed in framework of the matrix theory but not in framework of the tensors theory. We will state some relationships between tensors and multidimensional matrices to achieve our goal.

I. Tensors and multidimensional matrices

Tensor is an object in linear finite-dimensional space. In tensor analysis, the so called Einstein summation convention is used: if an index is repeated in some term of the expression then the term must be summed with respect to that index for all admissible values of the index. For example, $x_i e_i$ is written instead of $\sum_{i=1}^{n} x_i e_i$, and $b_j = x_j^i e_i$ means the equality $b_j = \sum_{i=1}^{n} x_j^i e_i$. The tensor definition is connected with the transformation of the basis (coordinate system). Let x_i be the initial coordinate system with the basis e_i and y_j be the new coordinate system with the basis $e_j^{'}$.

A tensor a of the order $p = r + s$ of the type r, s (r time covariant and s time contravariant) is the geometrical object which 1) is defined by n^{r+s} components $a_{i_1,\dots,i_r}^{k_1,\dots,k_s}$ in each basis e_i, $i = 1, 2, ..., n$, of the real n-dimensional linear space L^n (indexes $i_1, ..., i_r, k_1, ..., k_s$ take the values $1, 2, ..., n$ independently) 2) has such a property that its components $\bar{a}_{i_1^{'},\dots,i_r^{'}}^{k_1^{'},\dots,k_s^{'}}$ in the basis $e_j^{'}$, $j = 1, 2, ..., n$, are connected with the components $a_{i_1,\dots,i_r}^{k_1,\dots,k_s}$ in the basis e_i by the relations

$$\bar{a}_{i_1^{'},\dots,i_r^{'}}^{k_1^{'},\dots,k_s^{'}} = \alpha_{i_1^{'}}^{i_1} \cdots \alpha_{i_r^{'}}^{i_r} \alpha_{k_1}^{k_1^{'}} \cdots \alpha_{k_s}^{k_s^{'}} a_{i_1,\dots,i_r}^{k_1,\dots,k_s}, \qquad (1)$$

in which $\alpha_{i^{'}}^{i}$ are elements of the transition from the basis e_i to the basis $e_j^{'}$ and $\alpha_k^{k^{'}}$ are the elements of the inverse transition from the basis $e_j^{'}$ to the basis e_i [5].

We will consider only covariant tensors $a_{i_1,\dots,i_r}$, i.e. we suppose in the definition (1) $s = 0$ and receive the following definition:

$$\bar{a}_{i_1^{'},\dots,i_r^{'}} = \alpha_{i_1^{'}}^{i_1} \cdots \alpha_{i_r^{'}}^{i_r} a_{i_1,\dots,i_r}. \qquad (2)$$

A multidimensional (p-dimensional) matrix is a system of numbers or variables $a_{i_1,i_2,\dots,i_p}$, $i_\alpha = 1, 2, ..., n_\alpha$, $\alpha = 1, 2, ..., p$, located at the points of the p-dimensional space defined by the coordinates $i_1, i_2, ..., i_p$. The p-dimensional matrix is denoted as

$$A = (a_{i_1,i_2,\dots,i_p}), \quad i_\alpha = 1, 2, ..., n_\alpha, \quad \alpha = 1, 2, ..., p,$$

or $A = (a_i)$, where $i = (i_1, i_2, ..., i_p)$ is a multiindex, $i_\alpha = 1, 2, ..., n_\alpha$, $\alpha = 1, 2, ..., p$.

The matrix $A^T = (a_{i_1,i_2,\dots,i_p}^T)$ the elements $a_{i_1,i_2,\dots,i_p}^T$ of which are connected with the elements $a_{i_1,i_2,\dots,i_p}$ of the matrix A by the equalities

$$a_{i_1,i_2,\dots,i_p}^T = a_{i_{\alpha_1},i_{\alpha_2},\dots,i_{\alpha_p}},$$

where $i_{\alpha_1}, i_{\alpha_2}, ..., i_{\alpha_p}$ is some permutation of the indices $i_1, i_2, ..., i_p$ is called the transposed matrix A according to the substitution

$$T = \begin{pmatrix} i_1, & i_2, & \cdots, & i_p \\ i_{\alpha_1}, & i_{\alpha_2}, & \cdots, & i_{\alpha_p} \end{pmatrix}.$$

If a p-dimensional matrix A is represented in the form of $A = (a_{i_1,i_2,\dots,i_p}) = (a_{l,s,c})$, where $l = (l_1, l_2, ..., l_\kappa)$, $s = (s_1, s_2, ..., s_\lambda)$, $c = (c_1, c_2, ..., c_\mu)$ are multiindexes, $\kappa+\lambda+\mu = p$, and a q-dimensional matrix B is represented in the form of $B = (b_{i_1,i_2,\dots,i_p}) = (b_{c,s,m})$, where $m = (m_1, m_2, ..., m_\nu)$ is a multiindex, $\lambda + \mu + \nu = p$, then the matrix $D = (d_{l,s,m})$ is called a (λ, μ)-folded product of the matrices A and B, if its elements are defined by the expression

$$d_{l,s,m} = \sum_c a_{l,s,c} b_{c,s,m}.$$

The (λ, μ)-folded product of the matrices A and B is denoted ${}^{\lambda,\mu}(AB)$. Thus,

$$D = {}^{\lambda,\mu}(AB) = (\sum_c a_{l,s,c} b_{c,s,m}) = (d_{l,s,m}).$$

In the case of the $(0,0)$-folded product we often omit the left upper indices and write AB instead of ${}^{0,0}(AB)$ and write A^k instead of ${}^{0,0}A^k$.

The matrix $E(\lambda, \mu)$ is called the (λ, μ)-identity matrix if the equalities

$${}^{\lambda,\mu}(AE(\lambda,\mu)) = {}^{\lambda,\mu}(E(\lambda,\mu)A) = A$$

are satisfied for any multidimensional matrix A. The matrix $E(\lambda, \mu)$ is $(\lambda + 2\mu)$-dimensional matrix whose elements are defined by the formula

$$E(\lambda,\mu) = (e_{c,s,m}) = \begin{pmatrix} 1, & c = m \\ 0, & c \neq m \end{pmatrix}.$$

II. Multidimensional-matrix representation of tensor

It is convenient to express a second order tensor $a_{i,j}$ in form of a matrix $a = (a_{i,j}), i, j = 1, 2, ..., n$ [6]. If we introduce the matrix $\alpha = (\alpha_{i,j})$ of the transition from the initial basis e_i to the new basis $e_i^{'}$ then the definition of the second order tensor (2) takes the form

$$\bar{a} = \alpha a \alpha^T. \tag{3}$$

The matrix representation is more convenient for the visual perception and computer calculations since the matrix algebra is very good represented in all programming systems.

It is noted in [6] that the matrix notation fails for tensors of higher order. However, this statement is refuted below. We give below the generalization of the expression (3) for the arbitrary order tensor based on the notion of the multidimensional matrix. We turn for this to the tensor definition (2) and introduce the two-dimensional matrix

$$\beta = (\beta_{j,i}), \quad \beta_{j,i} = \alpha_j^i, \tag{4}$$

of the transition from the initial basis e_i to the new basis $e_j^{'}$ and the r-dimensional matrices $a = (a_{i_1,...,i_r})$, $\bar{a} = (\bar{a}_{j_1,...,j_r})$ combined from the elements of the tensors $a_{i_1,...,i_r}$ and $\bar{a}_{j_1,...,j_r}$.

Theorem 1. The tensor definition (2) has the following multidimensional-matrix representation:

$$\bar{a} = {}^{0,r}(\bar{\beta} a), \tag{5}$$

where

$$\bar{\beta} = (\beta^r)^{T_{2r}}, \tag{6}$$

β is the matrix (4), T_{2r} is the following transpose substitution

$$T_{2r} = \begin{pmatrix} 1, & 2, & 3, & \cdots, & 2r-1, & 2r \\ 1, & 3, & 5, & \cdots, & 2r-2, & 2r \end{pmatrix}.$$

The known expression (3) is the particular case of the expression (5) provided $r = 2$. We can write the following expression instead of (3):

$$\bar{a} = {}^{0,r}((\beta^2)^{T_4} a).$$

Theorem 2. If $\xi = (\xi_1, ..., \xi_n)$ be a random vector in Euclidean space then the elements $\nu_{r,i_1,...,i_r}$ of the initial moment of the order r $\nu_r = E(\xi^r) = (\nu_{r,i_1,...,i_r})$ [3] are the components of the order r tensor.

Theorem 3. If the transition matrix $\beta = (\beta_{j,i})$ (4) from the basis e_i to the basis $e_j^{'}$ is orthogonal then the $2r$-dimensional matrix $\bar{\beta}$ (6) is orthogonal as well, i.e. it satisfies the following expression [7]:

$${}^{0,r}(\bar{\beta}\bar{\beta}^{B_{2r,r}}) = {}^{0,r}(\bar{\beta}^{B_{2r,r}}\bar{\beta}) = E(0,r),$$

where $E(0,r)$ is the $(0,r)$-identity matrix [2,3] and $B_{2r,r}$ is the transpose substitution of the type "onward" [3].

III. Conclusion

Thus, the article asserts the illegality of using a tensor as a multidimensional matrix. The use of the term tensor without taking into account its properties is unacceptable. Besides, the multidimensional-matrix notation has brightener possibility compared with the tensor analysis and can be used in tensor analysis. Particularly, the multidimensional-matrix representation of the arbitrary order tensor is received in this article. This representation is the generalization of the known matrix representation of the second order tensor. It is showed that the elements of the multidimensional matrix of the initial moments of the random vector are the components of the tensor of the respective order. The theorem on the orthogonality of the transformation matrix of the arbitrary order tensor is proved. The received results open the path to generalization the notion tensor to the multidimensional spaces.

IV. References

1. Mukha V.S., Kako N.F. Model-based optimization of the regression objects // Information Technologies and Systems 2021 (ITS 2021). Proceeding of the International Conference (BSUIR, Minsk, Belarus, 24th November 2021). – Pp. 234–235.
2. Sokolov N.P. Introduction to the Theory of Multidimensional Matrices. Kiev, Naukova dumka, 1972. 176 p. In Russian.
3. Mukha V.S. Analysis of Multidimensional Data. Minsk, Technoprint, 2004. 368 p. In Russian.
4. Cichocki A., Lee N., Oseledets I., Phan A.-H., Zhao Q. and Mandic D.P. (2016). Tensor Networks for Dimensionality Reduction and Large-scale Optimization: Part 1. Low-Rank Tensor Decompositions. Foundations and Trends in Machine Learning: Vol. 9: No. 4–5, pp. 249–429.
5. Ilyin V.A., Poznyak E.G. Linear algebra. Moscow.: Nauka, 1974. 296 p. In Russian.
6. Young, Eutiquio C. [1992] Vector and Tensor Analysis / Eutiquio C. Young – 2nd ed., rev. and expanded. – Marcel Dekker, Inc. – 511 p.
7. Mukha V.S. Multidimensional-matrix principal components method // Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series, 2013, no. 3, pp. 31–37. In Russian.