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TOTAL PROBABILITY AND BAYES FORMULAE FOR JOINT MULTIDIMENSIONAL-MATRIX GAUSSIAN DISTRIBUTIONS

Abstract. This paper is devoted to the development of a mathematical tool for obtaining the Bayesian estimations of the parameters of multidimensional regression objects in their finite-dimensional multidimensional-matrix description. Such a need arises, particularly, in the problem of dual control of regression objects when multidimensional-matrix mathematical formalism is used for the description of the controlled object. In this paper, the concept of a one-dimensional random cell is introduced as a set of multidimensional random matrices (in accordance with the "cell array" data type in the Matlab programming system), and the definition of the joint multidimensional-matrix Gaussian distribution is given (the definition of the Gaussian one-dimensional random cell). This required the introduction of the concepts of one-dimensional cell of the mathematical expectation and two-dimensional cell of the variance-covariance of the one-dimensional random cell. The integral connected with the joint Gaussian probability density function of the multidimensional matrices is calculated. The two formulae of the total probability and the Bayes formula for joint multidimensional-matrix Gaussian distributions are given. Using these results, the Bayesian estimations of the unknown coefficients of the multidimensional-matrix polynomial regression function are obtained. The algorithm of the calculation of the Bayesian estimations is realized in the form of the computer program. The results represented in the paper have theoretical and algorithmic generality.

Keywords: random cell, Gaussian random cell, multidimensional-matrix regression, Bayesian estimations

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ФОРМУЛЫ ПОЛНОЙ ВЕРОЯТНОСТИ И БАЙЕСА ДЛЯ СОВМЕСТНЫХ МНОГОМЕРНО-МАТРИЧНЫХ ГАУССОВСКИХ РАСПРЕДЕЛЕНИЙ

Аннотация. Работа посвящена разработке математического аппарата для получения байесовских оценок параметров многомерных регрессионных объектов в их конечномерном многомерно-матричном описании. Такая потребность возникает, в частности, в задаче дуального управления регрессионными объектами, когда для описания многомерного управляемого объекта применяется многомерно-матричный математический аппарат. В статье вводится понятие одномерной случайной ячейки как совокупности многомерных случайных матриц (в соответствии с данными типа «массив ячеек» в системе программирования Матлаб) и дается определение совместного гауссовского распределения многомерных случайных матриц (определение гауссовской одномерной случайной ячейки). Это потребовало введения понятия одномерной ячейки математического ожидания и понятия двумерной ячейки вариаций-ковариаций одномерной случайной ячейки. Далее вычисляется один интеграл, связанный с функцией совместной гауссовской плотности вероятности многомерных случайных матриц. Приводятся две формулы полной вероятности и формула Байеса для совместных многомерно-матричных гауссовских распределений. На основе этих результатов получены байесовские оценки неизвестных коэффициентов многомерно-матричной полиномиальной функции регрессии. Алгоритм расчета байесовских оценок реализован в виде компьютерной программы. Представленные результаты обладают теоретической и алгоритмической общностью.

Ключевые слова: случайная ячейка, гауссовская случайная ячейка, многомерно-матричная регрессия, байесовские оценки

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Introduction. This work is devoted to the problem of obtaining the parameters of multidimensional regression objects in their finite-dimensional multidimensional-matrix description. The first results in this direction were published in 1974. In paper [1], three integrals related to the probability density function of the vector Gaussian distribution were calculated. In paper [2], the Bayesian estimations were obtained for the unknown coefficients of the multiple regression function, which is linear in its coefficients, provided joint Gaussian priory distribution of the coefficients. In papers [3, 4], the extended version of these results, which is necessary for further generalization, was published. In paper [5], the more general case related to the joint vector Gaussian distribution and allowing us to obtain the Bayesian estimations of the parameters of regression objects with many vector parameters was considered.

In some cases, it is necessary to consider not only the set of random vectors with joint Gaussian distribution, but also the set of random multidimensional matrices with joint Gaussian distribution. Let us consider this case that arises, particularly, in the problem of the dual control of multivariate regression objects [6, 7] when the object is described by the multidimensional-matrix mathematical tool [8, 9]. In this paper, we generalize the results of work [5] for the case of the joint multidimensional-matrix Gaussian distribution.

Let $\Xi_1, \Xi_2, ..., \Xi_m$ be $q_1, q_2, ..., q_m$-dimensional random matrices, respectively. Let us combine them into a set $\Xi = \{\Xi_1, \Xi_2, ..., \Xi_m\} = \{\Xi_i\}$, ordered by the index $i = 1, 2, ..., m$, which we will name a random cell, in accordance with the “cell array” data type in the Matlab programming system. In contrast to the elements of the multidimensional matrix, we will enclose the elements of the cell into curly brackets, as it is accepted in the Matlab programming system: $\Xi = \{\Xi_i\}$. Let us note that a cell is not a multidimensional matrix, so it should be considered as a purly mathematical object.

We will denote the mathematical expectation of the random q_i-dimensional matrix Ξ_i as $\nu_{\Xi_i} = E(\Xi_i)$, $i = 1, 2, ..., m$, and the covariance matrix of the random q_i and q_j-dimensional matrices Ξ_i, Ξ_j as

$$d_{\Xi,i,j} = E\left({}^{0,0}(\overset{\circ}{\Xi}_i \overset{\circ}{\Xi}_j) \right), \quad i, j = 1, 2, ..., m, \quad \overset{\circ}{\Xi}_i = \Xi_i - \nu_{\Xi_i}, \ \overset{\circ}{\Xi}_j = \Xi_j - \nu_{\Xi_j}, \quad {}^{0,0}\left(\overset{\circ}{\Xi}_i \overset{\circ}{\Xi}_j \right)$$

is the (0,0)-folded product of the matrices $\overset{\circ}{\Xi}_i$, $\overset{\circ}{\Xi}_j$ [9]. The $d_{\Xi,i,j}$ is the $(q_i + q_j)$-dimensional matrix.

Let us combine the mathematical expectations ν_{Ξ_i} into a set $\nu_\Xi = \{\nu_{\Xi_1}, \nu_{\Xi_2}, ..., \nu_{\Xi_m}\}$ $(i = 1, 2, ..., m)$ which we will name as a mathematical expectation of the random cell $\Xi = \{\Xi_1, \Xi_2, ..., \Xi_m\}$.

Let us combine the covariance matrices $d_{\Xi,i,j}$ into a set $d_\Xi = \{d_{\Xi,i,j}\}$ ordered by two indices $i, j = 1, 2, ..., m$ which we will name as a covariance cell of the random cell $\Xi = \{\Xi_1, \Xi_2, ..., \Xi_m\}$.

We will name the cell denoted by $d_\Xi^{-1} = \left\{ d_\Xi^{i,j} \right\}$ as the $(q_i + q_j)$-inverse to the covariance cell $d_\Xi = \{d_{\Xi,i,j}\}$, $i, j = 1, 2, ..., m$, if its elements $d_\Xi^{i,j}$ satisfy the following equalities:

$${}^{0,(p_i+p_j)}\left\{ d_{\Xi,i,j} d_\Xi^{i,j} \right\} = {}^{0,(p_i+p_j)}\left\{ d_\Xi^{i,j} d_{\Xi,i,j} \right\} = \begin{cases} E(0, q_i), & \text{at } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

where $E(0,q_i)$ is the $(0,q_i)$-identity matrix ($2q_i$-dimensional matrix) and 0 is the $(q_i + q_j)$-dimensional zero matrix.

In this notations, we can define the Gaussian probability density function $f(\xi)$ of the random cell $\Xi = \{\Xi_1, \Xi_2, ..., \Xi_m\}$ by the following expression:

$$f(\xi) = M_\Xi \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_i}\left((\xi_i - \nu_{\Xi_i}) \, {}^{0,q_j}\left(d_\Xi^{i,j} (\xi_j - \nu_{\Xi_j}) \right) \right) \right) =$$

$$= M_\Xi \exp\left(-\frac{1}{2} \, {}^{0,q_i}\left(\sum_{i=1}^{m} (\xi_i - \nu_{\Xi_i}) \, {}^{0,q_j}\left(\sum_{j=1}^{m} d_\Xi^{i,j} (\xi_j - \nu_{\Xi_j}) \right) \right) \right) =$$

$$= M_\Xi \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_i}\left(\xi_i \, {}^{0,q_j}\left(d_\Xi^{i,j} \xi_j \right) \right) + \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_i}\left(\nu_{\Xi_i} \, {}^{0,q_j}\left(d_\Xi^{i,j} \xi_j \right) \right) - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_i}\left(\nu_{\Xi_i} \, {}^{0,q_j}\left(d_\Xi^{i,j} \nu_{\Xi_j} \right) \right) \right) =$$

$$=M_{\Xi}\exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\xi_i d_{\Xi}^{i,j}\right)\xi_j\right)+\right.$$

$$\left.+\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i} d_{\Xi}^{i,j}\right)\xi_j\right)-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i} d_{\Xi}^{i,j}\right)\nu_{\Xi_j}\right)\right), \tag{1}$$

where $\xi=\{\xi_1,\xi_2,...,\xi_m\}$ is the cell-argument of the probability density function corresponding to the random cell $\Xi=\{\Xi_1,\Xi_2,...,\Xi_m\}$, $|d_{\Xi}|$ is the determinant of the covariance cell d_{Ξ}, k_{Ξ} is the number of the scalar elements of the cell Ξ, $M_{\Xi}=\dfrac{1}{\sqrt{(2\pi)^{k_{\Xi}}|d_{\Xi}|}}$.

1. Integral connected with the joint Gaussian distribution of the multidimensional random matrices. The following theorem relative to the joint Gaussian distribution of the multidimensional random matrices (1) is fulfilled.

Theorem 1 (the integral connected with the joint Gaussian distribution of the multidimensional random matrices). *If* $A=\{A_{i,j}\}$, $i,j=1,2,...,m$, *is a two-dimensional symmetric positive definite cell,* $A^{-1}=\{A^{i,j}\}$, $i,j=1,2,...,m$, *is the cell inverse to the cell A,* $|A^{-1}|$ *is the determinant of the cell* A^{-1}, $\xi=\{\xi_i\}$, $I=1,2,\ldots,m$, *is a one-dimensional cell composed of the* q_i*-dimensional matrices* ξ_i *allowing the multiplication* ${}^{0,q_i}\left({}^{0,q_j}(A_{i,j}\xi_j)\xi_i\right)$, k_i *is the number of the scalar components of the matrix* ξ_i, $k_1+k_2+...+k_m=k_{\Xi}$, $B=\{B_i\}$ *is a one-dimensional cell composed of the* q_i*-dimensional matrices* B_i *allowing the multiplication* ${}^{0,q_i}(B_i\xi_i)$, $E^{k_{\Xi}}$ *is the* k_{Ξ}*-dimensional Euclidean space, then the following equality is fulfilled:*

$$\int_{E^{k_{\Xi}}}\exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_i+q_j}\left(A_{i,j}\,{}^{0,0}(\xi_i\xi_j)\right)+\sum_{j=1}^{m}{}^{0,q_j}(B_j\xi_j)\right)d\xi_1\cdots d\xi_m=$$

$$=\sqrt{(2\pi)^{k_{\Xi}}|A^{-1}|}\exp\left(\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_i+q_j}\left(A^{i,j}\,{}^{0,0}(B_iB_j)\right)\right). \tag{2}$$

This theorem is the generalization of theorem 1 from paper [5] (the integral connected with the joint Gaussian distribution of the random vectors) to the multidimensional-matrix case. We obtain it applying the theorem on the associated multidimensional matrices [9] to theorem 1 of paper [5].

2. Total probability formulae for the joint Gaussian distribution of the multidimensional random matrices. Theorem 2 (total probability formula 1 for the joint Gaussian distribution of the multidimensional random matrices). *Let* $\Xi=\{\Xi_i\}$, $i=1,2,\ldots,m$, *be a one-dimensional random cell, composed of the* q_i*-dimensional* matrices Ξ_i, k_i the number of the scalar components of the matrix Ξ_i, $f(\xi)$ the probability density function of the cell Ξ, $k_{\Xi}=k_1+k_2+...+k_m$ *the number of the scalar components of the cell* Ξ, $f(y/\xi)$ *the condition probability density function of a p-dimension matrix Y,* k_Y *the number of the scalar components of the matrix Y,* $E^{k_{\Xi}}$ *the* k_{Ξ}*-dimensional Euclidean space. If in the total probability formula*

$$f(y)=\int_{E^{k_{\Xi}}}f(y/\xi)f(\xi)d\xi \tag{3}$$

the conditional probability density function $f(y/\xi)$ *is represented in the form*

$$f(y/\xi)=\frac{1}{\sqrt{(2\pi)^{k_Y}|d_Y|}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}(\xi_iS_{i,j})\xi_j\right)+\sum_{j=1}^{m}{}^{0,q_j}(V_j\xi_j)-\frac{1}{2}W\right\}, \tag{4}$$

where $S_{i,j}$ *are the elements of a symmetric positive definite cell* $S=\{S_{i,j}\}$, $i,j=1,2,...,m$, *and the probability density function* $f(\xi)$ *is represented in the form*

$$f(\xi) = M_\Xi \exp(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}\left((\xi_i - \nu_{\Xi_i}) d_\Xi^{i,j}\right)(\xi_j - \nu_{\Xi_j})\right) =$$

$$= M_\Xi \exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(\xi_i d_\Xi^{i,j}\right)\xi_j\right) + \right.$$

$$\left. + \sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i} d_\Xi^{i,j}\right)\xi_j\right) - \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i} d_\Xi^{i,j}\right)\nu_{\Xi_j}\right)\right), \tag{5}$$

$$M_\Xi = \frac{1}{\sqrt{(2\pi)^{k_\Xi} \, |d_\Xi|}},$$

then integral (3) *(the total probability formula) is defined by the following expression*

$$f(y) = \int_{E^{k_\Xi}} f(y/\xi) f(\xi) d\xi = \frac{1}{\sqrt{(2\pi)^{k_Y} \, |d_\Xi| \, |d_Y| \, |A|}} \exp\left(\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}(B_i A^{i,j}) B_j\right) - \frac{1}{2} C\right),$$

where

$$A = (A_{i,j}) = \left(d_\Xi^{i,j} + S_{i,j}\right), \quad i, j = \overline{1,m},$$

$$B = (B_i) = \left(\sum_{j=1}^{m} d_\Xi^{i,j} \nu_{\Xi,j} + V_i\right), \quad i = \overline{1,m},$$

$$C = \sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left(\nu_{\Xi,i} \, {}^{0,q_i}(d_\Xi^{i,j} \nu_{\Xi,j})\right) + W.$$

This theorem is the generalization of theorem 2 from work [5] (the total probability formula for the joint vector Gaussian distribution) to the multidimensional-matrix case. The proof is performed by multiplication $f(y/\xi)f(\xi)$ in formula (3) which is followed by integration using formula (2), or by applying the theorem on the associated multidimensional matrices [9] to theorem 2 of work [5].

The functions $f(y/\xi)$ and $f(\xi)$ in total probability formula (3) are usually given not in the form of expressions (4), (5), but in the natural for the Gaussian distribution form. In this case, theorem 2 takes the form of theorem 3. Such representation is more convenient for practical utilization.

T h e o r e m 3 (total probability formula 2 for the joint Gaussian distribution of the multidimensional random matrices). *Let* $\Xi = \{\Xi_i\}$, $i = 1, 2, ..., m$, *be a one-dimensional random cell, composed of the* q_i*-dimensional matrices* Ξ_i, k_i is *the number of the scalar components of the matrix* Ξ_i, $f(\xi)$ *the probability density function of the cell* Ξ, $k_\Xi = k_1 + k_2 + ... + k_m$ *the number of the scalar components of the cell* Ξ, $f(y/\xi)$ *the condition probability density function of a p-dimensional matrix Y,* k_Y *the number of the scalar components of the matrix Y,* E^{k_Ξ} *the* k_Ξ*-dimensional Euclidean space. If in the total probability formula*

$$f(y) = \int_{E^{k_\Xi}} f(y/\xi) f(\xi) d\xi \tag{6}$$

the conditional probability density function $f(y/\xi)$ *has the following form*

$$f(y/\xi) = \frac{1}{\sqrt{(2\pi)^{k_Y} \, |d_Y|}} \exp\left(-\frac{1}{2} \, {}^{0,p}\left(d_Y^{-1}\left(y - \sum_{i=1}^{m} {}^{0,q_i}(h_i \xi_i)\right)^2\right)\right),$$

where h_i *is a* $(p + q_i)$*-dimensional matrix, allowing the multiplication* ${}^{0,q_i}(h_i \xi_i)$, *and the probability density function* $f(\xi)$ *has the following form*

$$f(\xi)=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|d_\Xi|}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left((\xi_i-\nu_{\Xi_i})d_\Xi^{i,j}\right)(\xi_j-\nu_{\Xi_j})\right)\right\},$$

then the integral (6) *(the total probability formula) is defined by the following expression:*

$$f(y)=\int_{E^{k_\Xi}}f(y/\xi)f(\xi)d\xi=\frac{1}{\sqrt{(2\pi)^{k_Y}\,|D_Y|}}\exp\left(-\frac{1}{2}{}^{0,p}\left(D_Y^{-1}\left(y-\sum_{i=1}^{m}{}^{0,q_i}(h_i\nu_{\Xi,i})\right)^2\right)\right),$$

where

$$D_Y=d_Y+\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}(h_i d_{\Xi,i,j})h_j\right).$$

This theorem is the generalization of theorem 3 from work [5] to the multidimensional-matrix case. It is obtained by applying the theorem on the associated multidimensional matrices [9] to theorem 3 of work [5].

3. Bayes formula for the joint multidimensional-matrix Gaussian distribution. Theorem 4 (the Bayes formula for the joint multidimensional-matrix Gaussian distribution). *Let $\Xi=\{\Xi_i\}$, $i=1,2,...,m$, be a one-dimensional random cell, composed of the q_i-dimensional matrices Ξ_i, k_i the number of the scalar components of the matrix Ξ_i, $f(\xi)$ the probability density function of the cell Ξ, $k_\Xi=k_1+k_2+...+k_m$ the number of the scalar components of the cell Ξ, $f(y/\xi)$ the condition probability density function of a p-dimensional matrix Y, k_Y the number of the scalar components of the matrix Y, E^{k_Ξ} the k_Ξ-dimensional Euclidean space. If in the Bayes formula*

$$f(\xi/y)=\frac{f(\xi)f(y/\xi)}{\int_{E^{k_\Xi}}f(\xi)f(y/\xi)d\xi} \tag{7}$$

the conditional probability density function f(y/ξ) is represented in the form

$$f(y/\xi)=\frac{1}{\sqrt{(2\pi)^{k_Y}\,|d_Y|}}\exp\left\{-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}(\xi_i S_{i,j})\xi_j\right)+\sum_{j=1}^{m}{}^{0,q_j}(V_j\xi_j)-\frac{1}{2}W\right\}, \tag{8}$$

where $S_{i,j}$ are the elements of a symmetric positive definite cell $S=\{S_{i,j}\}$, $i,j=1,2,...,m$, and the probability density function f(ξ) is represented in the form

$$f(\xi)=M_\Xi\exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left((\xi_i-\nu_{\Xi_i})d_\Xi^{i,j}\right)(\xi_j-\nu_{\Xi_j})\right)\right)=$$
$$=M_\Xi\exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\xi_i d_\Xi^{i,j}\right)\xi_j\right)+\right.$$
$$\left.+\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i}d_\Xi^{i,j}\right)\xi_j\right)-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}{}^{0,q_j}\left({}^{0,q_i}\left(\nu_{\Xi_i}d_\Xi^{i,j}\right)\nu_{\Xi_j}\right)\right), \tag{9}$$

$$M_\Xi=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|d_\Xi|}},$$

then the posterior probability density function $f(\xi/y)$ of the random cell Ξ defined by the Bayes formula (7) has the following form

$$f(\xi/y)=\frac{1}{\sqrt{(2\pi)^{k_\Xi}\,|D_\Xi|}}\exp\left(-\frac{1}{2}{}^{0,2}\left\{D_\Xi^{-1}\,{}^{0,0}\{\xi-\mathrm{N}_\Xi\}^2\right\}\right), \tag{10}$$

where the cell D_{Ξ}^{-1} is defined by the expression

$$D_{\Xi}^{-1} = \left\{D_{\Xi}^{i,j}\right\} = \left\{d_{\Xi}^{i,j} + S_{i,j}\right\}, \quad i,j = 1,2,...,m,$$

$D_{\Xi} = \{D_{\Xi,i,j}\}$ is the two-dimensional cell inverse to the cell D_{Ξ}^{-1},

$$B = \{B_i\} = \sum_{j=1}^{m} {}^{0,q_j}\left(d_{\Xi}^{i,j}\nu_{\Xi,j}\right) + V_i,$$

$$\mathrm{N}_{\Xi} = \{\mathrm{N}_{\Xi,i}\} = {}^{0,1}\{A^{-1}B\} = \left\{\sum_{j=1}^{m} {}^{0,p+q_j}(D_{\Xi,i,j}B_j)\right\}, \quad i = 1,2,...,m. \tag{11}$$

The expression ${}^{0,0}\{\xi - \mathrm{N}_{\Xi}\}^2$ means the (0,0)-folded square of the one-dimensional cell $\xi - \mathrm{N}_{\Xi}$. In this case the elements of the cells are multiplied in the sense of the (0,0)-folded production. If one denotes $\xi - \mathrm{N}_{\Xi} = \overset{\circ}{\xi}$, then

$${}^{0,0}\{\xi - \mathrm{N}_{\Xi}\}^2 = {}^{0,0}\{\overset{\circ}{\xi}\}^2 = \left\{{}^{0,0}(\overset{\circ}{\xi}_i\,\overset{\circ}{\xi}_j)\right\} = \eta = \{\eta_{i,j}\}, \quad i,j = 1,2,...,m.$$

Finally, the expression ${}^{0,2}\left\{D_{\Xi}^{-1}\,{}^{0,0}\{\xi - \mathrm{N}_{\Xi}\}^2\right\} = {}^{0,2}\{A\eta\}$ means the (0,2)-folded product of the two-dimensional cell $D_{\Xi} = \{D_{\Xi,i,j}\}$ and the two-dimensional cell $\eta = \{\eta_{i,j}\}$, which is defined by the following formula:

$${}^{0,2}\{D_{\Xi}\eta\} = \sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_i+q_j}(D_{\Xi,i,j}\eta_{i,j}).$$

R e m a r k. The cell $d_{\Xi} = \{d_{\Xi,i,j}\}$, $i,j = 1,2,...,m$, in formula (9) is the prior covariance cell of the random cell $\Xi = \{\Xi_1, \Xi_2, ..., \Xi_m\}$, and the cell $\nu_{\Xi} = \{\nu_{\Xi,i}\}$, $i = 1,2,...,m$, is the prior mathematical expectation of the random cell $\xi = \{\xi_1, \xi_2, ..., \xi_m\}$. The two-dimensional cell $d_{\Xi}^{-1} = (d_{\Xi}^{i,j})$, $i,j = \overline{1,m}$, is the cell inverse to the cell d_{Ξ}. The one-dimensional cell $\mathrm{N}_{\Xi} = \{\mathrm{N}_{\Xi,i}\}$, $i = 1,2,...,m$, (11) is the posterior mathematical expectation of the random cell Ξ $\left(\mathrm{N}_{\Xi,i} = E(\Xi_i / y)\right)$, and the two-dimensional cell $D_{\Xi} = \{D_{\Xi,i,j}\}$ is the posterior covariance cell of the random cell Ξ. Owing to this, the matrix $D_{\Xi,i,j}$ is the posterior covariance matrix of the random matrices Ξ_i and Ξ_j $\left(D_{\Xi,i,j} = \mathrm{cov}(\Xi_i, \Xi_j / y)\right)$.

Theorem 4 is the generalization of theorem 4 of work [5] (Bayes formula for the joint vector Gaussian distribution) to the multidimensional case. The proof is realized multiplicating $f(y/\xi)f(\xi)$ in the numerator of formula (7), integrating in the denominator of formula (7), and dividing the numerator by the denominator. The proof can be performed also by applying the theorem on the associated multidimensional matrices [9] to theorem 4 of work [5].

4. Bayesian estimations of the coefficients of the polynomial multidimensional-matrix regression. Let us consider an object with the q-dimensional matrix of the input variable $x = (x_j)$, $j = (j_1, j_2, ..., j_q)$, $j_\alpha = 1,2,...,k_{x,\alpha}$, $\alpha = 1,2,...,q$, the p-dimensional matrix of the output variable $\eta = (\eta_i)$, $i = (i_1, i_2, ..., i_p)$, $i_\beta = 1,2,...,k_{y,\beta}$, $\beta = 1,2,...,p$, [9], and suppose that the output variable η has the stochastic dependence on the input variable x so that the conditional probability density function $f(\eta/x)$ exists. We will denote $y = \varphi(x)$ as the regression function of η on x and suppose that the dependence of η on x could be represented in the form $\eta = \varphi(x) + \varepsilon$, where ε is the p-dimensional random matrix with the zero mean value. Let the values $y_{o,1}, y_{o,2}, ..., y_{o,n}$ (measurements) of the output variable be obtained for the values $x_1, x_2, ..., x_n$ of the input variable in the form

$$y_{o,\mu} = \varphi(x_\mu) + z_\mu, \quad \mu = 1,2,...,n, \tag{12}$$

where z_μ is the value of the random matrix ε, which we will call the measurement error. We will consider the distribution of the random matrix ε to be Gaussian with the zero mean value and the covariance matrix d_Y.

We will agree to use below the following notations for indexing the elements of the multidimensional matrices: i_1, i_2, … are separate indices; $\overline{i}_{(p)} = (i_1, i_2, ..., i_p)$ is a set of p indices (p-multi-index); $\overline{\overline{i}}_{(p,k)} = (\overline{i}_{(p),1}, \overline{i}_{(p),2}, ..., \overline{i}_{(p),k})$ is a set of k p-multi-indices.

Let us suppose that the hypothetical regression function $\varphi(x)$ is represented in the form of a polynomial of the degree m [9]:

$$\varphi(x) = \sum_{k=0}^{m} {}^{0,kq}\left(c_k x^k\right) = \sum_{k=0}^{m} {}^{0,kq}\left(x^k c_{t,k}\right), \quad m = 0,1,2,..., \tag{13}$$

where c_k is the $(p + kq)$-dimensional matrix of the coefficients (parameters),

$$c_k = \left(c_{\overline{i}_{(p)}, \overline{\overline{j}}_{(q,k)}}\right), \quad \overline{i}_{(p)} = (i_1, i_2, ..., i_p), \quad \overline{\overline{j}}_{(q,k)} = \left(\overline{j}_{(q)1}, \overline{j}_{(q)2}, ..., \overline{j}_{(q)k}\right),$$

symmetric for $k \geq 2$ relative the q-multi-indices $\overline{j}_{(q)1}, \overline{j}_{(q)2}, ..., \overline{j}_{(q)k}$, $c_{t,k}$ is the $(kq + p)$-dimensional matrix of the coefficients (parameters),

$$c_{t,k} = \left(c_{t, \overline{\overline{j}}_{(q,k)}, \overline{i}_{(p)}}\right), \quad \overline{i}_{(p)} = (i_1, i_2, ..., i_p), \quad \overline{\overline{j}}_{(q,k)} = \left(\overline{j}_{(q)1}, \overline{j}_{(q)2}, ..., \overline{j}_{(q)k}\right),$$

$c_{t,k} = (c_k)^{T_k}$, $T_k = B_{p+kq,kq}$ is the transpose substitution of the type “onward” [10], ${}^{0,kq}(c_k x^k) = {}^{0,kq}(x^k c_{t,k})$ is the $(0,kq)$-folded product of the matrices c_k and x^k, $x^k = {}^{0,0}x^k$ is the (0,0)-folded k-th degree of the matrix x [9]. Let the $c_t = \{c_{t,k}\}$, $k = 0,1,2,...,m$, be the one-dimensional cell of the coefficients of regression function (13), then the matrix of separate measurement $y_{o,\mu}$ (12) will have the probability density function of the following form

$$f(y_{o,\mu} / x_\mu, c_t) = M_y \exp\left(-\frac{1}{2} {}^{0,2p}\left(d_Y^{-1}\left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(x_\mu^k c_{t,k}\right)\right)^2\right)\right), \quad \mu = 1,2,...,n, \tag{14}$$

where $M_y = \left(\sqrt{(2\pi)^{n_y} |d_Y|}\right)^{-1}$ is the normalizing constant, d_Y^{-1} is the matrix $(0,p)$-inverse to the matrix d_Y [9], n_y is the number of the scalar elements of the matrix y.

The problem consists of finding the Bayesian estimations $\widehat{c}_{t,0}, \widehat{c}_{t,1}, ..., \widehat{c}_{t,m}$ of the unknown coefficients $c_{t,0}, c_{t,1}, ..., c_{t,m}$ of multidimensional-matrix regression (13) on the basis of the independent measurements $(x_1, y_{o,1}), (x_2, y_{o,2}), ..., (x_n, y_{o,n})$.

Let the random cell $c_t = \{c_{t,k}\}$, $k = 0,1,2,...,m$, have the Gaussian probability density function

$$f(c_t) = M_\Xi \exp\left(-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} {}^{0,q_j}\left({}^{0,q_i}\left((c_{t,i} - \nu_{c_t,i}) d_{c_t}^{i,j}\right)(\xi_j - \nu_{c_t,j})\right)\right) =$$

$$= M_{c_t} \exp\left(-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(c_{t,i} d_{c_t}^{i,j}\right) c_{t,j}\right) + \right.$$

$$\left. + \sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(c_{t,i} d_{c_t}^{i,j}\right)\nu_{c_t,j}\right) - \frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,q_j}\left({}^{0,q_i}\left(\nu_{c_t,i} d_{c_t}^{i,j}\right)\nu_{c_t,j}\right)\right),$$

$$M_{c_t} = \frac{1}{\sqrt{(2\pi)^{n_c} |d_{c_t}|}}, \quad q_i = p + iq, \quad i = 0,1,2,...,m,$$

in which $d_{c_t} = \{d_{c_t,i,j}\}$, $i, j = 0,1,2,...,m$, is the covariance cell of the random cell c_t,

$$d_{c_t,i,j} = E\left({}^{0,0}\left((c_{t,i} - \nu_{c_t,i})(c_{t,j} - \nu_{c_t,j})\right)\right)$$

is the $\left((iq + p) + (jq + p)\right)$-dimensional matrix, $d_{c_t}^{-1} = \left\{d_{c_t}^{i,j}\right\}$, $i, j = 0,1,2,...,m$, is the cell inverse to the covariance cell d_{c_t}, $\nu_{c_t} = \{\nu_{c_t,0}, \nu_{c_t,1}, ..., \nu_{c_t,m}\} = \{\nu_{c_t,i}\}$, $i = 0,1,2,...,m$, is the one-dimensional cell of the mathematical expectation of the random cell c_t (i. e. $\nu_{c_t,i} = E(c_{t,i})$ is the $(iq + p)$-dimensional matrix), n_c is the number of the scalar elements of the cell c_t.

The probability density function of the set of all having measurements $\vec{y}_o = \{y_{o,1}, y_{o,2}, ..., y_{o,n}\}$ is defined by the following expression

$$f(\vec{y}_o / \vec{x}, c) = \prod_{\mu=1}^{n} f(y_{o,\mu} / x_\mu, c) = M_y^n \exp\left(-\frac{1}{2} \sum_{\mu=1}^{n} {}^{0,2p}\left(d_Y^{-1} \left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k \right) \right)^2 \right) \right), \tag{15}$$

where we denote $\vec{x} = \{x_1, x_2, ..., x_n\}$.

We should transform expression (15) to form (8) (when the c replaces the ξ) for applying the Bayes formula (10). This transformation is given in the Appendix. As a result, in accordance with the Bayes formula (10), the posterior probability density function of the cell $c_t = \{c_{t,i}\}$, $i = 0,1,2,...,m$, is defined by the following expression:

$$f(c_t / \vec{y}_o) = \frac{1}{\sqrt{(2\pi)^{n_y} |D_{c_t}|}} \exp\left(-\frac{1}{2} {}^{0,2}\left\{ D_{c_t}^{-1} \, {}^{0,0}\{c_t - \mathrm{N}_{c_t}\}^2 \right\} \right) =$$

$$= \frac{1}{\sqrt{(2\pi)^{n_y} |D_{c_t}|}} \exp\left(-\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} {}^{0,q_i}\left((c_{t_i} - \mathrm{N}_{c_{t,i}}) \, {}^{0,qj}\left(D_{c_t}^{i,j} (c_{t_j} - \mathrm{N}_{c_{t,j}}) \right) \right) \right), \tag{16}$$

in which

$$D_{c_t}^{-1} = \left\{ D_{c_t}^{i,j} \right\} = \left\{ d_{c_t}^{i,j} + S_{i,j} \right\} = \left\{ d_{c_t}^{i,j} + {}^{0,0}\left(d_Y^{-1} S_{x^i x^j} \right)^{T_{i,j}} \right\}, \quad i,j = 0,1,2,...,m, \tag{17}$$

$$B = \{B_i\} = \left\{ \sum_{j=0}^{m} {}^{0,jq+p}\left(d_{c_t}^{i,j} \nu_{c_t,j} \right) + {}^{0,p}\left(d_Y^{-1} S_{yx^i} \right)^{T_i} \right\}, \quad i = 0,1,2,...,m, \tag{18}$$

$$\mathrm{N}_{c_t} = {}^{0,1}\{D_{c_t} B\} = \left\{ \sum_{j=0}^{m} {}^{0,p+jq}(D_{c_t,i,j} B_j) \right\} = \{\mathrm{N}_{c_t,i}\}, \quad i = 0,1,2,...,m, \tag{19}$$

$$S_{x^k x^\lambda} = \sum_{\mu=1}^{n} {}^{0,0}\left(x_\mu^k x_\mu^\lambda \right), \quad S_{yx^\lambda} = \sum_{\mu=1}^{n} {}^{0,0}\left(y_\mu x_\mu^\lambda \right), \tag{20}$$

$$\vec{y}_o = (y_{o,1}, y_{o,2}, ..., y_{o,n}).$$

The transpose substitutions $T_{i,j}$ in (17) and T_i in (18) are defined by the expressions:

$$T_{i,j} = \begin{pmatrix} \bar{i}_1, \bar{i}_2, ..., \bar{i}_i, \bar{\lambda}, \bar{j}_1, \bar{j}_2, ..., \bar{j}_j, \bar{\mu}, \\ \bar{\lambda}, \bar{\mu}, \bar{i}_1, \bar{i}_2, ..., \bar{i}_i, \bar{j}_1, \bar{j}_2, ..., \bar{j}_j \end{pmatrix}, \quad i,j = 0,1,2,...,m,$$

$$T_i = \begin{pmatrix} \bar{i}_1, \bar{i}_2, ..., \bar{i}_i, \bar{\mu} \\ \bar{\mu}, \bar{i}_1, \bar{i}_2, ..., \bar{i}_i \end{pmatrix}, \quad i = 0,1,2,...,m,$$

where the multi-indices $\bar{j}_1, \bar{j}_2, ..., \bar{j}_j, \bar{i}_1, \bar{i}_2, ..., \bar{i}_i$ contain by q indexes each, and the multi-indices $\bar{\lambda}, \bar{\mu}$ contain by p indices each.

The two-dimensional cell $D_{c_t}^{-1} = \left\{ D_{c_t}^{i,j} \right\}$, $i,j = 0,1,2,...,m$, (17) has the same size as the two-dimensional cell $d_{c_t} = \{d_{c_t,i,j}\}$, i. e. $D_{c_t}^{i,j}$ is the $\left((iq + p) + (jq + p) \right)$-dimensional matrix.

The element B_i of the one-dimensional cell $B = \{B_i\}$, $i = 0,1,2,...,m$, (18) is the $(iq + p)$-dimensional matrix.

Thus, the Bayesian estimation of the coefficient cell of multidimensional-matrix polynomial regression (13) is defined by one-dimensional cell (19),

$$\widehat{c}_t = \{\mathrm{N}_{c_{t,i}}\} = {}^{0,1}\{D_{c_t}B\} = \left\{\sum_{j=0}^{m} {}^{0,p+jq}(D_{c_{t,i,j}}B_j)\right\}, \quad i = 0,1,2,...,m, \tag{21}$$

and the two-dimensional cell $D_{c_c} = \{D_{c_{c,i,j}}\}$ in (16) is defined by the posterior covariance of the multidimensional-matrix coefficients $c_{t,k}$, $k = 0,1,2,...,m$.

Let us note that we obtained the estimations of the coefficients $c_{t,k}$ for the regression function in the form $\varphi(x) = \sum_{k=0}^{m} {}^{0,kq}\left(x^k c_{t,k}\right)$. It means that we should prepare the priory characteristics just for these coefficients (not for the coefficients c_k).

The use of the single measurement to update the estimation is often of interest. Such a case takes place in the dual control theory [6–8]. In this case, one should use, instead of expressions (20), the following expressions defining the single measurement (x_n, y_n):

$$S_{x_n^k x_n^\lambda} = {}^{0,0}\left(x_n^{k+\lambda}\right), \quad S_{y_n x_n^\lambda} = {}^{0,0}\left(y_n x_n^\lambda\right).$$

Conclusion. The calculation of the Bayesian estimations of the coefficients of the polynomial multidimensional-matrix regression by formulae (17), (18), (21) requires the involvement of the computer equipment. This algorithm was implemented in the form of the m-file-function of the Matlab programming system. The efficiency of the algorithm was confirmed for the examples of the one-dimensional and two-dimensional matrix-matrix quadratic regressions.

The results obtained in the paper possess the theoretical and algorithmic generality.

Appendix. Let us transform the probability density function

$$f(\vec{y}_o / \vec{x}, c) = M_y^n \exp\left(-\frac{1}{2}\sum_{\mu=1}^{n} {}^{0,2p}\left(d_Y^{-1}\left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)^2\right)\right) \tag{1A}$$

to the form (8) when the c_{t_i} replaces the ξ_i, i. e. to the following form:

$$f(y_o / x, c) = M_y^n \exp\left\{-\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{m} {}^{0,p_i}\left({}^{0,p_j}(c_{t,i}S_{i,j})c_{t,j}\right) + \sum_{j=0}^{m} {}^{0,p_j}(V_j c_{t,j}) - \frac{1}{2}W\right\}, \tag{2A}$$

where $p_j = jq + p$ is the dimensionality of the j-th coefficient $c_{t,j}$ in (13), to obtain the expressions for the $S_{i,j}$ and the V_j, as it requires theorem 4.

Let us take into account, that $\sum_{k=0}^{m} {}^{0,kq}\left(c_k x^k\right) = \sum_{k=0}^{m} {}^{0,kq}\left(x^k c_{t,k}\right)$. Then in (1A)

$$\left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)^2 = \left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)\left(y_{o,\mu} - \sum_{l=0}^{m} {}^{0,lq}\left(x_\mu^l c_{t,l}\right)\right) =$$

$$= \left(y_{o,\mu}y_{o,\mu}\right) - \left(y_{o,\mu}\left(\sum_{l=0}^{m} {}^{0,lq}\left(x_\mu^l c_{t,l}\right)\right)\right) - \left(\left(\sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)y_{o,\mu}\right) + \left(\left(\sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)\left(\sum_{l=0}^{m} {}^{0,lq}\left(x_\mu^l c_{t,l}\right)\right)\right) =$$

$$= \left(y_{o,\mu}y_{o,\mu}\right) - \left(\sum_{l=0}^{m} {}^{0,lq}\left(\left(y_{o,\mu}x_\mu^l\right)c_{t,l}\right)\right) - \left(\sum_{k=0}^{m} {}^{0,kq}\left(c_k\left(x_\mu^k y_{o,\mu}\right)\right)\right) + \left(\left(\sum_{k=0}^{m}\sum_{l=0}^{m} {}^{0,kq}\left(c_k {}^{0,lq}\left(\left(x_\mu^k x_\mu^l\right)c_{t,l}\right)\right)\right)\right).$$

After summation by μ, as it is supposed in (1A), we will get

$$\sum_{\mu=1}^{n}\left(y_{o,\mu} - \sum_{k=0}^{m} {}^{0,kq}\left(c_k x_\mu^k\right)\right)^2 = S_{y^2} - \left(\sum_{l=0}^{m} {}^{0,lq}(S_{yx^l}c_{t,l})\right) - \left(\sum_{k=0}^{m} {}^{0,kq}(c_k S_{x^k y})\right) + \left(\sum_{k=0}^{m}\sum_{l=0}^{m} {}^{0,kq}\left(c_k {}^{0,lq}(S_{x^k x^l}c_{t,l})\right)\right),$$

where

$$S_{y^2}=\sum_{\mu=1}^{n} y_{o,\mu}^2,\quad S_{yx^l}=\sum_{\mu=1}^{n} y_{o,\mu}x_\mu^l,\quad S_{x^kx^l}=\sum_{\mu=1}^{n} x_\mu^k x_\mu^l=\sum_{\mu=1}^{n} x_\mu^{k+l},$$

and after multiplication by d_Y^{-1} we will have

$$^{0,2p}\left(d_Y^{-1}\sum_{\mu=1}^{n}\left(y_{o,\mu}-\sum_{k=0}^{m}{}^{0,kq}\left(c_k x_\mu^k\right)\right)^2\right)=$$

$$={}^{0,2p}\left(d_Y^{-1}S_{y^2}\right)-2\,{}^{0,2p}\left(d_Y^{-1}\left(\sum_{l=0}^{m}{}^{0,lq}(S_{yx^l}c_{t,l})\right)\right)+{}^{0,2p}\left(d_Y^{-1}\left(\sum_{k=0}^{m}\sum_{l=0}^{m}{}^{0,kq}\left(c_k\,{}^{0,lq}(S_{x^kx^l}c_{t,l})\right)\right)\right). \quad (3A)$$

Now we transform the second summand in (3A):

$$^{0,2p}\left(d_Y^{-1}\left(\sum_{l=0}^{m}{}^{0,lq}(S_{yx^l}c_{t,l})\right)\right)=\sum_{l=0}^{m}{}^{0,2p}\left(d_Y^{-1}\,{}^{0,lq}(S_{yx^l}c_{t,l})\right)=\sum_{l=0}^{m}{}^{0,p+lq}\left({}^{0,p}\left(d_Y^{-1}S_{yx^l}\right)c_{t,l}\right). \quad (4A)$$

Let us proceed in (4A) to the element form of the representation:

$$^{0,2p}\left(d_Y^{-1}\,{}^{0,lq}(S_{yx^l}c_{t,l})\right)=\sum_{\bar\lambda,\bar\mu} d^{\bar\lambda,\bar\mu}\sum_{\bar i_1,\dots,\bar i_l} s_{\bar\lambda,\bar i_1,\dots,\bar i_l}c_{t,\bar i_1,\dots,\bar i_l,\bar\mu}=\sum_{\bar\mu}\sum_{\bar\lambda,\bar i_1,\dots,\bar i_l} d^{\bar\mu,\bar\lambda}s_{\bar\lambda,\bar i_1,\dots,\bar i_l}c_{t,\bar i_1,\dots,\bar i_l,\bar\mu}=$$

$$=\sum_{\bar\mu}\sum_{\bar i_1,\dots,\bar i_l}\sum_{\bar\lambda} d^{\bar\mu,\bar\lambda}s_{\bar\lambda,\bar i_1,\dots,\bar i_l}c_{t,\bar i_1,\dots,\bar i_l,\bar\mu}=\sum_{\bar i_1,\dots,\bar i_l,\bar\mu}\sum_{\bar\lambda} d^{\bar\mu,\bar\lambda}s_{\bar\lambda,\bar i_1,\dots,\bar i_l}c_{t,\bar i_1,\dots,\bar i_l,\bar\mu}=\sum_{\bar i_1,\dots,\bar i_l,\bar\mu} z_{\bar\mu,\bar i_1,\dots,\bar i_l}c_{t,\bar i_1,\dots,\bar i_l,\bar\mu}, \quad (5A)$$

where

$$(z_{\bar\mu,\bar i_1,\dots,\bar i_l})=\left(d^{\bar\mu,\bar\lambda}s_{\bar\lambda,\bar i_1,\dots,\bar i_l}\right)={}^{0,p}\left(d_Y^{-1}S_{yx^l}\right)=Z.$$

Then instead of (5A) we will get the expression

$$^{0,2p}\left(d_Y^{-1}\,{}^{0,lq}(S_{yx^l}c_{t,l})\right)=\left(\sum_{\bar i_1,\dots,\bar i_l,\bar\mu} z^{T_3}_{\bar i_1,\dots,\bar i_l,\bar\mu}c^{T}_{\bar i_1,\dots,\bar i_l,\bar\mu}\right)={}^{0,lq+p}\left({}^{0,p}\left(d_Y^{-1}S_{yx^l}\right)^{T_{3,l}}c_{t,l}\right),$$

from which the expression for the matrix V_l of interest follows:

$$V_l={}^{0,p}\left(d_Y^{-1}S_{yx^l}\right)^{T_{3,l}}.$$

It remains to find the transpose substitution $T_{3,l}$. Because $z^{T_3}_{\bar i_1,\dots,\bar i_l,\bar\mu}=z_{\bar\mu,\bar i_1,\dots,\bar i_l}$, then

$$T_{3,l}=\begin{pmatrix}\bar i_1,\dots,\bar i_l,\bar\mu\\ \bar\mu,\bar i_1,\dots,\bar i_l\end{pmatrix}=H_{lq+p,p}=T_l,$$

where the multi-index $\bar\mu$ contains p indices, and multi-indices $\bar i_1,\bar i_2,\dots,\bar i_l$ contain q indices each. It is seen, that this substitution coincides with the substitution in the definition of regression function (13), what is reflected in its notation T_l.

Let us transform now the third summand in (3A):

$$^{0,2p}\left(d_Y^{-1}\sum_{k=0}^{m}\sum_{l=0}^{m}{}^{0,kq}\left(c_k\,{}^{0,lq}(S_{x^kx^l}c_{t,l})\right)\right)=\sum_{k=0}^{m}\sum_{l=0}^{m}{}^{0,2p}\left(d_Y^{-1}\,{}^{0,kq}\left(c_k\,{}^{0,lq}(S_{x^kx^l}c_{t,l})\right)\right), \quad (6A)$$

particularly, proceed in (6A) from the matrix form to the element form of representation. We will get

$$^{0,2p}\left(d_Y^{-1}\,{}^{0,kq}\left(c_k\,{}^{0,lq}(S_{x^kx^l}c_{t,l})\right)\right)=\left(\sum_{\bar\lambda}\sum_{\bar\mu}d^{\bar\lambda,\bar\mu}\sum_{\bar i_1,\dots,\bar i_k}c_{\bar\lambda,\bar i_1,\dots,\bar i_k}\sum_{\bar j_1,\dots,\bar j_l}s_{\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l}c_{t,\bar j_1,\dots,\bar j_l,\bar\mu}\right)=\left(\sum_{\bar\lambda}\sum_{\bar\mu}d^{\bar\lambda,\bar\mu}w_{\bar\lambda,\bar\mu}\right)=$$

$$=\left(\sum_{\bar\lambda}\sum_{\bar\mu}\sum_{\bar i_1,\dots,\bar i_k} c_{\bar\lambda,\bar i_1,\dots,\bar i_k}\sum_{\bar j_1,\dots,\bar j_l} d^{\bar\lambda,\bar\mu} s_{\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l} c_{t,\bar j_1,\dots,\bar j_l,\bar\mu}\right)=$$

$$=\left(\sum_{\bar i_1,\dots,\bar i_k,\bar\lambda}\sum_{\bar j_1,\dots,\bar j_l,\bar\mu} d^{\bar\lambda,\bar\mu} s_{\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l} c_{t,\bar j_1,\dots,\bar j_l,\bar\mu} c_{t,\bar i_1,\dots,\bar i_k,\bar\lambda}\right).$$

We denote

$$(q_{\bar\lambda,\bar\mu,\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l})=\left(d^{\bar\lambda,\bar\mu} s_{\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l}\right)={}^{0,0}\left(d_Y^{-1}S_{x^k x^l}\right)=Q_{k,l},\quad k,l=0,1,2,\dots,m.$$

Then

$$\left(\sum_{\bar i_1,\dots,\bar i_k,\bar\lambda}\sum_{\bar j_1,\dots,\bar j_l,\bar\mu} d^{\bar\lambda,\bar\mu} s_{\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l} c_{t,\bar j_1,\dots,\bar j_l,\bar\mu} c_{t,\bar i_1,\dots,\bar i_k,\bar\lambda}\right)=$$

$$=\left(\sum_{\bar i_1,\dots,\bar i_k,\bar\lambda}\sum_{\bar j_1,\dots,\bar j_l,\bar\mu} q_{\bar\lambda,\bar\mu,\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l} c_{t,\bar j_1,\dots,\bar j_l,\bar\mu} c_{t,\bar i_1,\dots,\bar i_k,\bar\lambda}\right)=$$

$$=\left(\sum_{\bar i_1,\dots,\bar i_k,\bar\lambda}\sum_{\bar j_1,\dots,\bar j_l,\bar\mu} q^T_{\bar i_1,\dots,\bar i_k,\bar\lambda,\bar j_1,\dots,\bar j_l,\bar\mu} c_{t,\bar j_1,\dots,\bar j_l,\bar\mu} c_{t,\bar i_1,\dots,\bar i_k,\bar\lambda}\right)={}^{0,(kq+p)}\left({}^{0,(lq+p)}\left(Q_{k,l}^{T_{k,l}}c_{t,l}\right)c_{t,k}\right).$$

As a result, we get for the third summand in (3A) the following expression

$${}^{0,2p}\left(d_Y^{-1}\left(\sum_{k=0}^{m}\sum_{l=0}^{m}{}^{0,kq}\left(c_k\,{}^{0,lq}(S_{x^k x^l}c_{t,l})\right)\right)\right)=\sum_{k=0}^{m}\sum_{l=0}^{m}{}^{0,(kq+p)}\left({}^{0,(lq+p)}\left({}^{0,0}\left(d_Y^{-1}S_{x^k x^l}\right)^{T_{k,l}}c_{t,l}\right)c_{t,k}\right),$$

so that

$$Q_{k,l}^{T_{k,l}}={}^{0,0}\left(d_Y^{-1}S_{x^k x^l}\right)^{T_{k,l}},\quad k,l=0,1,2,\dots,m, \tag{7A}$$

where $Q_{k,l}={}^{0,0}\left(d_Y^{-1}S_{x^k x^l}\right)$. It remains to find the transpose substitution $T_{k,l}$ in (7A). Since $q^{T_{k,l}}_{\bar i_1,\dots,\bar i_k,\bar\lambda,\bar j_1,\dots,\bar j_l,\bar\mu}=q_{\bar\lambda,\bar\mu,\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l}$, then

$$T_{k,l}=\begin{pmatrix}\bar i_1,\dots,\bar i_k,\bar\lambda,\bar j_1,\dots,\bar j_l,\bar\mu\\ \bar\lambda,\bar\mu,\bar i_1,\dots,\bar i_k,\bar j_1,\dots,\bar j_l\end{pmatrix},\quad k,l=0,1,2,\dots,m, \tag{8A}$$

where the multi-indices $\bar j_1,\bar j_2,\dots,\bar j_l,\bar i_1,\bar i_2,\dots,\bar i_k$ contain q indices each, and multi-indices $\bar\lambda,\bar\mu$ contain p indices each. Thus,

$$S_{k,l}={}^{0,0}\left(d_Y^{-1}S_{x^k x^l}\right)^{T_{k,l}},\quad k,l=0,1,2,\dots,m,$$

where $T_{k,l}$ is defined by expression (8A).

The required expressions for the matrices V_k and $S_{k,l}$ are obtained.

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