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КРАТКОЕ ИЗЛОЖЕНИЕ ДИСЦИПЛИНЫ «ТЕОРИЯ ЭЛЕКТРИЧЕСКИХ ЦЕПЕЙ». ПРИМЕРЫ РЕШЕНИЯ ЗАДАЧ

В двух частях
Часть 2

АНАЛИЗ ЦЕПЕЙ ПЕРЕМЕННОГО ТОКА

BRIEF ELECTRICAL CIRCUIT THEORY AND PRACTICAL PROBLEMS

In two parts
Part 2

AC ANALYSIS

Рекомендовано УМО по образованию в области информатики и радиоэлектроники в качестве пособия для специальностей 1 -45 01 01 «Инфокоммуникационные технологии» (по направлениям), 1-98 01 02 «Защита информации в телекоммуникациях»

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Содержит основные теоретические положения по расчёту линейных электрических цепей переменного тока, примеры расчёта цепей данного типа, а также задачи для самостоятельного решения.

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Content

PREFACE

The technical manual was composed in accordance with Electric Circuit Theory syllabus. The part 2 includes the AC circuit analyses with using complex numbers and phasor diagrams. Series and parallel resonance are considered.

For every topic the solutions of the tasks were included that should help clarify the approach and gain a better understanding. The part 2 includes further problems with answers at the end for independent solution.

The technical manual is designed for use by international students getting an education in English.

1. INTRODUCTION TO AC CIRCUIT ANALYSIS

An alternating voltage (AC voltage) is one that periodically changes polarity, and an alternating current (AC current) is an electric current that periodically changes direction.

The variation of an AC voltage or current versus time is called its waveform.

1.1. Sinusoidal AC Voltage and Current

Expressions for the AC sinusoidal voltage and current are:

$$v(t) = V_m \sin(\omega t + \psi_u);$$
$$i(t) = I_m \sin(\omega t + \psi_i),$$

where $v(t)$ and $i(t)$ – **instantaneous voltage and instantaneous current values** that are the values of voltage and current at any instant of time.

V_m, I_m – the largest value reached in a half cycle is called the **maximum** value or the **amplitude** of the waveform.

ω – angular velocity measured in radians per second, related to the frequency f by $\omega = 2\pi f$.

A sine wave or sinusoidal AC waveform is shown in Fig. 1.1.

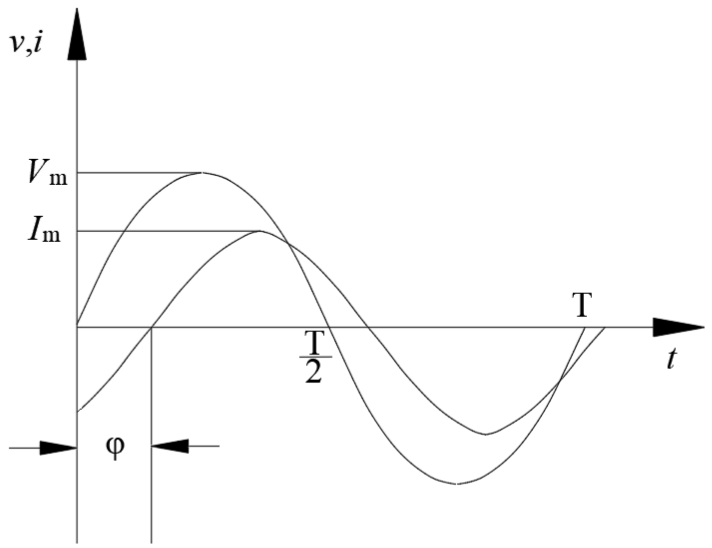


Fig. 1.1. Instantaneous values: $v(t)$ and $i(t)$

Starting at zero, the voltage increases to a positive maximum, decreases to zero, changes polarity, increases to a negative maximum, then returns again to zero.

One complete variation is called a **cycle**. **Period (*T*)** or the periodic time is the time of one cycle.

The number of complete cycles completed in one second is called the **frequency (*f*)** and is measured in cycles per second, which is called the Hertz (Hz):

$$f = \frac{1}{T} \text{ Hz.}$$

The time axis may be converted into an angle axis simply by multiplying by ω.

SUMMARY:

– the sine wave is a time-varying periodic waveform;

– alternating current changes direction in response to changes in the polarity of the source voltage;

– one cycle of an alternating sine wave consists of a positive alternation and a negative alternation;

– full cycle of a sine wave is 360°, or 2π radians. A half-cycle is 180°, or π radian. A quarter-cycle is 90°, or π/2 radians.

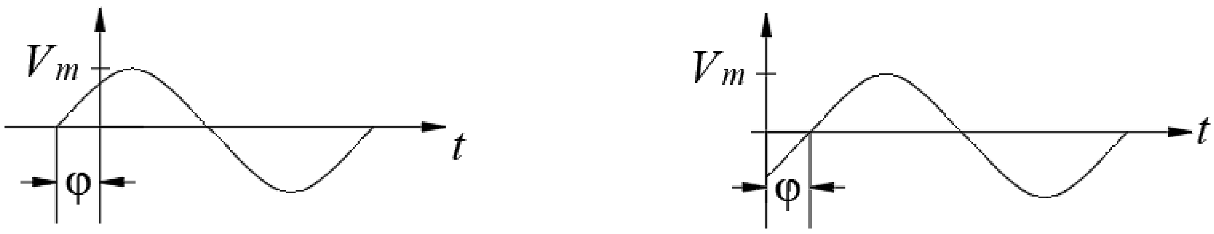
SELF-ASSESSMENT TEST:

1. Describe one cycle of a sine wave.
2. How many maximum points does a sine wave have during one cycle?
4. How is the period of a sine wave measured?
5. Define frequency, and state its unit.
6. Determine f when T = 5 μs.
7. Determine T when f = 110 Hz.

1.2 Phase Shifts: ψ_v and ψ_i

The phase of a sine wave is an angular measurement that shows the position of that sine wave relative to a reference. Fig. 1.1 shows a voltage sine wave that may be used as the reference.

If a sine wave does not pass through zero at t = 0 sec as in Fig. 1.2, it has a phase shift. Waveforms may be shifted to the left (see Fig. 1.2, a) or to the right (see Fig. 1.2, b).



a b

Fig. 1.2. Waveforms have a phase shift:
a – positive phase shift; b – negative phase shift

For a waveform shifted left as in (a): $v(t) = V_m \sin(\omega t + \psi_v)$.

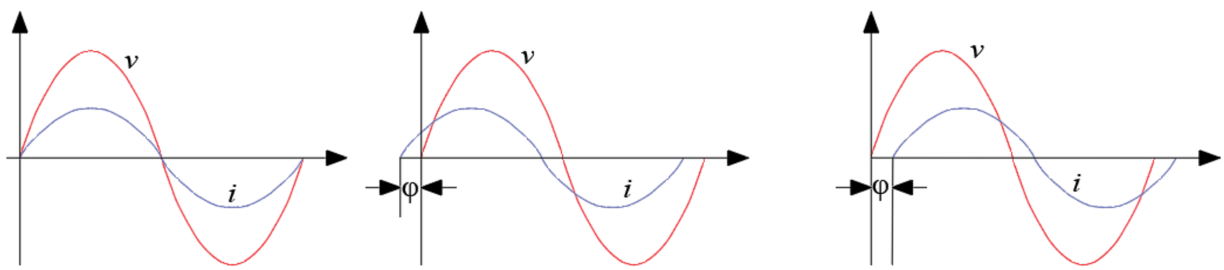
For a waveform shifted right as in (b): $v(t) = V_m \sin(\omega t - \psi_v)$.

1.3. Phase Difference: φ

Phase difference refers to the angular displacement between different waveforms of the same frequency:

$$\varphi = \psi_v - \psi_i.$$

View the following figures:



a b c

Fig. 1.3. The angular displacement between two sine waves:
a – in phase; b – current leads; c – current lags

For Fig. 1.3, a: if the angular displacement is 0° the waveforms are said to be in phase: $\psi_v = \psi_i$ and therefore $\varphi = 0$. For Fig. 1.3, b: the current waveform leads the voltage one: $\psi_v < \psi_i$ so $\varphi = \psi_v - \psi_i < 0$. For Fig. 1.3, c: the voltage waveform leads the current one: $\psi_v > \psi_i$ so $\varphi = \psi_v - \psi_i > 0$.

In other words, in cases «b» and «c» current and voltage are **out of phase**.

Example 1.1

Four sinusoidal alternating quantities are represented by:

a) $i(t) = 5\sin\omega t$ A; b) $i(t) = 15\sin(\omega t - 30°)$ A;

c) $v(t) = 10\sin(\omega t + 60°)$ V; d) $v(t) = 5\sin 2\omega t$ V,

where $\omega = 314$ rad/sec.

1. Comment on the relative magnitudes and frequencies of these quantities.
2. Determine the frequency of quantity «d» and its instantaneous value at $t = 0$: d(0).
3. State the period of quantity «b» and its instantaneous value at $t = 0$: $i(0)$.
4. State the phase relationship of:

– «a» with respect to «b»;

– «a» with respect to «c»;

– «b» with respect to «c».

***Solution*:**

1. The coefficient of the sine function represents the magnitude of the quantity: a) $I_m = 5$ A; b) $I_m = 15$ A; c) $V_m = 10$ V; d) $V_m = 5$ V.

The frequency of quantities «a», «b» and «c» is the same $f = \dfrac{\omega}{2\pi}$, whereas that of quantity «d» is double at $f = \dfrac{2\omega}{2\pi}$.

2. The frequency of «d» is (2·314)/(2·3,14) = 100 Hz.
3. The period T of quantity «b» is the reciprocal of the frequency:

$$T = \frac{2\pi}{\omega} = \frac{2 \cdot 3{,}14}{314} = \frac{2}{100} = 0{,}02 \text{ sec},$$

instantaneous current value at $t = 0$: $i(0) = 15\sin(\omega \cdot 0 - 30°) = 15\sin(-30°) = -7{,}5$ A.

4. a) «a» leads «b» by 30°; b) «a» lags «c» by 60°;

c) $\varphi_{bc} = \psi_b - \psi_c = -30° - 60° = -90°$.

Example 1.2

Find: the phase relationship between $i(t) = -4\sin(\omega t + 50°)$, A and $v(t) = 120\sin(\omega t - 60°)$,V.

***Solution*:**

Current $i(t) = -4\sin(\omega t + 50°)$ has phase shift $\psi_i = 50°$, which can be represented as $\psi_i = 50° - 180° = -130°$. We can rewrite the current in such way: $i(t) = 4\sin(\omega t - 130°)$.

The voltage has phase shift $\psi_v = -60°$.

The phase difference $\varphi = \psi_v - \psi_i = -60° - (-130°) = 70°$ and voltage leads.

1.4. RMS Value (the Root Mean Square Value)

The term **RMS** stands for root mean square. The RMS value represents the ability of the waveform to do useful work. For this reason RMS value **is often called effective value**. The concept of effective value is an important one; in practice, most ac voltages and currents are expressed as effective values.

The RMS value of sinusoidal current is actually a measure of the heating effect of the sine wave. The RMS value of sinusoidal current is equal to the dc current that produces the same amount of heat in a resistance as does the sinusoidal current.

The effective value of AC current can be found by using the following equation

$$I = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}\,. \tag{1.1}$$

Note: The RMS value of sinusoidal current, voltage or EMF will be represented by *I*, *V* and *E*. These symbols are the same as those used for DC current, voltage or EMF.

Find the RMS value of sinusoidal current using formula (1.1):

$$I = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} I_m^2 \sin^2(\omega t)dt} = I_m\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\frac{1-\cos 2\omega t}{2}dt} = I_m\sqrt{\frac{1}{4\pi}\cdot t\Big|_0^{2\pi} - 0} = \frac{I_m}{\sqrt{2}} = 0,707 I_m,$$

or

$$I = \frac{I_m}{\sqrt{2}} = 0,707 I_m. \tag{1.2}$$

Effective values for voltage are found in the same way:

$$V = \frac{V_m}{\sqrt{2}} = 0,707 V_m;$$

$$E = \frac{E_m}{\sqrt{2}} = 0,707 E_m.$$

Effective values for sinusoidal waveforms **depend only on magnitude**.

To obtain peak values from effective values, rewrite equations (1.2).
Thus

$$I_m = I\sqrt{2} = 1,414I;$$
$$V_m = V\sqrt{2} = 1,414V; \qquad (1.3)$$
$$E_m = I\sqrt{2} = 1,414E.$$

Example 1.3
Determine the effective values of:
a) $i(t)$ = 10sin ωt A;
b) $v(t)$ = 50sin(ωt – 20°) V.
***Solution*:**
a) I = 0,707·10 = 7,07 A;
b) V = 0,707·50 = 35,35 V.

SUMMARY:
– amplitude is the maximum value of a voltage or current;
– phase angle is the difference in degrees or radians between a given sine wave and a reference sine wave;
– the RMS value is the value of a sinusoidal voltage (or current) that indicates its heating effect, also known as **the effective** value. It is equal to 0,707 times the peak value.

SELF-ASSESSMENT TEST:
1. A sinusoidal current has a period of 25 ms and an amplitude of 0,8 A. Write its equation in the form of $i = I_m \sin \omega t$, with numerical values for I_m and ω.
2. If phase difference is a positive value, does a current lead or lag?
3. If a voltage and a current are in phase what is the phase difference between them?
4. Determine equations for sine waves with the following:
a) V_m = 170 V, f = 60 Hz; b) I_m = 40 mA, T = 10 ms;
c) T = 120 ms, v = 10 V at t = 12 ms.
5. Determine the effective values of each of the following:
a) v =100 sin(ωt) V; b) i =8 sin(377t) A;
c) v = 40 sin(ωt + 40°) V; d) i =120 cos ωt mA.

2. RESPONSE OF BASIC *R*, *L* AND *C* ELEMENTS TO A SINUSOIDAL VOLTAGE OR CURRENT

R, *L* and *C* circuit elements each have quite different electrical properties. Resistance, for example, opposes current, while inductance opposes changes in current, and capacitance opposes changes in voltage. These differences result in quite different voltage – current relationships.

2.1. Resistance and Sinusoidal AC

In a purely resistive circuit current is directly proportional to voltage.

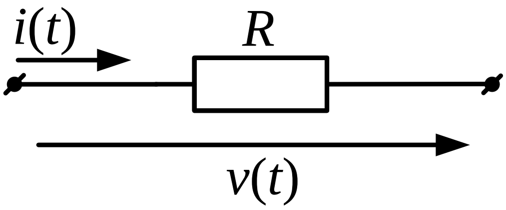


Fig. 2.1. Resistance

The voltage is represented by

$$v(t) = V_m \sin \omega t.$$

Define $i(t)$ using Ohm's law

$$i(t) = \frac{v(t)}{R} = \frac{V_m \sin \omega t}{R} = I_m \sin \omega t,$$

where $I_m = \dfrac{V_m}{R}$ and $V_m = I_m R$.

Similarly for RMS values

$$I = \frac{V}{R} \quad \text{and} \quad V = IR.$$

Peak and RMS values are related by Ohm's Law. Comparison of formulas for $v(t)$ and $i(t)$ shows **that phase difference between voltage and current is equal to zero**:

$$\varphi = \psi_v - \psi_i = 0. \quad (1.4)$$

Conclusion: For purely resistive element, the voltage across and the current through the element **are in phase**. It means that current variations follow voltage variations, reaching their peak when voltage reaches its peak, changing direction when voltage changes polarity, and so on (Fig. 2.2).

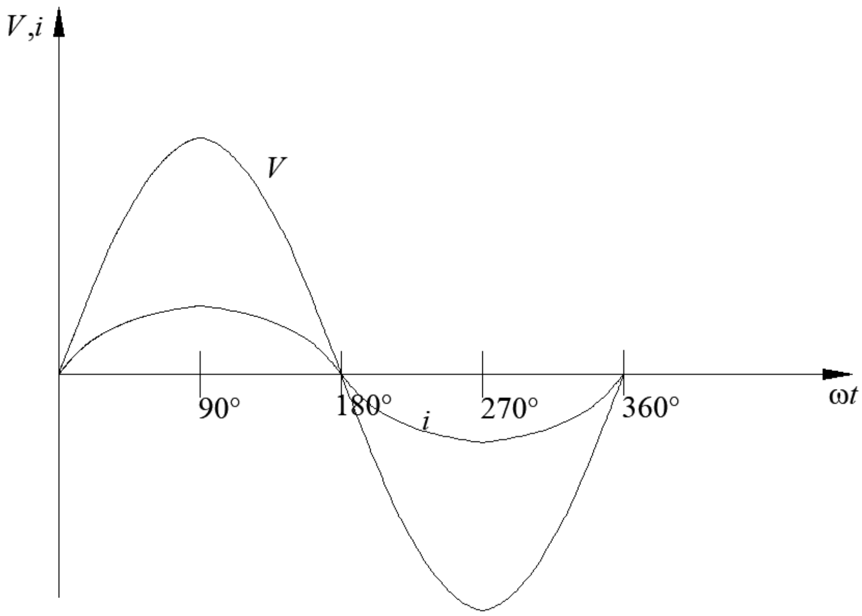


Fig. 2.2. The voltage across and the current through the resistor are in phase

Example 2.1

For the circuit of Fig. 2.1 find: $v_R(t)$ if $R = 5\ \Omega$ and $i(t) = 0{,}5\sin(\omega t - 25°)$ A.

Solution: $v_R(t) = Ri(t) = 5 \cdot 0{,}5\sin(\omega t - 25°) = 2{,}5\sin(\omega t - 25°)$ V.

2.2. Inductance and Sinusoidal AC

Current through inductor is equal to

$$i(t) = I_m \sin \omega t\,. \quad (1.5)$$

The relationship between voltage across inductor (Fig. 2.3) and current through it is

$$v_L = L\frac{di_L}{dt}. \quad (1.6)$$

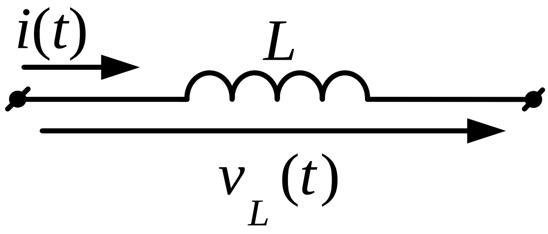


Fig. 2.3. Inductance

Substitution the expression (1.5) into the formula (1.6) gives:

$$v_L = L\frac{di_L}{dt} = L\frac{dI_m \sin\omega t}{dt} = \omega L I_m \cos\omega t = \omega L I_m \sin(\omega t + 90^\circ)$$

or

$$v_L = V_m \sin(\omega t + 90^\circ). \quad (1.7)$$

where $V_m = \omega L I_m$ – peak value of voltage; $V = \omega L I$ – RMS value of voltage.

The quantity ωL, called **the inductive reactance** (from the word **reaction**) of an inductor, is symbolically represented by X_L and is measured in ohms, that is,

$$X_L = \omega L \ \Omega. \quad (1.8)$$

Inductive reactance **is the opposition** to the flow of current. Inductive reactance does not dissipate electrical energy. In this element the continual interchange of energy between the source and magnetic field of an inductor has place.

The reciprocal of inductive reactance is called **inductive conductance**, that is,

$$b_L = \frac{1}{X_L} = \frac{1}{\omega L} \ \text{S}. \quad (1.9)$$

From comparison of formulas (1.5) and (1.7) follows that **phase difference between voltage across inductor and current** through it is

$$\varphi = \psi_v - \psi_i = 90°.$$

For a purely inductive circuit, **current lags voltage by 90°**. Alternatively, **Voltage leads current by 90°** (Fig. 2.4).

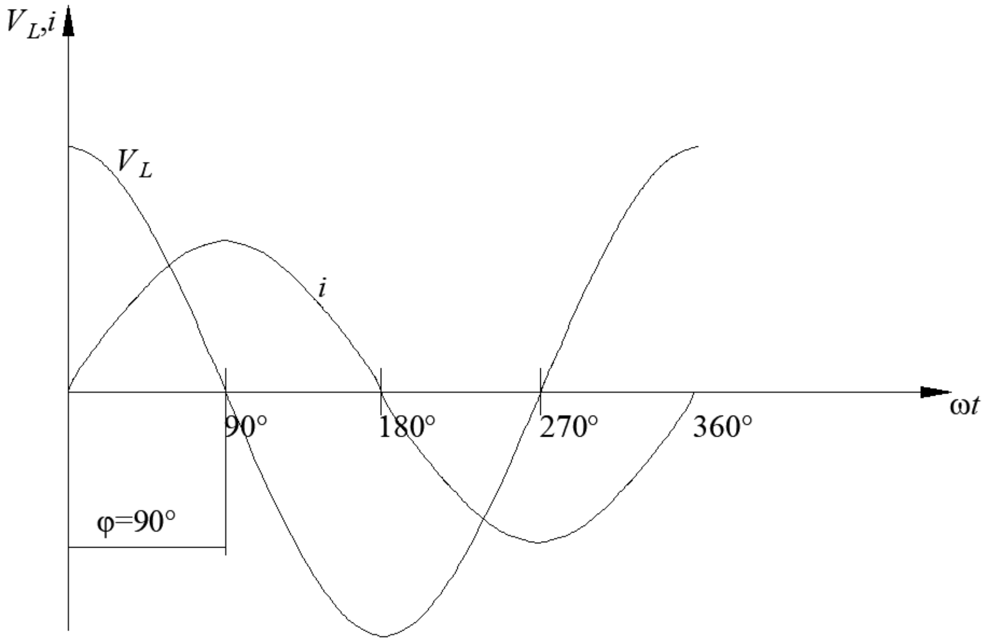


Fig. 2.4. Voltage leads current by 90°

If a phase angle is included in the sinusoidal expression for i_L, such as

$$i(t) = I_m \sin(\omega t \pm \psi_i),$$

then

$$v_L = V_m \sin(\omega t \pm \psi_i + 90^\circ),$$

where $\psi_v = \psi_i + 90°$.

Example 2.2

Given: The voltage across a 0,2 H inductance is $v_L = 100\sin(400t + 70°)$ V. Determine i_L.

***Solution*:**

ω = 400 rad/s. Therefore, $X_L = \omega L = 400 \cdot 0{,}2 = 80\ \Omega$.

$$I_m = \frac{V_m}{R} = \frac{100}{80} = 1{,}25 \quad \text{A} .$$

The current lags the voltage by 90°. Therefore $i(t) = 1{,}25\sin(\omega t - 20°)$ A.

Variation of Inductive Reactance with Frequency. Since $X_L = \omega L = 2\pi f L$, inductive reactance **is directly proportional to frequency** (Fig. 2.5). Thus, if frequency is doubled, reactance doubles, while if frequency is halved, reactance halves, and so on. In addition, X_L **is directly proportional to inductance**. Thus, if inductance is doubled, X_L is doubled, and so on.

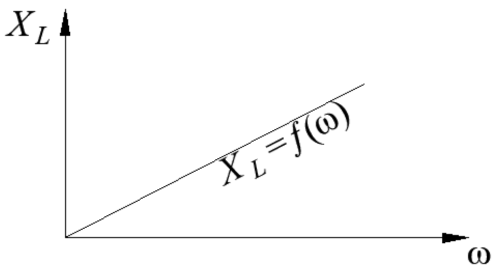


Fig. 2.5. Inductive reactance versus angular velocity

Note, that at $f = 0$, $X_L = 2\pi\cdot 0 = 0\ \Omega$. This means that **inductance looks like a short circuit to DC** (Fig. 2.6).

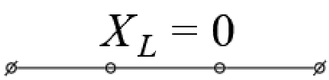


Fig. 2.6. Short circuit

2.3. Capacitance and Sinusoidal AC

Voltage across capacitor is equal to

$$v(t) = V_m \sin \omega t . \tag{1.10}$$

The fundamental equation relating the voltage across a capacitor (Fig. 2.7) to the current is:

$$i_C = C\frac{dv_C}{dt}. \tag{1.11}$$

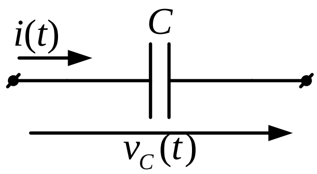


Fig. 2.7. Capacitance

Substitute expression (1.10) into formula (1.11) and obtain:

$$i_C = C\frac{dv_C}{dt} = C\frac{dV_m \sin\omega t}{dt} = \omega C V_m \cos\omega t = \omega C V_m \sin(\omega t + 90°)$$

or

$$i_L = I_m \sin(\omega t + 90°), \tag{1.12}$$

where $I_m = \omega C V_m$ – peak current value through capacitor; $I = \omega CV$ – RMS current value.

The quantity ωC, called **the capacitive conductance**, is symbolically represented by b_C and it is measured in Siemens, that is,

$$b_C = \omega C \text{ S}. \tag{1.13}$$

The reciprocal of the capacitive conductances is called **capacitive reactance:**

$$X_C = \frac{1}{\omega C} \; \Omega. \tag{1.14}$$

Capacitive reactance **is the opposition** to the flow of charge. Capacitive reactance does not dissipate energy in any form. In this element the continual interchange of energy between the source and the electric field of capacitor has place.

From comparison formulas (1.10) and (1.12) follows that **phase difference between voltage across capacitor and current** through it is

$$\varphi = \psi_v - \psi_i = -90°.$$

In a purely capacitive circuit, **current leads voltage by 90°** (Fig. 2.8). Alternatively, **Voltage lags current by 90°**. Current leads voltage by 90°.

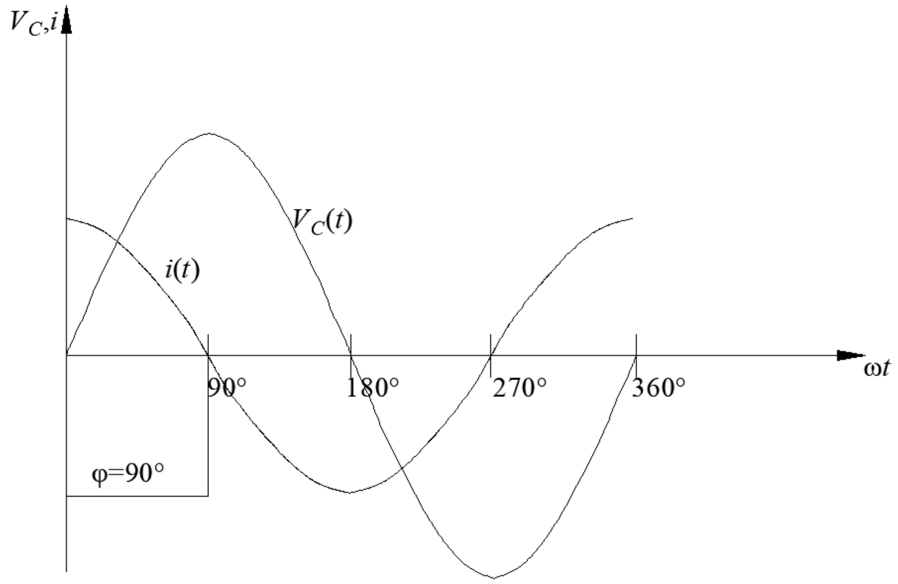


Fig. 2.8. Current leads voltage by 90°

If a phase angle is included in the sinusoidal expression for v_C, such as

$$v_C(t) = V_m \sin(\omega t \pm \psi_v),$$

then

$$i_C = I_m \sin(\omega t \pm \psi_v + 90^\circ),$$

where $\psi_i = \psi_v + 90°$.

Example 2.3

The voltage across a 10 mF capacitance is $v_C = 100\sin(\omega t - 40°)$ V and $f = 1000$ Hz. Determine i_C.

***Solution*:**

$$\omega = 2\pi f = 2\pi \cdot 1000 = 6283 \text{ rad/s};$$

$$X_C = \frac{1}{\omega C} = \frac{1}{6283 \cdot 10 \cdot 10^{-6}} = 15{,}92 \ \ \Omega;$$

$$I_m = \frac{V_m}{X_C} = \frac{100}{15{,}92} = 6{,}28 \ \ \text{A}.$$

Since current leads voltage by 90°, $i_C(t) = 6.28\sin(6283t + 50°)$ A.

Variation of Capacitive Reactance with Frequency. Since $X_C = 1/\omega C = 1/2\pi fC$, the capacitive reactance **varies inversely with frequency**. This means that the higher the frequency, the lower the reactance, and vice versa (Fig. 2.9).

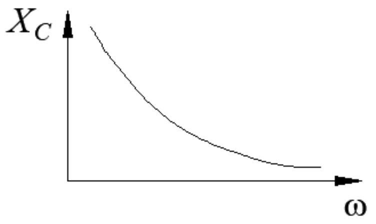


Fig. 2.9. Capacitive reactance versus angular velocity

Note: That at $f = 0$ (i. e., DC), X_C is *infinite*.

This means that *capacitance* **looks like an open circuit to DC** (Fig. 2.10).

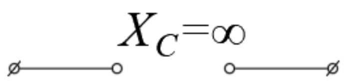


Fig. 2.10. Open circuit

Note: That X_C is also **inversely proportional to capacitance**. Thus, if capacitance is doubled, X_C is halved, and so on.

Fig. 2.11 illustrates effect of high and low frequencies on the circuit model of an inductor and a capacitor.

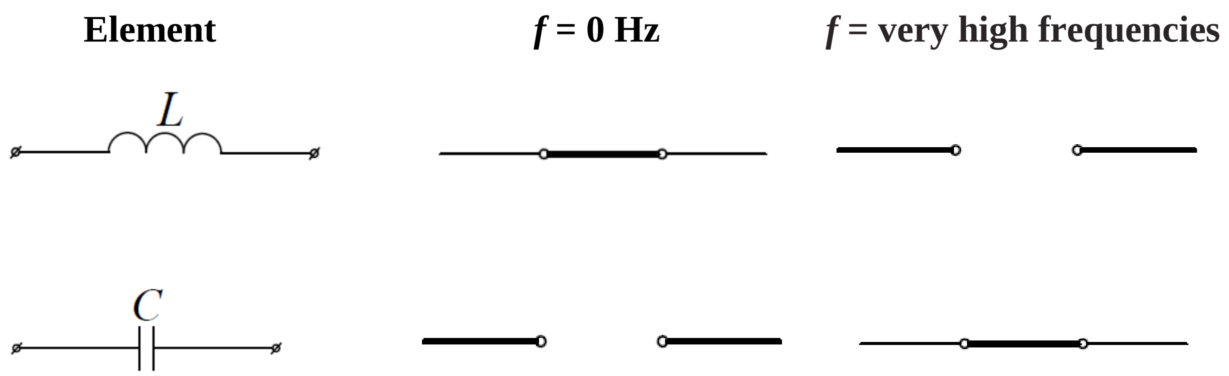


Fig. 2.11. Effect of high and low frequencies on the circuit model of an inductor and a capacitor

SUMMARY:

– Ohm’s Law is true relationship between instantaneous voltage across resistor and instantaneous current through it;

– voltage across resistor and current in it are in phase;

– voltage leads current by 90° in an inductor;

– inductive reactance, X_L, is directly proportional to frequency and inductance;

– the true power in an inductor is zero; that is, no energy is lost in an ideal inductor due to conversion to heat, only in its winding resistance;

– the amount of induced voltage is directly proportional to the inductance and to the rate of change in current;

– energy is stored by an inductor in its magnetic field;

– current leads voltage by 90° in a capacitor;

– capacitive reactance, X_C, is inversely proportional to frequency and capacitance;

– the true power in a capacitor is zero; that is, no energy is lost in an ideal capacitor due to conversion to heat.

SELF-ASSESSMENT TEST:

1. The voltage across a resistor is indicated. Find the sinusoidal expression for the current i(t) if the resistor is 10 Ω:

a) $v_R(t)$ = 100sin 377t V;

b) $v_R(t)$ = 25sin (377t +60°) V.

2. The current through a 0.1 H coil is provided. Find the sinusoidal expression for the voltage across the coil:

a) $i(t)$ = 10sin 377t A.

b) $i(t)$= 7sin (377t – 70°) A.

3. The current through a 100 μF capacitor is given. Find the sinusoidal expression for the voltage across the capacitor:

a) $i(t)$ = 40sin 500t A.

b) $i(t)$ = 12sin (377t +70°) A.

4. Give the relationship between voltage and current:

a) In a purely resistive circuit;

b) In a purely inductive circuit;

c) In a purely capacitive circuit.

5. State the phase relationship between current and voltage in a resistor.

6. State the phase relationship between current and voltage in an inductor.

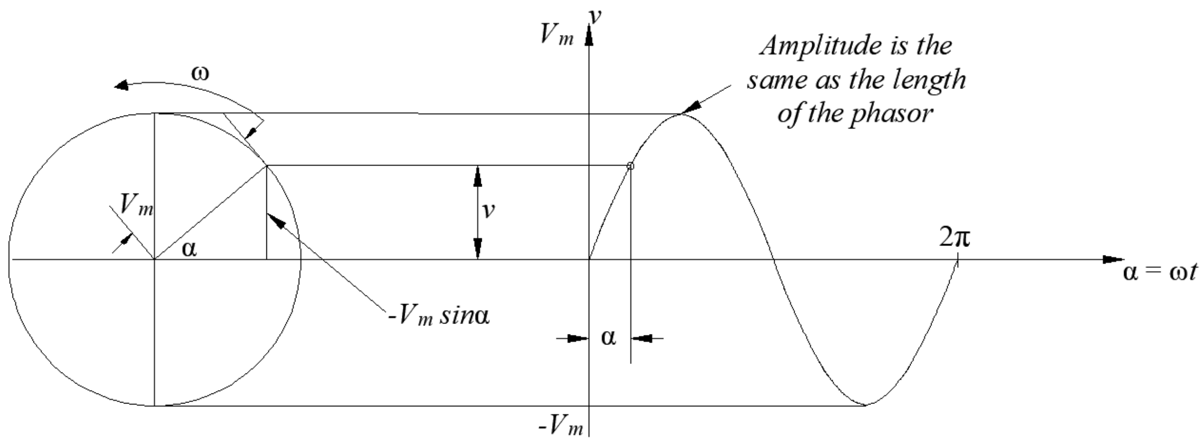
7. State the phase relationship between current and voltage in a capacitor.

8. Define inductive and capacitive reactance versus frequency.

3. COMPLEX NUMBER IN AC ANALYSIS

3.1. Introduction to Phasors

A **phasor** is a rotating vector whose projection on a vertical axis can be used to represent sinusoidally varying quantities.



a b

Fig. 3.1. Rotating anticlockwise vector and sine wave:
a – phasor; b – resulting sine wave

Fig. 3.1, a illustrates rotating anticlockwise vector or phasor. The vertical projection of the phasor (indicated in dotted red) is V_m sinα. Now, assume that the phasor rotates at angular velocity of ω rad/s. Then, $\alpha = \omega t$, and its vertical projection is $V_m \sin \omega t$ (Fig. 3.1, b). If we designate this projection as v, we get $v = V_m \sin \omega t$, which is the familiar sinusoidal voltage equation.

Example 3.1

Given: the sinusoidal voltage is $v(t) = 35\sin\omega t$ V. Draw the phasor that represents this waveform.

***Solution*:** the phasor of Fig. 3.2 has length of 35 V in any scale. As voltage $v(t)$ has *zero phase* shift, the phasor is drawn at its $t = 0$ position with *zero degrees*.

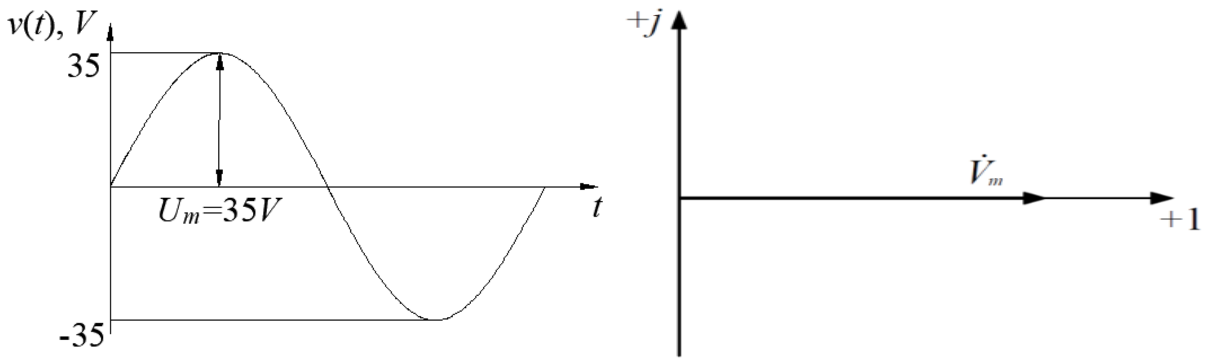


Fig. 3.2. Sinusoidal current and it's phasor representation

Example 3.2

Given: the sinusoidal current is $i(t)$ = 1,5sin (ωt + 45°) A. Draw the phasor that represents this sinusoidal current.

Solution**:** the phasor of Fig. 3.3 has length of 1,5 A in any scale. So its *phase shift* equal to 45°, the phasor is drawn at its t = 0 position with *angle 45°*.

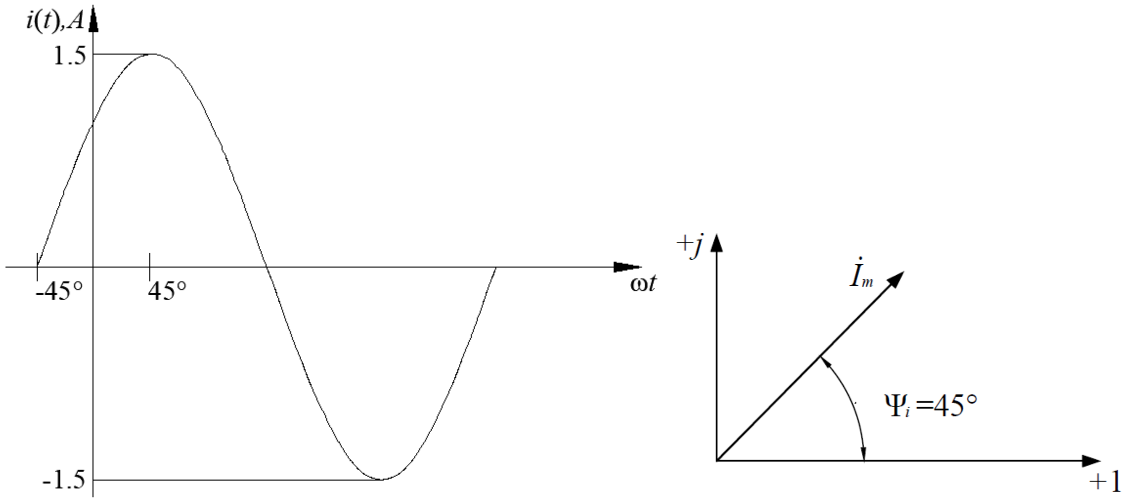


Fig. 3.3. Sinusoidal current and it's phasor representation

Conclusion: A sinusoidal waveform can be created by plotting the vertical projection of a phasor that rotates in the counterclockwise direction at constant angular velocity ω.

Note: Phasors apply only *to sinusoidal* waveforms. Phasor algebra for sinusoidal quantities is applicable only for waveforms having the same frequency.

3.2. Representing AC Voltages and Currents by Complex Numbers

Previously it was shown that AC voltages and currents may be represented as phasors. Since phasors have magnitude and angle, they may be represented as complex numbers. Let consider the current through the load of Fig. 3.4, a. Its phasor equivalent (b) has magnitude I_m and angle ψ_i. Therefore it may be viewed as the complex number.

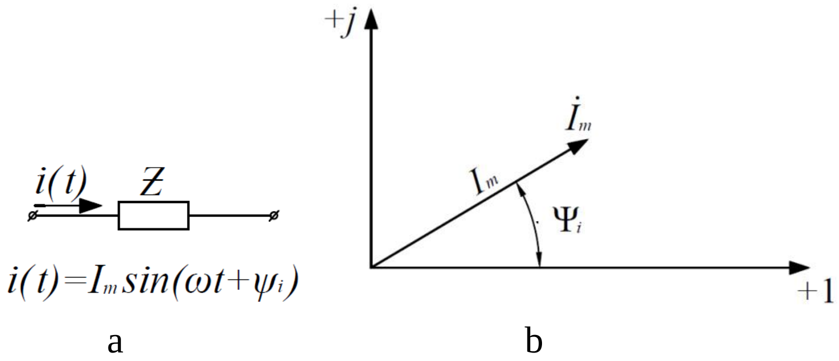


Fig. 3.4. Sinusoidal current and it's phasor representation

Representation of a sinusoidal current as a complex number. From this viewpoint, the sinusoidal current $i(t)$ = 2,5sin(ωt + 30°) of Fig. 3.5, a can be represented by its phasor equivalent, $\dot{I}_m = 2{,}5e^{j30°}$ A, as in (b).

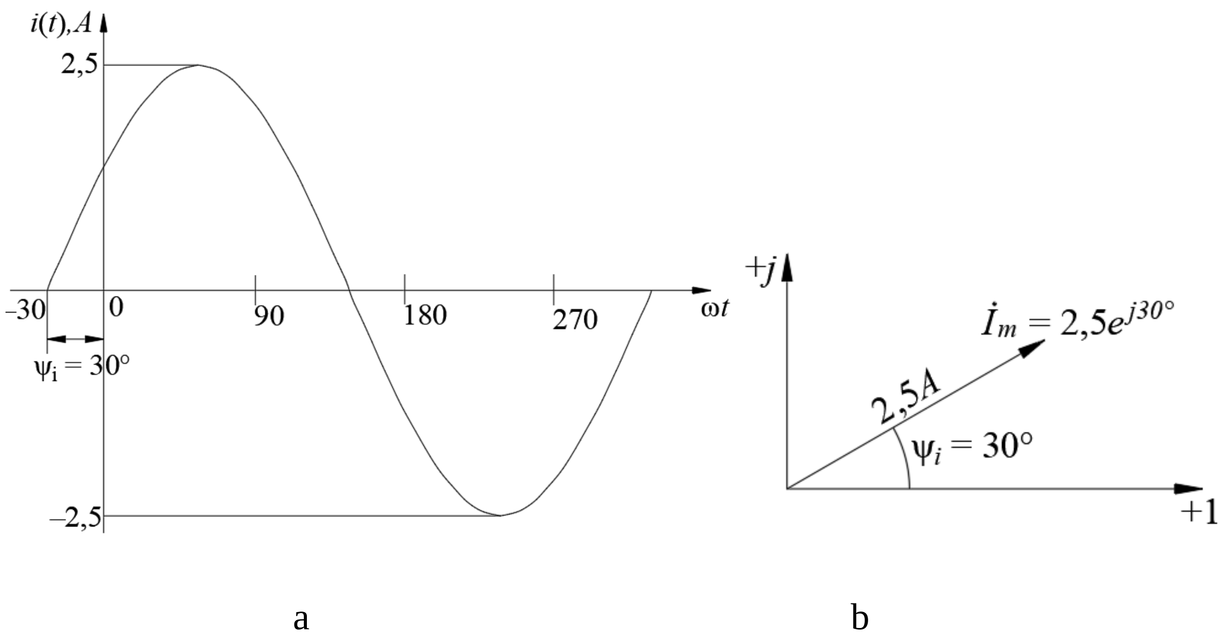


Fig. 3.5. Sinusoidal current and it's phasor representation

We can take the advantage of this equivalence:

– rather than show current (or voltage) as a time – varying quantity *i*(*t*) (or *v*(*t*)) that we later convert to a phasor, we can represent this current (or voltage) by its phasor equivalent right from the start;

– by replacing the time function *i*(*t*) or *v*(*t*) with its phasor equivalent $\dot{I}$ *or* $\dot{V}$, we have transformed the current or the voltage from *the time domain* to *the phasor domain.*

SUMMARY:

– a complex number represents a phasor quantity;

– a phasor is effective technique for representation of sinusoidal quantities;

– a phasor length is equal to amplitude or RMS of a sinusoidal function;

– a phasor angle position is equal to phase shift of a sinusoidal function.

SELF-ASSESSMENT TEST:

1. What are the two characteristics of a quantity indicated by a complex number?

2. Write the phasor interpretation for the following sinusoidal quantities:

a) $i(t)$ = 3,84sinωt A; b) $v(t) = 42\sin(\omega t + 38°)$ V;

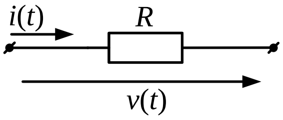
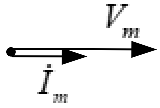
c) $v(t) = 95\sin(\omega t - 120°)$ V; d) $e(t) = 220\sin(\omega t + 115°)$ V.

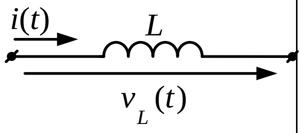
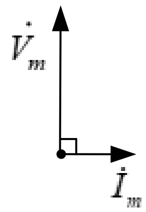
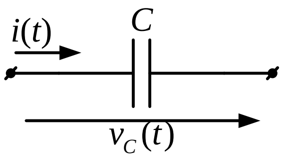
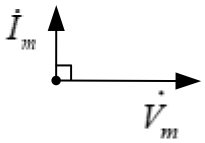
4. OHM'S LAW IN COMPLEX FORM

Simple AC circuits may be analysed by using complex numbers and phasor diagrams.

The table 4.1 illustrates Ohm's Law for time – varying quantities *i*(*t*) or *v*(*t*), their phasor representation and Ohm's Law in complex notation for pure resistive, pure inductive and pure capacitive circuits.

Table 4.1

A passive part of the circuit	A time – varying quantity *i*(*t*) or *v*(*t*)	Phasor diagrams	Ohm's Law in complex form
1	2	3	4
Pure resistive circuit 	$v(t) = V_m\sin\omega t$; $i_R = \frac{v(t)}{R} = I_m \sin\omega t$		$\dot{I} = Ie^{j\psi_i}$; $\dot{V}_R = RIe^{j\psi_i} = R\dot{I}$; $\dot{V}_R = \dot{I}R$

1	2	3	4
Pure inductive circuit 	$i(t) = I_m \sin\omega t;$ $v_L = L\dfrac{di_L}{dt} =$ $= V_m \sin(\omega t + 90^\circ)$		$\dot{I} = Ie^{j\psi_i};$ $\dot{V}_L = \omega LIe^{j(\psi_i + 90°)} =$ $= \omega LIe^{j\psi_i}e^{j90°} =$ $= j\omega L\dot{I};$ $\dot{V}_L = \dot{I}jX_L$
Pure capacitive circuit 	$v(t) = v_m \sin\omega t;$ $i_C = C\dfrac{dv_C}{dt} =$ $= \omega CV_m \sin(\omega t + 90°)$		$\dot{I} = Ie^{j\psi_i};$ $\dot{V}_C = \dfrac{1}{\omega C}Ie^{j(\psi_i - 90°)} =$ $= \dfrac{1}{\omega C}Ie^{j\psi_i}e^{-j90°} =$ $= -j\dfrac{1}{\omega C}\dot{I};$ $\dot{V}_C = \dot{I}(-jX_C)$

Example 4.1

Given: $i(t)$ = 2,3sin (314t – 35°) A; X_L = 55 Ω.

Calculate the complex voltage across resistor using Ohm's Law in complex notation. Write answer in the time domain.

***Solution*:** The current amplitude in complex form is $\dot{I}_m = 2.3e^{-j35°}$. Apply Ohm's Law:

$$\dot{V}_m = \dot{I}_m \cdot jX_L = 2{,}3e^{-j35^\circ} \cdot j55 = 2{,}3e^{-j35^\circ} \cdot 55e^{j90^\circ} = 126{,}5e^{j(-35^\circ + 90^\circ)} = 126{,}5e^{j55^\circ} \text{ V.}$$

The voltage in the time domain is $v(t)$ = 126,5sin(314t + 55°) V.

The Fig. 4.1 below demonstrates the phasor diagram for voltage and current.

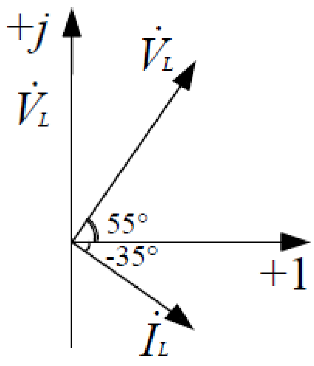


Fig. 4.1. Phase diagram

Example 4.2

Given: $v(t)$ = 380sin (314t – 27°) V; X_C = 38 Ω.

Calculate the complex current through capacitor using Ohm's Law in complex notation. Write answer in the time domain.

Solution: The voltage amplitude in complex form is $\dot{V}_m = 380e^{-j27°}$. Apply Ohm's Law:

$$\dot{I}_m = \frac{\dot{V}_m}{-jX_C} = \frac{380e^{-j27°}}{-j38} = \frac{380e^{-j27°}}{38e^{-j90°}} = 10e^{j(-27°+90°)} = 10e^{j63°} \quad \text{A.}$$

The current in the time domain is $i(t)$ = 10sin(314t + 63°) A.

Fig. 4.2 demonstrates the phasor diagram of voltage and current.

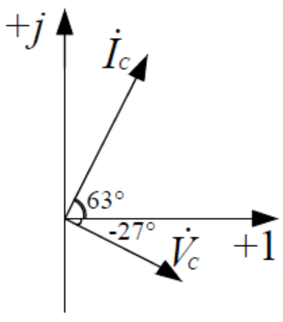


Fig. 4.2. Phasor diagram of voltage and current

SUMMARY:

– an angle between voltage and current on the phasor diagram for pure resistive element is zero (voltage and current phasors have the same directions), that represents zero phase difference in a pure resistive circuit (voltage and current are in phase);

– an angle between voltage and current on the phasor diagram for pure inductive element is equal to 90°, that represents phase difference of 90° for a pure inductive circuit (in terms of electrical circuits: voltage leads current by 90°);

– an angle between voltage and current on the phasor diagram for pure capacitive element is equal to (– 90°), that represents phase difference of – 90° for a pure capacitive circuit (in terms of electrical circuits: voltage lags current by 90°).

SELF-ASSESSMENT TEST:

1. Voltage across the pure resistive element R = 5,7 **Ω** is $v(t)$ = 17,1sin(ωt + + 30°) V. Define current, using Ohm's Law in complex form. What is the current phase shift?

2. Current through the pure inductive element L = 0,1 H is $i(t)$ = 17,1sin(50t – – 15°) A. Define complex voltage across inductor and draw the vector diagram.

3. Current through the pure capacitive element C = 0,1 pF is $i(t)$ = 14,1sin(50t – – 45°) A. Define complex voltage across capacitor and draw the vector diagram.

4. Define complex voltage across inductor and draw the vector diagram if the current $i(t)$ = 17,1sin(100t + 75°) A.

5. IMPEDANCE CONCEPT

Practically each circuit may be represented by its impedance. **Impedance** is the opposition that circuit element presents to current in phasor domain. The symbol for impedance is the letter Z and the unit is the ohm (**Ω**).

In general, *the impedance* of the element *is the ratio of voltage phasor* across it *to its current phasor*:

$$\dot{Z} = \frac{\dot{V}}{\dot{I}} \text{ Ohm.} \tag{5.1}$$

Formula (5.1) presents *Ohm's Law in complex notation for AC circuit.*

Since voltage and current in expression (5.2) are complex, $\dot{Z}$ *is also complex*:

$$\dot{Z} = \frac{\dot{V}}{\dot{I}} = \frac{Ve^{j\psi_v}}{Ie^{j\psi_i}} = ze^{j(\psi_v - \psi_i)} = ze^{j\varphi} \quad \Omega. \tag{5.2}$$

where φ – is *phase difference* between voltage and current.

Impedance for the basic circuit elements R, L, C:

$$\dot{Z}_R = R;$$
$$\dot{Z}_L = j\omega L;$$
$$\dot{Z}_C = -j(1/\omega C).$$

Note: Although $\dot{Z}$ is a complex number, it is not a phasor since it does not represent a sin wave quantity.

SUMMARY:

– impedance is the opposition to current flow in an AC circuit;

– impedance is complex number, but is not the sinusoidal quantity;

– the argument of impedance phasor is phase shift between voltage across any element and current through it.

SELF-ASSESSMENT TEST:

1. Formulate the general formula for Ohm's Law for AC circuit in complex notation.
2. Express impedance in complex form.
3. Call impedance for purely resistive, purely inductive and purely capacitive elements.

6. AC SERIES AND PARALLEL CIRCUITS

6.1. Series Configuration. The KVL in the Complex Form

We can apply KVL for series connection of resistor, inductor and capacitor, since the overall properties of series AC circuits (Fig. 6.1) are the same as those for *dc* circuits:

$$\dot{V}_t = \dot{V}_R + \dot{V}_L + \dot{V}_C\,. \qquad (6.1)$$

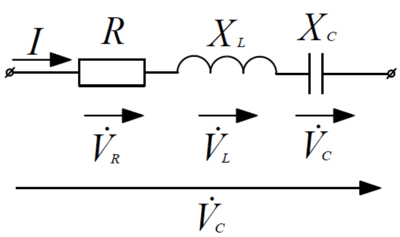


Fig. 6.1. The series *R-L-C* circuit

After substitution Ohm's Law in complex form for resistor, inductor and capacitor into expression 6.1 we obtain the following formula:

$$\dot{V}_t = \dot{I} \cdot R + \dot{I} \cdot jX_L + \dot{I} \cdot (-jX_C) = \dot{I} \cdot [R + jX_L - jX_C], \tag{6.2}$$

where expression in square brackets (see formula (6.2)) $\dot{Z} = R + jX_L - jX_C = R + j(X_L - X_C)$ is called **the total impedance** of series circuit;

the term $X = X_L - X_C$ is called **the total reactance**.

Thus, the total impedance

$$\dot{Z} = R \pm jX. \tag{6.3}$$

Formula (6.3) represents the total impedance in rectangular form, in that the real part R **is the total of all resistance** looking into the input terminal of the circuit; the imaginary part X is the difference between the total inductive and capacitive reactance.

Fig. 6.2 illustrates the phasor diagram for series *R-L-C* circuit.

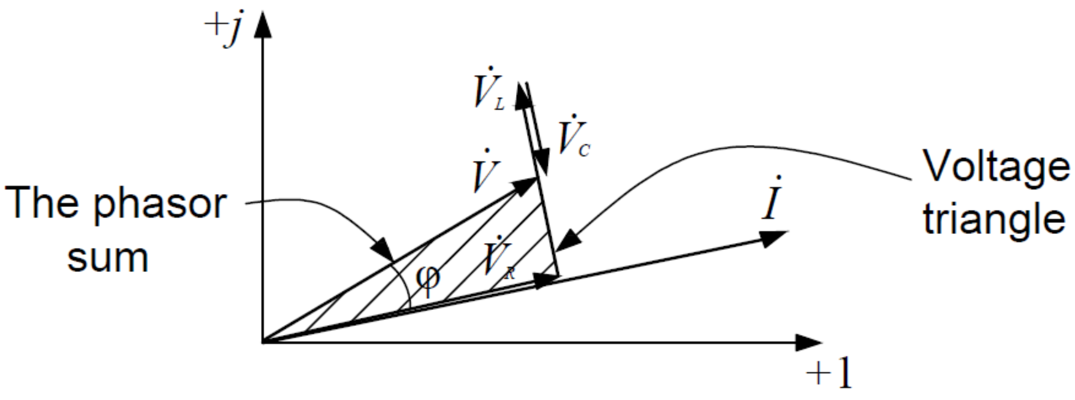


Fig. 6.2. The phasor diagram for series *R-L-C* circuit

The total reactance *will be positive* ($X > 0$) if the inductive reactance is greater than the capacitive reactance $X_L > X_C$;

The total reactance *will be negative*($X < 0$) if the inductive reactance is less than the capacitive reactance $X_L < X_C$.

Note: In an *AC* series circuit the current is common to each element of one and therefore it is taken as a reference phasor.

The phasor diagram indicates that *current* $\dot{I}$ *is in phase* with *the voltage across the resistor*, *lags the voltage across the inductor by* 90° and *leads the voltage across the capacitor by* 90°.

The hatched area on Fig. 6.2 is called **voltage triangle**.

If each side of the voltage triangle will be divided by the current $\dot{I}$ the diagram of Fig. 6.3 is obtained, which is called an **impedance triangle**.

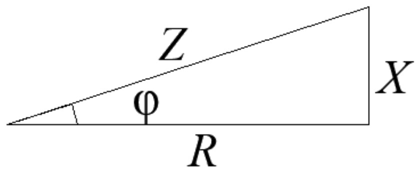


Fig. 6.3. An impedance triangle

The angle φ (Fig. 6.3) **is the phase difference** between the input voltage and current. From geometry of the impedance triangle it is seen the following relationships:

$$Z = \sqrt{R^2 + X^2};$$
$$\left.\begin{aligned} R &= Z\cos\varphi \\ X &= Z\sin\varphi \end{aligned}\right\}; \qquad (6.4)$$
$$\varphi = \mathrm{arctg}\frac{X}{R}.$$

Example 6.1

Given: for series *R-L-C* circuit: $R = 5\ \Omega$, $X_L = 17\ \Omega$, $X_C = 7\ \Omega$.

a) Express the total impedance.

b) Determine its magnitude and phase angle.

***Solution*:**

a) $\dot{Z} = R + jX_L - jX_C = 5 + j17 - j7 = 5 + j10\ \Omega$. This expression is the total impedance in the rectangular form.

b) To determining its magnitude it is possible to convert complex number from rectangular to polar form or use formulas (6.4):

$$Z = \sqrt{R^2 + X^2} = \sqrt{5^2 + 10^2} = \sqrt{25 + 100} = 11{,}18\ \ \Omega.$$

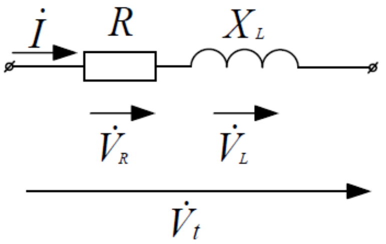
Then find phase angle φ:

$$\varphi = \text{arctg}\frac{X}{R} = \text{arctg}\frac{10}{5} = \text{arctg}2 = 63,43°.$$

Note the total impedance in polar form:

$$\dot{Z} = 11{,}18e^{j63{,}43°}\ \Omega.$$

Example 6.2
Given: R = 2,5 Ω; X_L = 7,5 Ω; $\dot{I} = 3{,}2e^{-j40°}$ A.
Find: $\dot{V}_t$; $\dot{V}_R$; $\dot{V}_L$ for the circuit below.



Draw phasor diagram.
***Solution*:**
1. Determine the total resistance: $\dot{Z} = R + jX_L = 2{,}5 + j7{,}5 = 7{,}9e^{j71{,}6°}$**Ω.**
2. Using Ohm's Law in complex notation define:
a) total voltage across the circuit:

$$\dot{V}_t = \dot{I}\dot{Z} = 3{,}2e^{-j40°} \cdot 7{,}9e^{j71.6°} = 25{,}28\ e^{j31{,}6°}\ \text{V};$$

b) voltage across resistor:

$$\dot{V}_R = \dot{I}R = 3{,}2e^{-j40°} \cdot 2{,}5 = 8\ e^{-j40°}\ \text{V};$$

c) voltage across inductor:

$$\dot{V}_L = \dot{I}jX_L = 3{,}2e^{-j40°} \cdot j7{,}5 = 3{,}2e^{-j40°} \cdot 7{,}5e^{j90°} = 24\ e^{j50°}\ \text{V}.$$

3. The phasor diagram is shown below (Fig. 6.4).

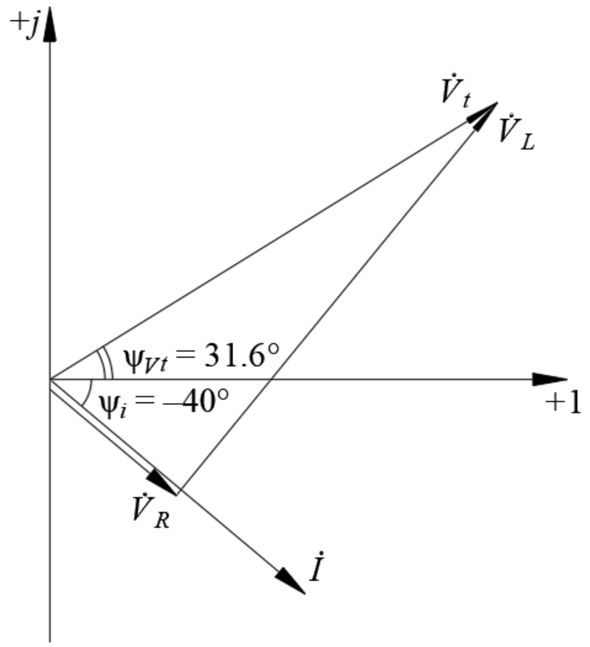


Fig. 6.4. The phasor diagram

SUMMARY:

– for AC series circuit the KVL is correctly written in complex notation;

– the total impedance is the complex sum of total resistance and total reactance looking into input circuit's terminals; in rectangular form the real part of that complex number is total resistance an d the imaginary part is the total reactance;

– the argument of the total impedance is phase angle between total voltage current through the circuit.

SELF-ASSESSMENT TEST:

1. Write down the expression of Ohm's Law for AC series circuit in complex form.

2. Write down the expression of the KVL for AC series circuit in complex form.

3. Draw phasor diagram for series *R-L-C* configuration.

4. Write down the total impedance for series circuit in complex form.

5. Draw an impedance triangle.

6. How to calculate impedance magnitude and the phase angle.

7. Note all known relationships for Z, R, X_L, X_C and φ, that are determined from impedance triangle geometry.

6.2. Parallel Configuration. The KCL in the Complex Form

We can apply KCL for parallel connection of resistor, inductor and capacitor, since the overall properties of parallel AC circuits (Fig. 6.5) are the same as those for DC circuits:

$$\dot{I}_t = \dot{I}_R + \dot{I}_L + \dot{I}_C. \qquad (6.5)$$

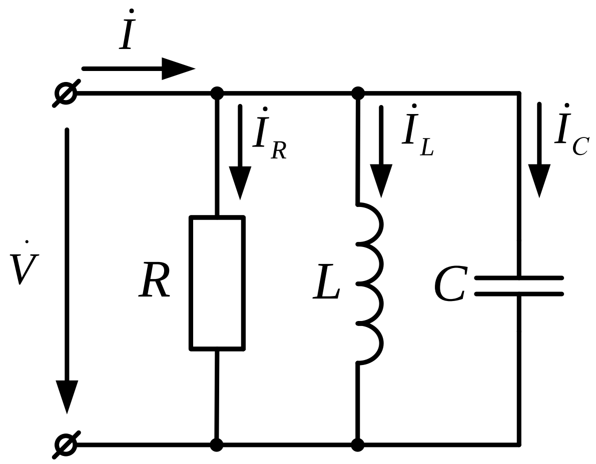


Fig. 6.5. The parallel *R*-*L*-*C* circuit

After substitution Ohm's Law in complex form for resistor, inductor and capacitor into expression (6.5) we obtain the following formula:

$$\dot{I}_t = \frac{\dot{V}}{R} + \frac{\dot{V}}{jX_L} + \frac{\dot{V}}{-jX_C} = \dot{V}\left(\frac{1}{R} + \frac{1}{jX_L} + \frac{1}{-jX_C}\right) = \dot{V}[g - jb_L + jb_C]. \qquad (6.6)$$

where expression in square brackets (see formula (6.6)) $\dot{Y} = g - jb_L + jb_C = g - j(b_L - b_C)$ is **the total conductance** of parallel circuit, often called **an admittance**;

g is the admittance of a resistor *R* and it is called **conductance**;

b_L and b_C are called **an inductive** and **a capacitive susceptance** correspondingly;

$b = b_L - b_C$ is **the total susceptance**.

Thus

$$Y = g \pm jb. \qquad (6.7)$$

Formula (6.7) represents the total admittance in rectangular form, in that the real part *g* is the total of all conductance looking into the input terminal of the circuit; the imaginary part *b* is the difference between the total inductive and capacitive susceptance. The total susceptance *will be positive* ($b > 0$) if the inductive susceptance is greater than the capacitive susceptance $b_L > b_C$. The total susceptance *will be negative* ($b < 0$) if the inductive susceptance is less than the capacitive susceptance $b_L < b_C$.

Fig. 6.6 illustrates phasor diagram for parallel circuit *R-L-C*.

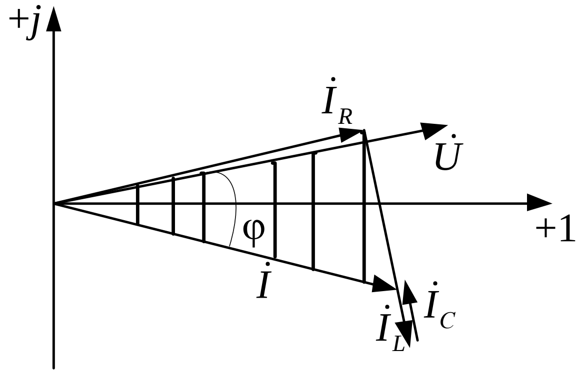


Fig. 6.6. The phasor diagram for parallel circuit *R-L-C*

Note: in an AC parallel circuit the voltage is common to each element of one and therefore taken as a reference phasor.

The phasor diagram indicates that *current* $\dot{I}$ is *in phase* with *the voltage across the resistor, lags the voltage across the inductor by 90°* and *leads the voltage across the capacitor by 90°*.

The hatched area on Fig. 6.6 is called **current triangle**. If each side of the current triangle will be divided by the voltage $\dot{V}$ the diagram of Fig. 6.7 is obtained, which is called an **admittance triangle**.

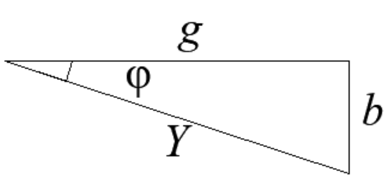


Fig. 6.7. An admittance triangle

The angle φ *is the phase difference* between the input voltage and total current. From geometry of the admittance triangle it is seen the following relationships:

$$Y = \sqrt{g^2 + b^2};$$

$$\left. \begin{array}{l} g = Y\cos\varphi \\ b = Y\sin\varphi \end{array} \right\}; \tag{6.8}$$

$$\varphi = \text{arctg}\frac{b}{g}.$$

SUMMARY:

– for AC parallel circuit the KCL is correctly written in complex notation;

– the total admittance is the complex sum of total conductance and total susceptance looking into input circuit's terminals; in rectangular form the real part of that complex number is total conductance an d the imaginary part is the total susceptance;

– the argument of the total admittance is phase angle between input voltage and input current.

SELF-ASSESSMENT TEST:

1. Write down the expression of Ohm's Law for AC parallel circuit in complex form.
2. Write down the expression of the KCL for AC parallel circuit in complex form.
3. Draw phasor diagram for parallel *R-L-C* configuration.
4. Write down the total admittance for parallel circuit in complex form.
5. Draw an admittance triangle.
6. How to calculate admittance magnitude and the phase angle.
7. Note all known relationships for Y, g, b_L, b_C and φ, that are determined from admittance triangle geometry.

7. POWER IN COMPLEX FORM. THE POWER TRIANGLE

When complex voltage $\dot{V} = V \cdot e^{j\psi_v}$ exists across load $\dot{Z}$ and complex current $\dot{I} = I \cdot e^{j\psi_i}$ flows through it (Fig. 7.1) the complex apparent power $\dot{S}$ is equal to:

$$\dot{S} = \dot{V}\overset{*}{I}, \tag{7.1}$$

where $\overset{*}{I} = I \cdot e^{-j\psi_i}$ is complex conjugate of current $\dot{I}$.

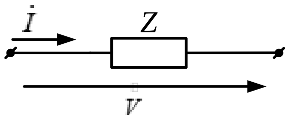


Fig. 7.1. Section of an electrical circuit

Therefore

$$\dot{S} = Ve^{j\psi_v} Ie^{-j\psi_i} = VIe^{j(\psi_v - \psi_i)} = Se^{j\varphi} \quad \text{VA}, \tag{7.2}$$

where φ – phase angle between an input voltage and an input current in a complex AC circuit.

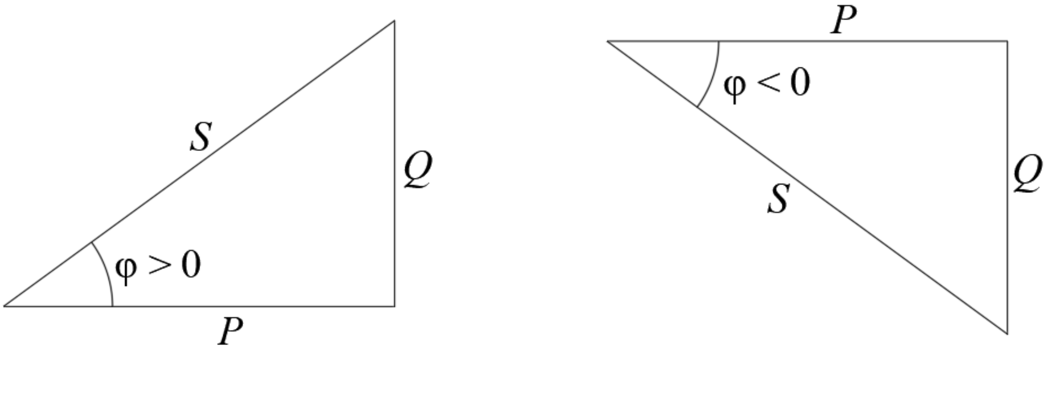
Formula (7.2) expresses complex power in polar form. Power relationship may be written in rectangular form:

$$\dot{S} = P \pm jQ \quad \text{V} \cdot \text{A}, \tag{7.3}$$

where P – is active power in an AC circuit; Q – is reactive power in an AC circuit.

If a circuit is inductive Q is positive and «+» is used before Q in the formula (6.7). If a circuit is capacitive Q is negative and «–» is used before Q in the expression (6.7).

If each side of the voltage triangle (see Fig. 6.2) is multiplied by $\dot{I}$, *the power triangle* will be obtained. Fig. 7.2 illustrates it. Each side of this triangle represents a particular type of power.



a b

Fig. 7.2. Power triangle for inductive and capacitive load:
a – power triangle for inductive load; b – power triangle for capacitive load

The phase angle is positive (φ > 0) in the case of inductive load and negative (φ < 0) in the case of capacitive load.

If a network has both capacitive and inductive elements, the reactive component of the power triangle will be determined by the difference between the reactive power delivered to each.

If $Q_L > Q_C$, the resultant power triangle will be similar to Fig. 7.2, a.

If $Q_C > Q_L$, the resultant power triangle will be similar to Fig. 7.2, b.

Fig. 7.2 (a) and (b) demonstrates that the three powers are related by the Pythagorean theorem, that is:

$$S = \sqrt{P^2 + Q^2} \quad \text{V} \cdot \text{A};$$

$$P = VI\cos\varphi = S\cos\varphi \quad \text{Wt}; \qquad (7.4)$$

$$Q = VI\sin\varphi = S\sin\varphi \quad \text{VAr}.$$

Phase angle φ is determined by:

$$\varphi = \text{arc tg}\frac{Q}{P}. \qquad (7.5)$$

Power factor is:

$$\cos\varphi = \frac{P}{S}. \qquad (7.6)$$

Example 7.1

Given: The current through the series *R-L-C* circuit is equal $\dot{I} = 0{,}15e^{j30^\circ}$ A; $R = 8$ Ω; $X_L = 6{,}5$ Ω; $X_C = 17$ Ω.

Find: a) active and reactive power for each element of the circuit; b) the apparent power supplied by the generator; c) the generator voltage; d) the phase angle and power factor; e) draw phasor diagram.

***Solution*:**

a) Power delivered in the resistive element:

$$P = I^2R = 0{,}15^2 \cdot 8 = 0{,}18 \quad \text{Wt}.$$

Reactive power in the inductance:

$$Q_L = I^2X_L = 0{,}15^2 \cdot 6{,}5 = 0{,}146 \quad \text{VAr}.$$

Reactive power in the capacitance:

$$Q_C = I^2 X_C = 0{,}15^2 \cdot 17 = 0{,}383 \quad \text{VAr}.$$

The total reactive power:

$$Q_t = Q_L - Q_C = 0{,}146 - 0{,}383 = -0{,}237 \quad \text{VAr}.$$

The negative sign of total reactive power indicates about capacitive character of the original circuit.

b) Apparent power is as follows in both complex forms:

$$\dot{S} = P - jQ_t = 0{,}18 - j0{,}237 = 0{,}298e^{-j53} \quad \text{V} \cdot \text{A}.$$

c) Determine the source voltage using two methods:

The 1st method: according Ohm's Law: $\dot{V} = \dot{I} \cdot \dot{Z}$, where

$$Z = R + j(X_L - X_C) = 8 + j(6{,}5 - 17) = 8 - j10{,}5 = 13{,}2e^{-j53} \quad \Omega;$$
$$\dot{V} = 0{,}15e^{j30} \cdot 13{,}2e^{-j53} = 1{,}98e^{-j23} \quad \text{V}.$$

The 2nd method: use formula (7.1):

$$\dot{V} = \frac{\dot{S}}{\overset{*}{I}},$$

where complex conjugate of current is $\overset{*}{I} = 0{,}15 \cdot e^{-j30}$.

$$\dot{V} = \frac{0{,}298e^{-j53}}{0{,}15e^{-j30}} = 1{,}98e^{-j23} \quad \text{V}.$$

d) The phase angle between the source voltage and the current is the argument of the both: total impedance and apparent power in the polar form. So φ = – 53°. The negative angle is referred for capacitive case of the circuit.

Power factor, according to (7.6):

$$\cos\varphi = \frac{P}{S} = \frac{0{,}18}{0{,}298} = 0{,}6\,.$$

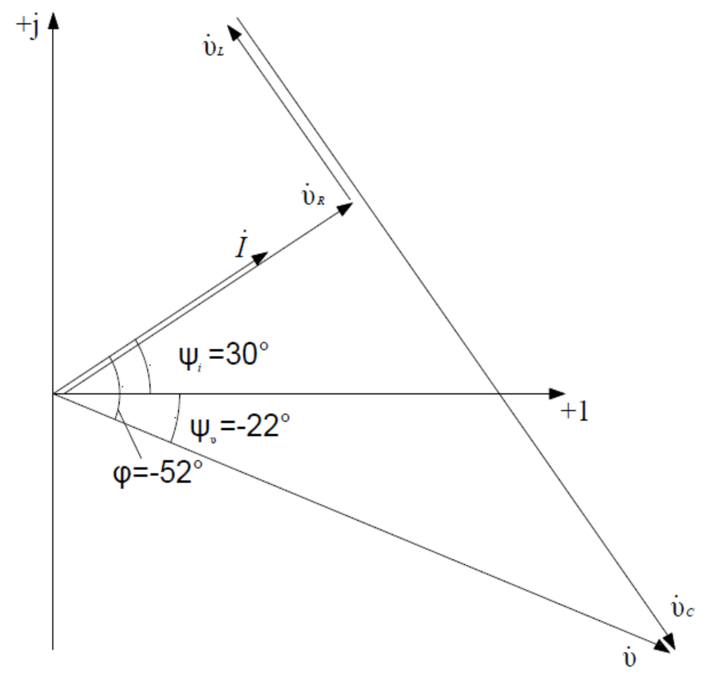
e) Draw phasor voltage diagram for series *R-L-C* circuit in chosen scale: m_V = 0,2 V/Sm. For this purpose calculate modules of V_R, V_L and V_C:

$$V_R = IR = 0{,}15 \cdot 8 = 1{,}2 \quad \text{V;}$$

$$V_L = IX_L = 0{,}15 \cdot 6.5 = 0{,}975 \quad \text{V;}$$

$$V_C = IX_C = 0{,}15 \cdot 17 = 2{,}55 \quad \text{V.}$$

The phasor diagram is shown in the figure below.



SUMMARY:

– the components of complex apparent power are average (or active) power and reactive power;

– the argument of complex apparent power is phase angle, that is positive for inductive load and is negative for capacitive load;

– the power triangle demonstrates relationships between power components.

SELF-ASSESSMENT TEST:

1. Write down two complex notations of apparent power: in polar and in rectangular forms.
2. Note the relationships between *P*, *Q* and *S*, using power triangle.
3. Define phase angle φ, using power triangle, if *P*, *Q*, *S* are given.

8. RESONANCE

8.1. Introduction

The resonance is very important mode of operation of electrical circuits, because resonant circuits are widely adopted in electrical and electronic systems today.

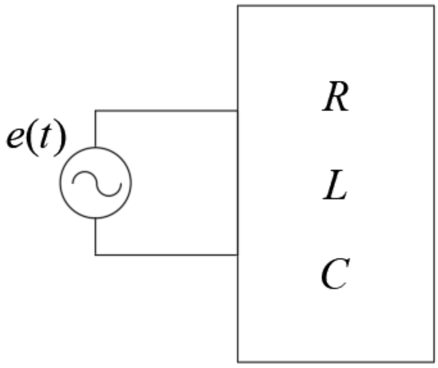


Fig. 8.1. The general representation of resonant circuit

The resonant electrical circuit must have both *inductance* and *capacitance*.

When the *voltage* $\dot{V}$ applied to an electrical network containing resistance, inductance and capacitance **is in phase** with **the resulting current** $\dot{I}$, the circuit is said to be **resonant**.

There are two types of resonant circuits: series and parallel. Each of them will be considered in detail.

8.2. Series Resonance. The Resonant Condition

Fig. 8.2 illustrates the basic configuration for series resonant circuit, that consists of a resistor, an inductor and a capacitor. The total impedance of this circuit is

$$\dot{Z} = R + jX_L - jX_C = R + j(X_L - X_C)\,.$$

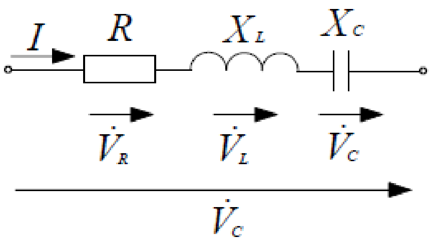


Fig. 8.2. The series resonant circuit

Resonance occurs when the total reactance of the circuit is equal to zero:

$$X = X_L - X_C = 0.$$

So **the resonant condition** is

$$X_L = X_C. \quad (8.1)$$

Conclusion:

– the *zero* reactive component of the total impedance means that *a total impedance is purely resistive*: $Z_t = R$;

– impedance $\dot{Z}$ has a minimum value at resonance.

8.3. Series Resonance. The Resonant Frequency

Since $X_L = 2\pi fL$ and $X_C = \dfrac{1}{2\pi fC}$ at resonance $2\pi f_0 L = \dfrac{1}{2\pi f_0 C}$.

Thus

$$f_0 = \frac{1}{2\pi\sqrt{LC}};$$
$$\omega_0 = \frac{1}{\sqrt{LC}}. \quad (8.2)$$

where f_0 – resonant frequency in Hertz; ω_0 – the resonant angular velocity in rad/s.

The subscript 0 in the above equations indicates that the frequency determined is *the series resonant frequency*.

Example 8.1

Calculate: a) the resonant frequency and b) total impedance in resonance of a series circuit consisting of a resistor 3 Ω, inductor 20 mH and capacitor 165 mF.

***Solution*:**

a) $f_0 = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{20\cdot 10^{-3}\cdot 165\cdot 10^{-6}}} = 87{,}6 \ \text{ Hz};$

$\omega_0 = 2\pi\cdot f_0 = 2\pi\cdot 87{,}6 = 550{,}5 \ \text{ rad/s}.$

б) The total impedance is resistive: $Z_t = R = 3$ Ω.

8.4. Series Resonance. Current at Resonance

Since impedance at resonance has a minimum value ($\dot{Z} = R$) so the current at resonance has *a maximum* value:

$$\dot{I} = \frac{\dot{V}}{R}. \tag{8.3}$$

Above and below resonance, the current decreases because the impedance increases. Fig. 8.3 (a and b) shows total impedance and current versus frequency.

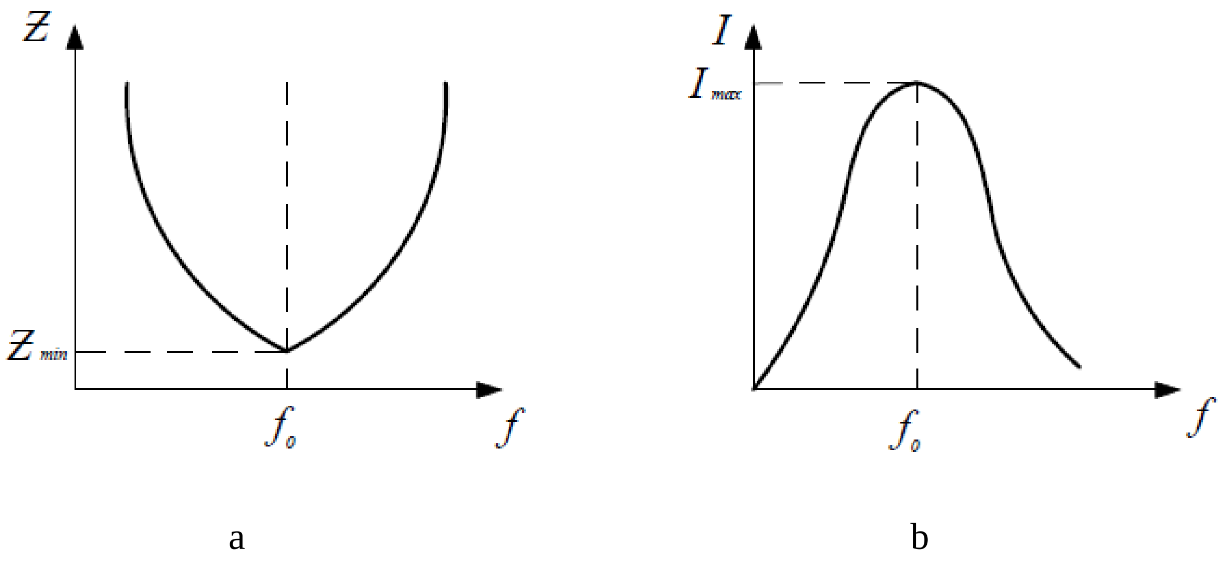


Fig. 8.3. Impedance (a) and current (b) versus frequency

8.5. Series Resonance. Voltages at Resonance. Phasor Diagram at Resonance

As the current is the same through each element of series circuit (including capacitor and inductor) and $X_L = X_C$, the voltage across inductor and across capacitor is equal in magnitude:

$$V_L = V_C.$$

The voltage across inductor and across capacitor in a complex notation are:

$$\dot{V}_L = \dot{I}\, jX_L = \dot{I}\, X_L \cdot e^{j90^\circ};$$

$$\dot{V}_C = \dot{I}(-jX_C) = \dot{I}\, X_C \cdot e^{-j90^\circ}.$$

Phasor diagram (Fig. 8.4) illustrates that $\dot{V}_L$ and $\dot{V}_C$ are opposite in phase.

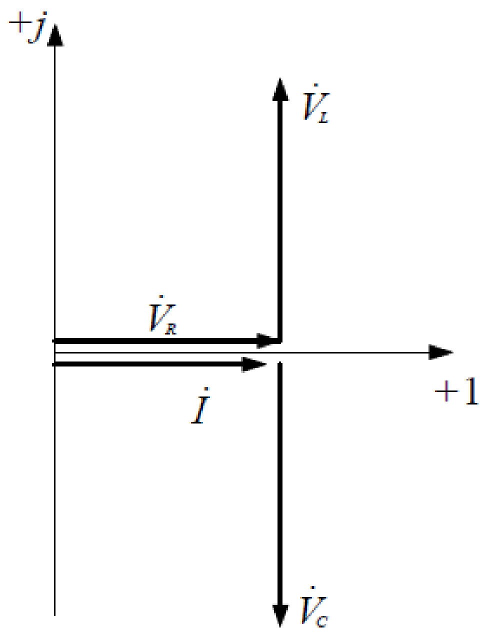


Fig. 8.4. Phasor diagram for the series resonance circuit

It is clearly from phasor diagram that:

– the input voltage of the series resonance circuit is equal to the voltage across resistor in the magnitude and phase angle: $\dot{V}_{in} = \dot{V}_R$;

– $\dot{V}_{in}$; $\dot{V}_R$ and $\dot{I}$ are in phase at resonance.

Example 8.2

A series circuit consists of a resistor 70 Ω, inductor 10 mH and capacitor 95 nF. Supply voltage is 49 V. Determine: a) the current; b) the voltage across the inductor and across the resistor.

***Solution*:**

a) Current at resonance is determined using Ohm's Law:

$$\dot{I} = \frac{\dot{V}}{R} = \frac{49}{70} = 0,7 \quad \text{A}.$$

At beginning calculate resonant frequency and inductive reactance:

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \cdot 10^{-3} \cdot 95 \cdot 10^{-9}}} = 3,24 \cdot 10^4 \quad \text{rad/s};$$

$$X_L = \omega_0 L = 3,24 \cdot 10^4 \cdot 10 \cdot 10^{-3} = 324,4 \quad \Omega.$$

Using Ohm's Law calculate the voltage across inductor:

$$V = IX_L = 0,7 \cdot 324,4 = 227 \quad \text{V}.$$

Note: This is 4.6 times greater than the supply voltage. The voltage across resistor is equal to the supply voltage $V_R = V_{supply}$ = 49 V.

8.6. X_L, X_C and Z_t Versus Frequency in the Series Resonant Circuit

Reactance X_L, X_C, total reactance X and impedance Z_t are functions of frequency. Their magnitudes versus frequency (and angular velocity) are:

$$X_L = \omega L = 2\pi f L;$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C};$$

$$X = X_L - X_C = \omega L - \frac{1}{\omega C} = 2\pi f L - \frac{1}{2\pi f C};$$

$$Z_t = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} = \sqrt{R^2 + (2\pi f L - \frac{1}{2\pi f C})^2}.$$

The curves of the Fig. 8.5 illustrates how these quantities change with frequency:

– resistance R isn't function of frequency;

– since X_L is *directly proportional* to frequency, so the curve of $X_L(f)$ is *straight lin*e;

– X_C is *inversely proportional* to frequency, so the curve of $X_C(f)$ is *hyperbola*;

– the point, where $X_L = X_C$, defines *the condition of resonance* and *resonant frequency*;

– as at resonant frequency $X_L = X_C$, the total reactance is 0 and the curve of the total reactance $X(f)$ passes through 0 at $f = f_0$, and $Z_t = R$ and has its minimum value.

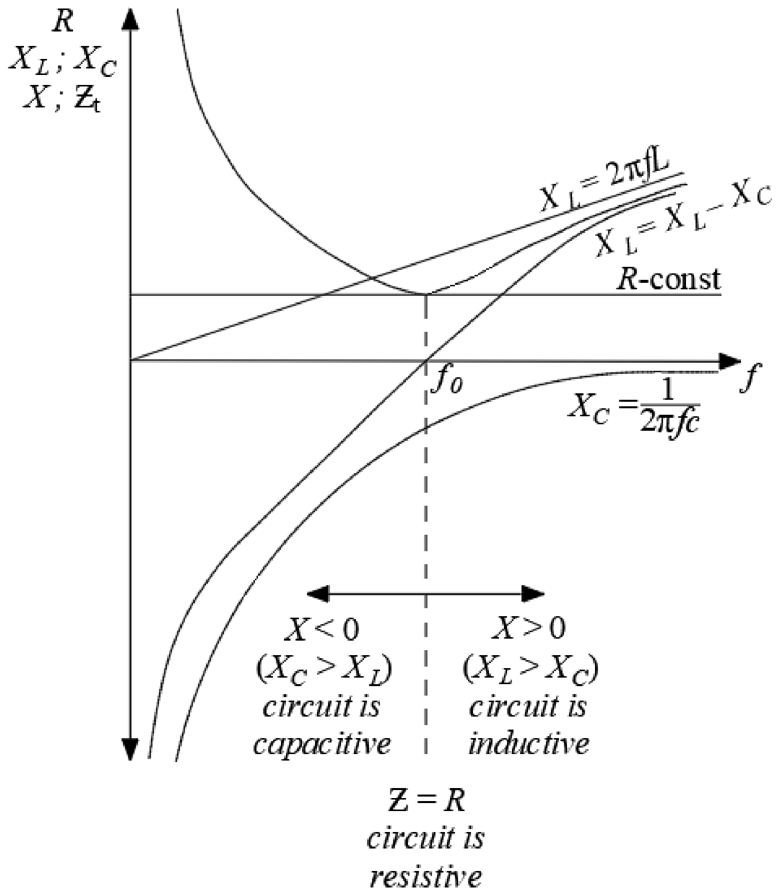


Fig. 8.5. X_L, X_C and Z_t versus frequency in the series resonant circuit.

At frequencies less than f_0, $X_L < X_C$ and *the circuit is capacitive*. At frequencies greater than f_0, $X_L > X_C$ and *the circuit is inductive*.

8.7. Phase Angle as a Function of Angular Frequency

At frequencies less then f_0, $X_C > X_L$, and the current leads the input voltage, as indicated in Figure above. The phase angle decreases as the frequency approaches the resonant value and is 0° at resonance. At frequencies more then f_0, $X_L > X_C$, and the current lags the source voltage. As the frequency goes higher, the phase angle approaches 90° (Fig. 8.6).

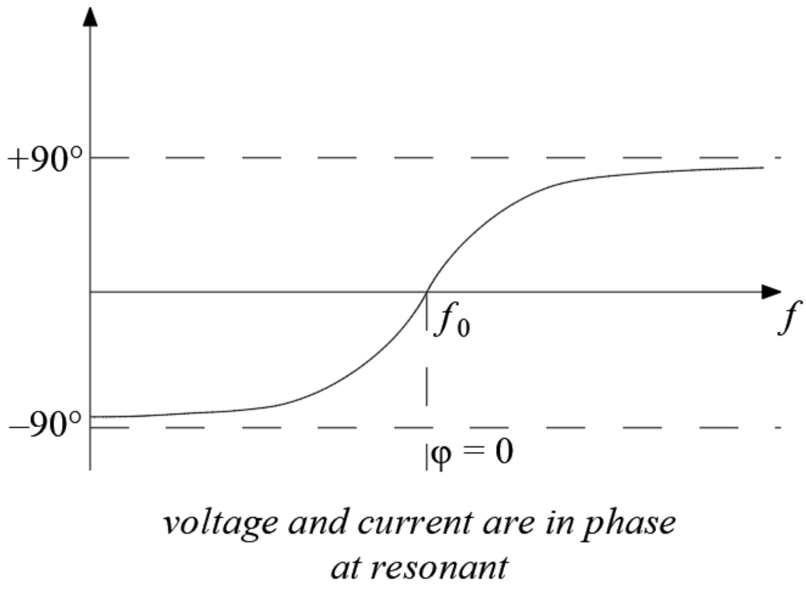


Fig. 8.6. Phase difference versus frequency

8.8. The Quality Factor (*Q*) or *Q*-factor

Q-factor is a **qualitative characteristic** of merit for a resonant device such as an *R-L-C* circuit. *Q*-factor is an abbreviation for **quality factor** and refers to the ‘goodness’ of a reactive component. The quality factor is also an indication of how much energy is placed in storage (continual transfer from one reactive element to the other) compared to that dissipated.

For any resonant circuit, *Q*-factor is defined as the ratio of reactive power to average power:

$$Q = \frac{Q_L}{P} = \frac{Q_C}{P} = \frac{I^2 X_L}{I^2 R} = \frac{I^2 X_C}{I^2 R} = \frac{X_L}{R} = \frac{X_C}{R}. \qquad (8.4)$$

Hence

$$Q = \frac{X_L}{R} = \frac{X_C}{R} = \frac{IX_L}{IR} = \frac{IX_C}{IR} = \frac{V_L}{V_R} = \frac{V_C}{V_R}. \qquad (8.5)$$

From formula (8.5) follows:

$$\begin{aligned} V_L &= V_R Q = V_{in} Q; \\ V_C &= V_R Q = V_{in} Q. \end{aligned} \qquad (8.6)$$

As Q-factor can have a value of several hundreds at resonance, so voltages V_L and V_C may be much greater than that of the supply voltage V_{in}. For this reason, Q-factor is often called **the circuit magnification factor**.

For series resonant circuits used in communication systems, Q-factor is usually greater than 1. Resonance is usually of interest only in circuits of Q-factor greater than about 10.

As follows from aforesaid one of practical application of series resonance circuit is as the voltage amplifier.

Example 8.3

In the series circuit at resonance condition $R = 10\ \Omega$; $X_L = X_C = 560\ \Omega$ and $V_{in} = 15$ V. Determine: Q-factor; V_L and V_C.

Solution: $Q = \frac{X_L}{R} = \frac{560}{19} = 56;\ V_L = V_C = V_{in} Q = 15 \cdot 56 = 840$ V.

8.9. V_R, V_L, V_C Versus Frequency

Current, voltages across resistor, inductor and capacitor versus frequency. Diagrams of Fig. 8.7 illustrates voltages across resistor, inductor and capacitor against frequency for series resonant circuit. Observation shows that:

– the curve $V_R(f)$ has the same shape as the curve $I(f)$ and at resonant frequency f_0 the value of V_R is equal to V_{in};

– the curve $V_L(f)$ changes from zero to its maximum at frequency after resonance. After reaching its peak value the voltage V_L will drop toward V_{in};

– the curve $V_C(f)$ changes from value equal to V_{in} at zero frequency. Its peak value will occur at a frequency just before resonance. After reaching its maximum the voltage V_C will drop in magnitude to zero.

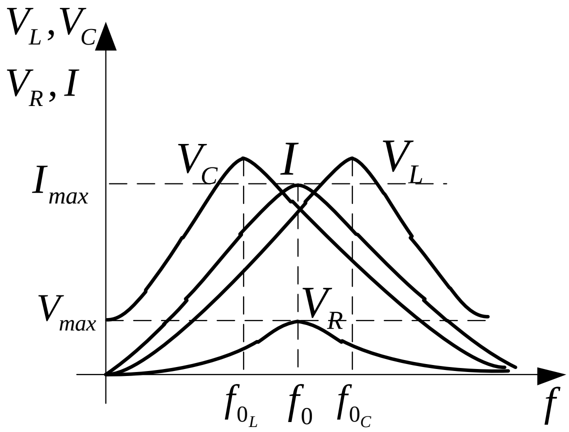


Fig. 8.7. Versus frequency

8.10. Characteristic Reactance

The value of inductive and capacitive reactance at resonant frequency is called **characteristic reactance** and is indicated by letter ρ and has unit of Ohm (Ω)

$$\rho = X_L = X_C.$$

Let substitute the formula for ω_0 to the expressions of inductive and capacitive reactance and obtain the following

$$X_L = \omega_0 L = \frac{1}{\sqrt{LC}} L = \sqrt{\frac{L}{C}};$$
$$X_C = \frac{1}{\omega_0 C} = \frac{\sqrt{LC}}{C} = \sqrt{\frac{L}{C}}.$$

Hence characteristic reactance:

$$\rho = \sqrt{\frac{L}{C}} \quad \Omega. \tag{8.7}$$

Substituting ρ in formula (8.5) we obtain the formula for *Q*-factor:

$$Q = \frac{\rho}{R}. \tag{8.8}$$

8.11. Bandwidth

Fig. 8.8 shows the range of frequencies at which the current is not greater than 0,707 of the maximum current. This definite range of frequencies is called **the bandwidth** (abbreviated BW).

The BW is the range of frequencies for which the current is equal or greater than 0,707 of its maximum value at resonance.

The frequencies corresponding to 0,707 of the maximum current are called critical frequencies, cut off frequencies, or half – power frequencies and indicated as f_1 and f_2.

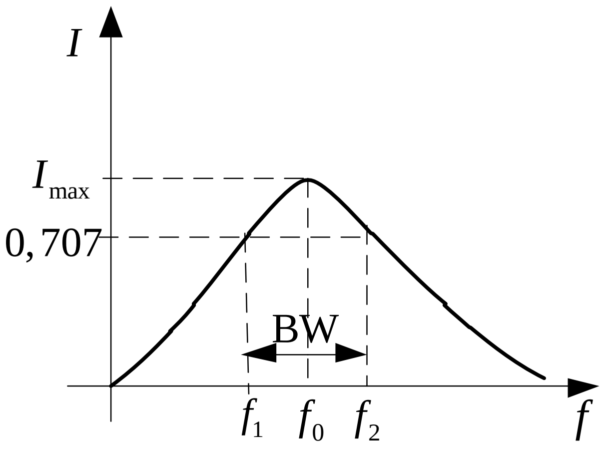


Fig. 8.8. Resonant current curve and bandwidth

The BW is the difference between f_2 and f_1:

$$\mathrm{BW} = f_2 - f_1. \tag{8.9}$$

Ideally the resonant frequency is located in the center between f_1 and f_2. So it can be calculated as follows:

$$f_0 = \frac{f_1 + f_2}{2}. \tag{8.10}$$

The BW is an important characteristic of a resonant circuit. A series resonance circuit that has a *narrow BW* is the circuit *with large Q-factor*.

Example 8.4

A certain series resonant circuit has a maximum current of 100 mA at f_0 and cut of frequency f_1 = 8 kHz and f_2 = 12 kHz.

Find: the value of current at this critical frequencies; BW; f_0.

***Solution*:**

The current at the critical frequencies:

$$I_{f1} = I_{f2} = 0{,}707 I_{max} = 70{,}7 \text{ mA};$$
$$\text{BW} = f_2 - f_1 = 12 - 8 = 4 \text{ kHz};$$
$$f_0 = \frac{f_1 + f_2}{2} = \frac{12+8}{2} = 10 \text{ kHz}.$$

8.12. Selectivity

The curve in Fig. 8.9 is called **selectivity curve**.

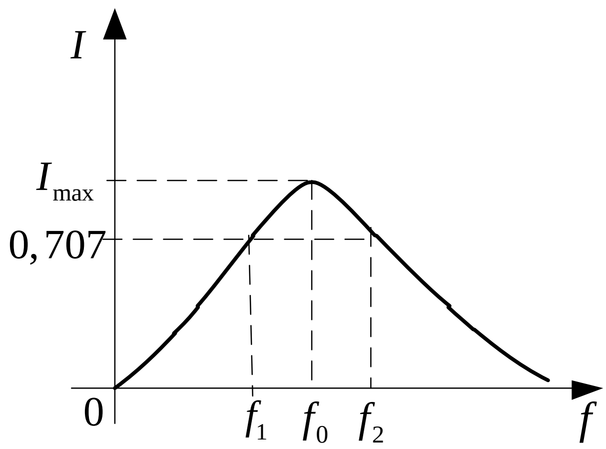


Fig. 8.9. The selectivity curve

This term is derived from the fact that only frequencies around f_0 will permit significant amounts of power to be dissipated by the circuit (examine the selectivity curve of Fig. 8.9).

Selectivity is the ability of a resonant circuit to respond to a signal of certain frequency and discriminate against all other.

The response becomes progressively weaker as the frequency departs from the resonant frequency. Discrimination against other signals becomes more pronounced as circuit losses are reduced, in other words, as the *Q*-factor is increased.

Thus

$$Q = \frac{f_0}{f_2 - f_1}$$

is *a measure of the circuit selectivity* in terms of the points on each side of resonance where the circuit current has fallen to 0,707 of its maximum value reached at resonance.

The higher the Q-factor, the narrower the bandwidth and the more selective is the circuit.

SUMMARY:

– at series resonance, the inductive and capacitive reactances are equal in magnitude;

– the impedance of a series R-L-C circuit is purely resistive at resonance;

– in a series R-L-C circuit the current is maximum and in phase with input voltage at resonance;

– the voltages V_L and V_C are equal in magnitude and 180° out of phase, therefore they cancel each other;

– *the bandwidth* of a series resonant circuit is the range of frequencies for which the current is 0,707 of $I_{\max}$ or greater;

– a higher Q-factor produces a narrower bandwidth.

SELF-ASSESSMENT TEST:

1. What is the condition for series resonance?
2. State the formula for resonant frequency.
3. Is the impedance minimum or maximum at series resonance?
4. Why is the current maximum at resonance?
5. State the relation between V_R, V_L, V_C, V_{in} at series resonance and plot their graphs: $V_R(f)$, $V_L(f)$, $V_C(f)$.
6. Explain the behavior of reactance X_L, X_C, X_t and Z versus frequency and plot their graphs.
7. State the formula for bandwidth.
8. Explain how the Q-factor affects the bandwidth.

8.13. Parallel Resonance. Resonance at an Ideal Parallel Circuit

Fig. 8.10 illustrates an ideal parallel resonant circuit and the phasor diagram for resonant state. In an ideal case resistors are not present at both inductive and capacitive branches. In other words, an inductor and a capacitor are considered as pure elements. Resonant condition for ideal parallel circuit is the same as for series one: $X_L = X_C$.

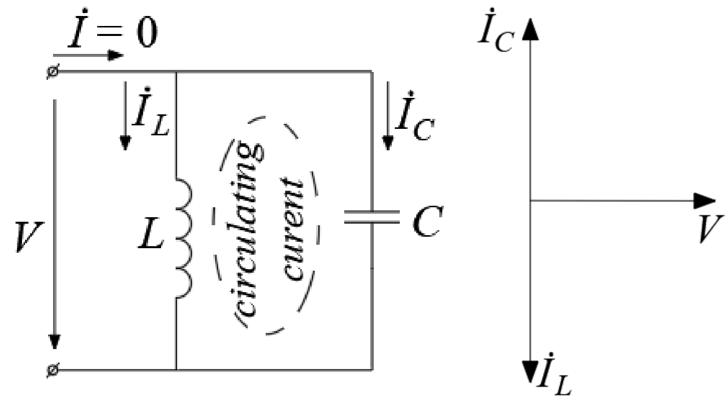


Fig. 8.10. An ideal parallel circuit and phasor diagram at resonance

Hence the resonant frequency is the same as in the series case

$$f_0 = \frac{1}{2\pi\sqrt{LC}};$$
$$\omega_0 = \frac{1}{\sqrt{LC}}. \qquad (8.11)$$

Conclusion:

– when the parallel resonance occurs, the two branch currents I_C and I_L *are equal in magnitude and 180°out of phase with each other*;

– so the total input current is equal to zero;

– at resonance current circulates within the closed *L*–*C* loop only;

– the case of $I_t = 0$ means that *the total impedance* of ideal parallel circuit *is infinitely large* at resonance. And the circuit's behavior is as an open circuit.

8.14. An Almost Ideal Parallel Circuit

Consider admittance Versus Frequency. Fig. 8.11 shows an almost ideal circuit, since one of the parallel branches contains the pure resistive element.

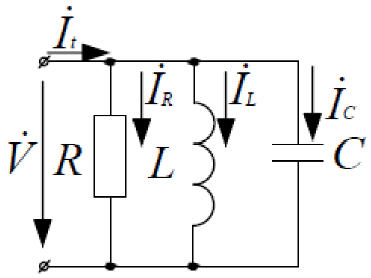


Fig. 8.11. An almost ideal parallel circuit

The admittance of each parallel branch is determined as following:

– resistive conductance:

$$g = \frac{1}{R};$$

– inductive admittance:

$$b_L = \frac{1}{jX_L} = -\frac{j}{\omega L};$$

– capacitive admittance:

$$b_C = \frac{1}{-jX_C} = j\omega C.$$

The total circuit admittance:

$$\dot{Y} = g + j(b_C - b_L) = g + j(\omega C - \frac{1}{\omega L}).$$

According to resonance definition, the circuit is in resonance when phase angle between input voltage and current is equal to zero. It has place when the imaginary part of the total circuit admittance is zero:

$$\omega C - \frac{1}{\omega L} = 0 \quad \text{or} \quad \omega C = \frac{1}{\omega L}.$$

Hence **the total circuit admittance is minimum** and **equal to the resistive conductance**

$$\dot{Y} = g = \frac{1}{R} \text{ is min}. \quad (8.12)$$

Formula (8.12) represents *the condition for parallel resonance* in an almost ideal circuit.

The condition of minimum admittance means the maximum of total impedance and consequently the minimum of the total input current of parallel circuit at the resonance

$$\dot{Z} = \frac{1}{g} = R \text{ is max};$$

$$I_t = \frac{V}{Z} = VY \text{ is min}.$$

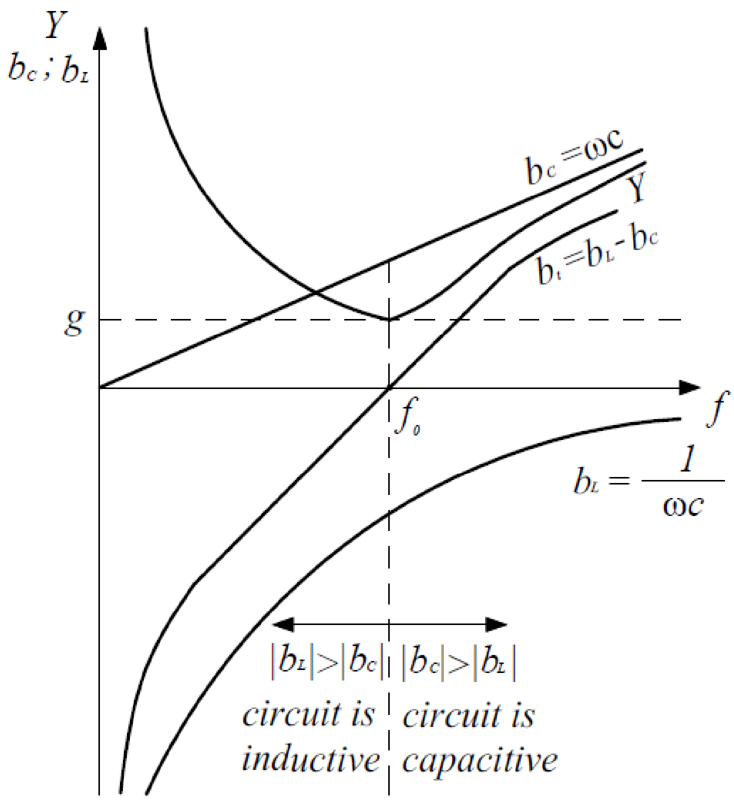


Fig. 8.12. Inductive, capacitive and the total circuit admittance versus frequency

Fig. 8.12 shows graphs of b_C, b_L, g and Y versus frequency. At the point of f_0, b_C and b_L are equal in magnitude and $Y = g$ (the resonant circuit behaves itself as a resistor element).

8.15. A Practical Parallel Resonant Circuit. Condition of Resonance

Fig. 8.13 illustrates the case of practical network that includes a real coil of inductance ($R_1 – L$) in parallel with capacitance and resistance ($R_2 – C$).

Consider an admittance of each parallel branch:

– admittance of branch 1:

$$\dot{Y}_1 = \frac{1}{\dot{Z}_1} = \frac{1}{R_1 + jX_L} = \frac{R_1 - jX_L}{R_1^2 + X_L^2} = \frac{R_1}{R_1^2 + X_L^2} - j\frac{X_L}{R_1^2 + X_L^2};$$

– admittance of branch 2:

$$\dot{Y}_2 = \frac{1}{\dot{Z}_2} = \frac{1}{R_2 - jX_C} = \frac{R_2 + jX_C}{R_2^2 + X_C^2} = \frac{R_2}{R_2^2 + X_C^2} + j\frac{X_C}{R_2^2 + X_C^2}.$$

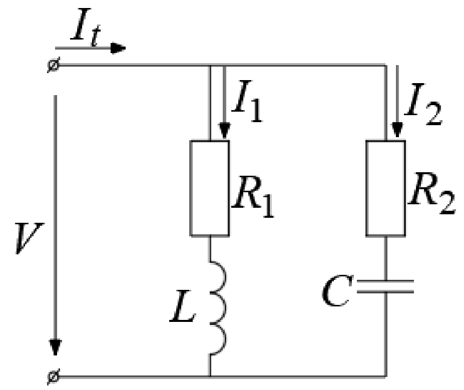


Fig. 8.13. The practical parallel resonance network

The total circuit admittance:

$$\dot{Y}_1 + \dot{Y}_2 = \frac{R_1}{R_1^2 + X_L^2} - j\frac{X_L}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} + j\frac{X_C}{R_2^2 + X_C^2} =$$

$$= \frac{R_1}{R_1^2 + X_L^2} + \frac{R_2}{R_2^2 + X_C^2} - j\left(\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2}\right).$$

For resonant condition the imaginary part of the total network admittance must be equal to zero, hence

$$\frac{X_L}{R_1^2 + X_L^2} - \frac{X_C}{R_2^2 + X_C^2} = 0,$$

or

$$\frac{X_L}{R_1^2 + X_L^2} = \frac{X_C}{R_2^2 + X_C^2}. \tag{8.13}$$

Expression (8.13) is *the condition of parallel resonance at practical network.*

At resonance

$$\frac{\omega_r L}{R_1^2 + (\omega_r L)^2} = \frac{\dfrac{1}{\omega_r C}}{R_2^2 + (\dfrac{1}{\omega_r C})^2}, \tag{8.14}$$

where ω_r – is the resonant angular velocity of parallel practical network.

Rearranging of (8.14) gives

$$\omega_r = \frac{1}{\sqrt{LC}} \sqrt{\frac{R_1^2 - \dfrac{L}{C}}{R_2^2 - \dfrac{L}{C}}} = \omega_0 \cdot \sqrt{\frac{R_1^2 - \rho^2}{R_2^2 - \rho^2}}, \tag{8.15}$$

where ω_0 and ρ – are resonant angular velocity of ideal network and characteristic reactance accordantly.

And resonant frequency

$$f_r = \frac{\omega_0}{2\pi} \sqrt{\frac{R_1^2 - \rho^2}{R_2^2 - \rho^2}}. \tag{8.16}$$

8.16. *Q*-factor and Bandwidth in a Parallel Network

As was mentioned in section 8.12 currents circulate within the parallel branches of a parallel resonant circuit: the current leavs the capacitor and produces the magnetic field of the inductance, then it collapsis and rechargs the capacitor, and so on. These currents are higher than the input current.

The *Q*-factor of a parallel resonant circuit is the ratio of the current circulating in the parallel branches of the circuit to the supply current, i. e. in a parallel circuit, *Q*-factor is a measure of the current magnification:

$$Q = \frac{I_1}{I_t} = \frac{I_2}{I_t}. \tag{8.17}$$

Currents in parallel branches (I_1 and I_2) may be several hundreds of times greater than the supply current at resonance. The formula of *Q*-factor for parallel circuit is the same as for series one.

The expression used for calculating of BW in the series circuit also applied to parallel circuit:

$$Q = \frac{f_r}{f_2 - f_1} = \frac{f_r}{\text{BW}} \quad \text{so} \quad \text{BW} = \frac{f_r}{Q},$$

where f_1 and f_2 – are cut off frequencies and are defined by the condition that the output voltage is 0,707 times the maximum value.

8.17. *V*, *I*

Fig. 8.14 shows that currents in parallel branches are more higher than supply current and are equal in magnitude at resonance. The total current reaches to its minimum value at resonant frequency.

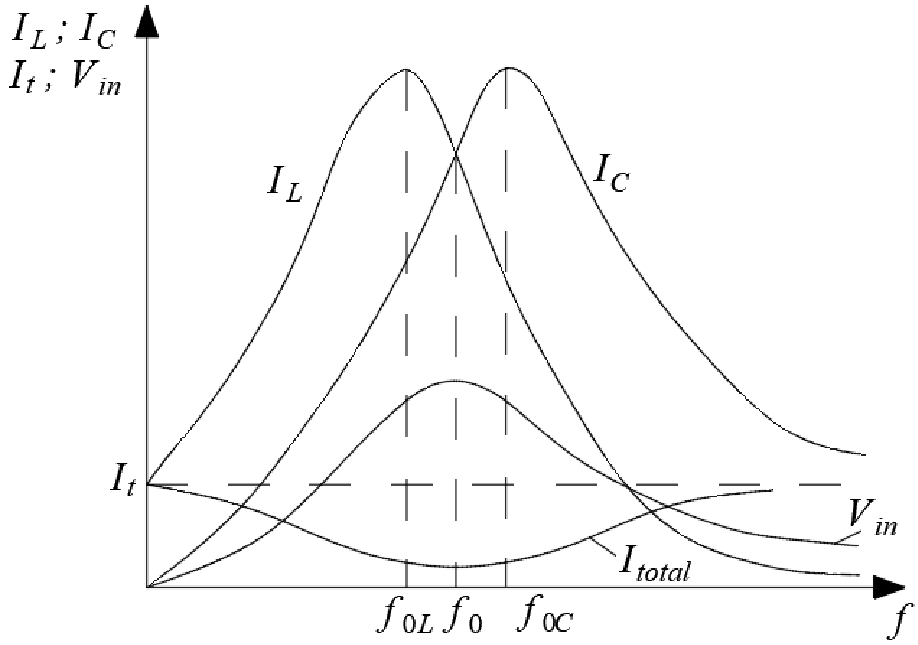


Fig. 8.14. The currents in parallel branches and voltage versus frequency

8.18. Phase Angle Versus Frequency

At low frequencies (Fig. 8.15), the capacitive reactance is quite high, and the inductive reactance is low. Since the elements are in parallel, the total impedance at low frequencies will therefore be inductive.

At high frequencies, the reverse is true, and the network is capacitive. At resonance (f_r), the network appears resistive.

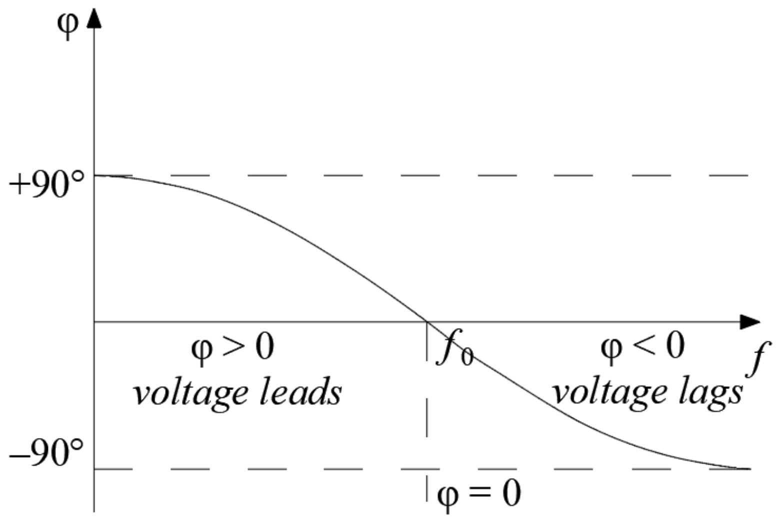
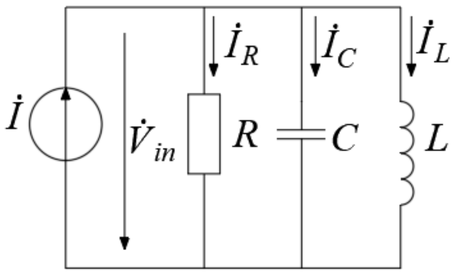


Fig. 8.15. Phase angle versus frequency

Example 8.5

Given: $\dot{I}$ = 4,5 mA; R = 750 Ω; C = 4 μF; L = 8 mH. Find: a) the resonant frequencies: ω_0 and f_0; b) $\dot{V}_{in}$; c) $\dot{I}_R$, $\dot{I}_L$, $\dot{I}_C$; d) Q factor; e) BW.



***Solution*:**

a) As the circuit of Figure above is *the almost ideal network* the resonant frequencies ω_0 and f_0 are determined by expressions (8.15):

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{8 \cdot 10^{-3} \cdot 4 \cdot 10^{-6}}} = 5,6 \cdot 10^3 \ \text{ rad/s;}$$

$$f_0 = \frac{\omega_0}{2\pi} = \frac{5,6 \cdot 10^3}{2\pi} = 891,7 \ \text{ Hz.}$$

b) In parallel circuit $\dot{V}_{in} = \dot{V}_R = \dot{V}_C = \dot{V}_L$, hence

$$\dot{V}_{in} = \dot{I} \cdot R = 4,5 \cdot 750 \cdot 10^{-3} = 3,37 \ \text{ V.}$$

c) At resonance: $\dot{I}_R = \dot{I}$ = 4,5 mA;

$$\dot{I}_L = \frac{\dot{V}_{in}}{Z_L} = \frac{\dot{V}_{in}}{jX_L} = \frac{3,37}{5,6 \cdot 10^3 \cdot 8 \cdot 10^{-3} e^{j90}} = 75e^{-j90} \ \text{ mA,}$$

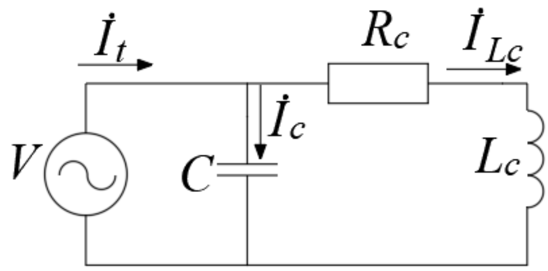
$$\dot{I}_C = 75e^{j90} \ \text{ mA.}$$

d) $Q = \dfrac{R}{X_L} = \dfrac{750}{44,8} = 16,7.$

e) $\text{BW} = \dfrac{f_0}{Q} = \dfrac{891,7}{16,7} = 53,2 \ \text{ Hz.}$

Example 8.6

Given: the coil of inductor and capacitor are connected in parallel; $\dot{V}$ = 18 V; R_c = 40 Ω; L_c = 0,45 H; C = 20 μF. Find: a) the resonant frequency f_r; b) Q-factor.



***Solution*:**

a) For the circuit of Figure above we will use formula (8.16):

$$f_r = \frac{1}{2\pi\sqrt{LC}}\sqrt{1-\frac{R_c^2}{\frac{L_c}{C}}} = \frac{1}{2\pi\sqrt{0,45\cdot 20\cdot 10^{-6}}}\sqrt{1-\frac{40^2}{\frac{0,45}{20\cdot 10^{-6}}}} = 51{,}13 \text{ Hz.}$$

b) $Q = \dfrac{2\pi f_r L}{R_c} = \dfrac{144{,}5}{40} = 3{,}6$.

Alternatively, Q-factor at resonance is a current magnification (for a parallel circuit) equal to I_C/I_R:

$$Q = \frac{I_C}{I_t} = \frac{V\omega C}{I_t} = \frac{18\cdot 0{,}00642}{0{,}032} = 3{,}6.$$

The same result was obtained above.

SUMMARY:

– at parallel resonance, the admittances of inductive and capacitive branches are equal in magnitude;

– the total impedance of an ideal parallel resonant circuit is infinite quantity at resonance;

– so the input current in an ideal circuit is zero at resonance;

– the total impedance of a practical parallel resonant circuit is resistive and has its maximum value at resonance;

– so the input current in a practical resonant circuit is minimum at $f = f_r$;

– *the bandwidth* and Q-factor of a parallel resonant circuit are determined using the same expressions as for series circuit;

– in a *parallel* circuit the Q-factor is a measure of *current magnification*, whereas in a series circuit it is a measure of *voltage magnification*.

SELF-ASSESSMENT TEST:

1. What is the condition for parallel resonance: a) in an ideal parallel circuit; b) in a practical parallel network?
2. State the formula for resonant frequency: a) in an ideal parallel circuit; b) in a practical parallel network?
3. Are currents in parallel branches of an ideal resonant circuit cancel each other?
4. Is the impedance minimum or maximum at parallel resonance?
5. Why is the input current equal to zero in an ideal parallel circuit at $f = f_0$?
6. What is the difference between f_0 and f_r in the case of parallel resonance?
7. Why is the current minimum at parallel resonance in a practical circuit?
8. State the relation between I_t, I_L, I_C, V_{in} at parallel resonance and plot their graphs: $V_{in}(f)$, $I_L(f)$, $I_C(f)$, $I_t(f)$.
9. Explain the behavior of admittances b_L, b_C, b_t and Y versus frequency and plot their graphs.
10. State the formula for Q-factor and bandwidth.

9. AC CIRCUIT. TASKS

Task №1

1.1. Determine the periodic time for the following frequencies:

a) 10 Hz; b) 250 Hz; c) 80 kHz.

1.2. Calculate the frequency for the following periodic times:

a) 2 ms; b) 50 μs; c) 0.04 s.

Answer: 1.1: a) 0,1 s; b) 4 ms; c) 12,5 μs.

1.2: a) 0,5 kHz; b) 20 kHz; c) 25 Hz.

Task №2

Two sinusoidal alternating quantities are represented by:

a) $i(t) = 0{,}17\sin(\omega t - 35°)$ A; b) $v(t) = 54\sin(3\omega t + 45°)$ V, where $\omega = 200$ rad/sec;

2.1. Comment on the relative magnitudes and frequencies of these quantities.

2.2. Determine the frequency of quantity «b» and its instantaneous value at $t = 0$: $v(0)$.

2.3. State the period of quantity «b».

2.4. State its phase shifts.

2.5. State the phase relationship of «b» with respect to «a».

2.6. Determine RMS value for current and voltage.

Answer: 2.1: a) $I_m = 0{,}17$ A, $\omega = 200$ rad/sec; b) $V_m = 54$ V, $\omega = 600$ rad/sec.

2.2: $f = 95{,}54$ Hz; $v(0) = 38{,}18$ V.

2.3: T= 0,01 s.

2.4: $\psi_i = -35°$; $\psi_v = 45°$.

2.5: $\varphi = \psi_v - \psi_i = 80°$.

2.6: $I = 0{,}12$ A, $V = 38{,}3$ V.

Task №3

3.1. The current through a 3,7 Ω resistor is indicated. Find the sinusoidal expression for the voltage $v(t)$ if $i(t) = 2{,}5\sin(314t - 110°)$ A;

3.2. The current through a 3,7 kΩ resistor is indicated. Find the sinusoidal expression for the voltage $v(t)$ if $i(t) = 0.025\sin(314t + 60°)$ A.

Answer: 3.1: $v_R(t) = 9{,}25\sin(314t - 110°)$ V;

3.2: $v_R(t) = 92{,}5\sin(314t + 60°)$ V.

Task №4

The voltage across a 0.5 *H* coil is provided. Find the sinusoidal expressions for the current across the coil:

4.1. $v_L(t) = 165\sin(100t - 37°)$ A;

4.2. $v_L(t) = 95\sin(100t + 115°)$ A.

Answer: 4.1: $i(t) = 3{,}3\sin(314t - 127°)$ A;

4.2: $i(t) = 1{,}9\sin(314t + 25°)$ A.

Task №5

The current through a 5,5 μF capacitor is given. Find the sinusoidal expressions for the voltage across the capacitor:

5.1. $i(t) = 0{,}54\sin(7272t + 55°)$ A;

5.2. $i(t) = 1{,}27\sin(7272t + 10°)$ A.

Answer: 5.1: $v_R(t) = 9{,}25\sin(314t - 110°)$ V;

5.2: $v_R(t) = 92{,}5\sin(314t + 60°)$ V.

Task №6

Solve tasks №3–5 using complex algebra. Draw phasor representation for complex voltages and currents.

Task №7

A coil of inductor with R = 10 Ω and L= 0,05 H is connected to voltage source, RMS of that is V = 120 V and f = 50 Hz.

Define: a) the total complex impedance of the coil; b) the complex current through the coil; c) phase difference between voltage and current; d) active, reactive and apparent power; e) draw phasor diagram.

Answer: a) $\dot{Z} = 18{,}6e^{\,j57^\circ}$ Ω; b) $\dot{I} = 6{,}45e^{\,-j57^\circ}$A; c) φ = 57°; d) P = 416 W, Q = 653 V·Ar, S = 773 V·A.

Task №8

The series connection of resistor R = 120 Ω and capacitor C = 30 μF is supplied by voltage source $v(t)$ = 311sin314t V.

Define: a) the total complex circuit impedance; b) RMS voltage's and current's values; c) phase difference between voltage and current; d) active and reactive power; e) draw phasor diagram.

Answer: a) $\dot{Z} = 160e^{\,-j41^\circ}$ Ω; b) V = 220 V, I = 1,37 A; c) φ = – 41°; d) P = 226 W, Q = – 210 V·Ar.

Task №9

A series circuit consists of a resistor, an inductor and a capacitor: R = 3 Ω, L = 8 mH, C = 15 μF. A voltage source V = 20 V, f = 500 Hz supplies that circuit.

Find: a) the complex current through the circuit; b) voltage through each element; c) active power; d) draw phasor diagram.

Answer: a) $\dot{I} = 4e^{\,j53^\circ}$A; b) V_R = 12 V, V_L = 101 V, V_C = 85 V; c) P = 48 W.

Task №10

Current through the series connection of resistor R = 12 Ω, inductor X_L = 20 Ω and capacitor X_C = 24 Ω is equal to $i(t)$ = 0,04sin314t A. Using complex algebra define voltage across each element and total input voltage. Write down each voltage as a function of time. Draw phasor diagram.

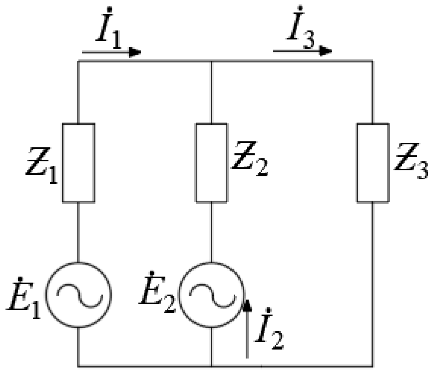
Answer: $v_R(t)$ = 0,48sin314t V; $v_L(t)$ = 0,8sin(314t + 90°) V;

$v_C(t)$ = 0,8sin(314t – 90°)V; $v_{input}(t)$ = 0,506sin(314t – 18,44°) V.

Task №11

Given: $\dot{E}_1 = 100$ V, $\dot{E}_2 = 100e^{-j30°}$ V, $\dot{Z}_1 = \dot{Z}_2 = 50 + j30$ Ω, $\dot{Z}_3 = 100$ Ω.

Determine all currents using the method of superposition and active power, delivered in the circuit.

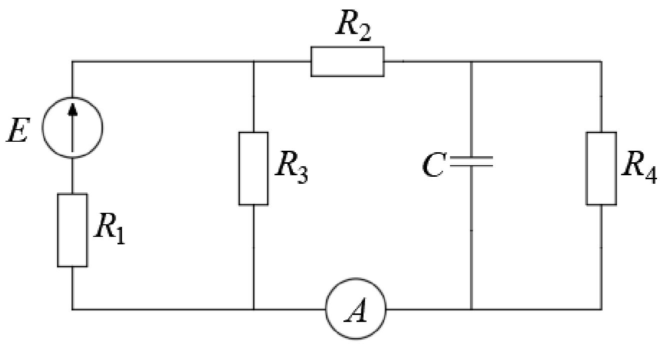


Answer: $\dot{I}_1 = 0{,}693e^{j13°}$A; $\dot{I}_2 = 0{,}45e^{-j85°}$A; $\dot{I}_3 = 0{,}77e^{-j21°}$A; $P = 93{,}2$ W.

Task №12

Given: $E = 40$ V, $R_1 = 200$Ω , $R_2 = 160$ Ω, $R_3 = 120$ Ω, $R_4 = 80$ Ω, $X_C = 60$ Ω.

Determine current of ammeter using Thevenin's Theorem.



Answer: $\dot{I}_A = 56$ mA.

Task №13

Given: $\dot{E} = 55$ V; $R = 5$ Ω; $R_c = 6$ Ω; $L = 10$ mH; $C = 1$ μF.

Find the following quantities for the circuit of Figure below:

a) total impedance at resonance;

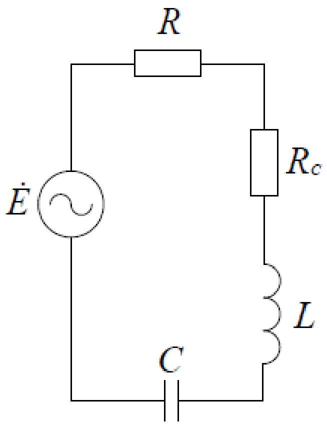
b) current at resonance;

c) resonant frequency expressed as ω (rad/sec);

d) voltages: V_L and V_C;

e) active and reactive powers: P, Q_C and Q_L;

f) Q-factor of the circuit.



Answer: $\dot{Z} = 11\ \Omega$; $\dot{I} = 5$ A; $\omega_0 = 10\,000$ rad/sec; $\dot{V}_L = 100e^{\,j90°}$ V; $\dot{V}_C = 100e^{\,-j90°}$ V; $P = 275$ W; $Q_L = Q_C = 2500$ VAr; Q-factor = 9,1.

Task №14

The series resonant circuit with $R_{coil} = 16\ \Omega$ and $L_{coil} = 158\ \mu$H is at resonance at frequency $f_0 = 1$ MHz, $V_{in} = 1{,}8$ V.

Define the following quantities for that circuit:

a) the capacitance of this resonant circuit;

b) current at resonance;

c) voltages: V_{coil} and V_C;

d) active power: P.

Answer: $C = 160$ pF; $I = 50$ mA; $V_L = V_C = 49{,}6$ V; $P = 40$ mW.

Task №15

The series circuit consists of a coil of inductance (R_{coil}, L_{coil}) and capacitor. The supply voltage $V_{in} = 35$ V.

Define the voltage across the coil of inductor, if the capacitor voltage is equal to 120 V.

Answer: $V_{coil} = 125$ V.

Task №16

For the series resonant circuit $R – L – C$: $R = 2\ \Omega$, $X_L = X_C = 10\ \Omega$

Find:

a) I, V_R, V_L, and V_C at resonance, if supply voltage $V_{in} = 10$ V.

b) What is the Q-factor of the circuit?

c) If the resonant frequency is 5000 Hz, find the bandwidth.

d) What is the power dissipated in the circuit at the half-power frequencies (HPF)?

Answer: I = 5 A; V_R = 10 V; V_L = V_C = 50 V; Q-factor = 5; BW = 1000; P_{HPF} = 25 W.

Task №17

The bandwidth of a series resonant circuit is 45 Hz; the resonant frequency is 1500 Hz.

Find:

a) the value of Q-factors;

b) the value of X_L at resonance if R = 7,5 Ω;

c) the inductance L and capacitance C of the circuit.

Answer: Q-factor = 33,3; X_L = 250 Ω; L = 26 mH; C = 0,42 μF.

Task №18

A series R-L-C circuit has a series resonant frequency of 16000 Hz. The circuit resistance is 15 Ω and inductive reactance X_L = 900 Ω.

Find:

a) the bandwidth;

b) the cut off frequencies.

Answer: BW = 266,7 Hz; f_1 = 15866,65 Hz and f_2 = 16133,35 Hz.

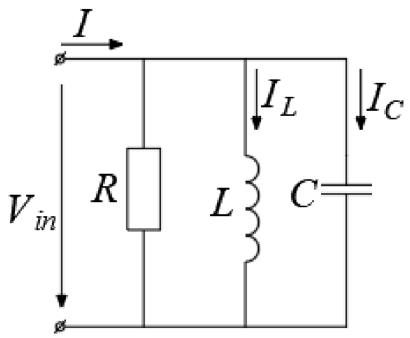
Task №19

Given: R = 16 Ω; L = 1,6 mH; C = 0,42 μF; V_{in} = 10 V.

Determine the following quantities for the circuit of Figure above:

a) the resonant frequency f_0;

b) currents I, I_L, I_C.



Answer: ω_0 = 5000 rad/s; I = 0,625 A; I_L = 1,25 A, I_C = 1,25 A.

Task №20

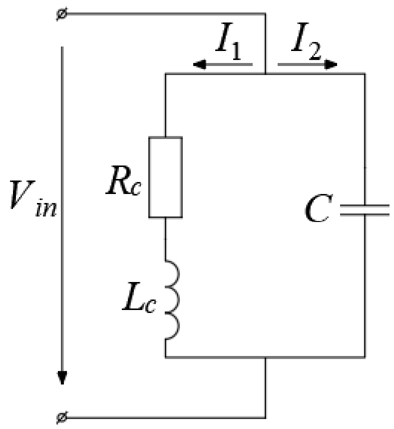
The coil of inductance (R_{coil} – L_{coil}) of Figure below is connected with capacitance C in parallel: R_{coil} = 11,2 Ω ; L_{coil} = 4 mH; C = 2,5 μF.

Determine for that circuit:

a) the resonant frequency f_r;

b) the total impedance at resonance;

c) draw the phasor diagram at resonance.



Answer: f_r = 9600 Hz; Z_t = 143 Ω.

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В двух частях
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АНАЛИЗ ЦЕПЕЙ ПЕРЕМЕННОГО ТОКА

BRIEF ELECTRICAL CIRCUIT THEORY AND PRACTICAL PROBLEMS

In two parts
Part 2

AC ANALYSIS

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