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ON BÄCKLUND TRANSFORMATIONS FOR SOME SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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We obtain second-order nonlinear differential equations (and the associated Bäcklund transformations) with an arbitrary analytic function of the independent variable. These equations (which are not of Painlevé type in general) under certain constraints imposed on an arbitrary analytic function can be reduced, in particular, to the second, third or fourth Painlevé equation. We consider the properties of the Bäcklund transformations for the second-order nonlinear differential equations generated by two systems of two first-order nonlinear differential equations with quadratic nonlinearities in derivatives of the unknown functions.

Keywords: Painlevé property, Painlevé equations, direct and inverse Bäcklund transformations

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1. Introduction

In review [1] (also see [2]), some important directions are outlined for studying the properties of solutions of Painlevé-type nonlinear ordinary differential equations whose general solutions have no movable singularities. These equations are usually called equations having the P-property of solutions or P-type equations. Not aspiring to the completeness of presentation, we also note the ongoing research into various properties of solutions of equations that are higher analogues of Painlevé-type equations [3]–[7], and studies of Painlevé-type non-Abelian equations [9]–[13] initiated in [8].

The aim of this paper is to study analytic properties of solutions of the differential equations

$$w_\alpha'' = 2w_\alpha^3 + \varphi w_\alpha + \alpha\varphi' + \frac{\varphi''}{2\varphi'}(2w_\alpha' - 2\varepsilon w_\alpha^2 - \varepsilon\varphi), \tag{1}$$

$$w'' = \frac{w'^2}{w} - \frac{\varphi'}{\varphi}w - \frac{1}{w} + \frac{1}{\varphi}(w^2 + \beta\varphi') + \frac{\varepsilon - \beta}{\varepsilon}\frac{\varphi''}{\varphi}w, \tag{2}$$

$$\varphi w w'' = \varphi w'^2 - \varphi' w w' + (\alpha + \varphi'\varepsilon - \varepsilon)w^3 + (\beta + \varphi'\sigma - \sigma)w + \varphi w^4 - \varphi, \tag{3}$$

$$2ww'' = w'^2 + 3w^4 + 8\varphi w^3 + 4\left(\varphi^2 + \varepsilon\left(\varphi' + p + \frac{q}{2}\right)\right)w^2 - q^2, \tag{4}$$

and the systems

$$y = -w + \varphi + \frac{[w' + (b-1)\varphi']^2}{2w^2}, \tag{5a}$$

$$w = -y + \varphi + \frac{[y' - b\varphi']^2}{2y^2} \tag{5b}$$

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and

$$y+\frac{M(z)(-2z+M(z))w}{M(z)(-2z+M(z))+2z(2+\beta+\alpha\varepsilon)w-(4+2\alpha\varepsilon)M(z)w}=0, \tag{6a}$$

$$w+\frac{N(z)(-2z+N(z))y}{N(z)(-2z+N(z))+2z(-2+\beta-2\varepsilon)y+(4+2\alpha\varepsilon)N(z)y}=0. \tag{6b}$$

In Eqs. (1)–(4), $\varphi=\varphi(z)$ is an arbitrary analytic function of the independent variable z; b, α, and β are arbitrary parameters; $\varepsilon^2=\sigma^2=1$, $q^2+2\beta=0$, and $p=-1-2\varepsilon-q/2$. In system (6), $M(z)=zw'+\varepsilon zw^2+(\alpha\varepsilon+1)w+z$ and $N(z)=zy'-\varepsilon zy^2-(\alpha\varepsilon+3)y+z$.

2. Analysis of Eq. (1)

Equation (1) can be represented as a system of equations

$$w_\alpha=-w_{\alpha-\varepsilon}-\varepsilon\frac{(2\alpha-\varepsilon)\varphi'}{2w'_{\alpha-\varepsilon}+2\varepsilon w^2_{\alpha-\varepsilon}+\varepsilon\varphi}, \tag{7a}$$

$$w_{\alpha-\varepsilon}=-w_\alpha+\varepsilon\frac{(2\alpha-\varepsilon)\varphi'}{2w'_\alpha-2\varepsilon w^2_\alpha-\varepsilon\varphi} \tag{7b}$$

with the unknown functions w_α, $w_{\alpha-\varepsilon}$ of the independent variable z and with an arbitrary analytic function $\varphi(z)$ ($\varphi'(z)\not\equiv 0$). Under the condition

$$(2\alpha-\varepsilon)\varphi'\neq 0, \tag{8}$$

it follows from system (7) that

$$w'_\alpha-\varepsilon w^2_\alpha+w'_{\alpha-\varepsilon}+\varepsilon w^2_{\alpha-\varepsilon}=0. \tag{9}$$

Eliminating the unknown function w_α from (9) under condition (8), we obtain the equation

$$w''_{\alpha-\varepsilon}=2w^3_{\alpha-\varepsilon}+\varphi w_{\alpha-\varepsilon}+(\alpha-\varepsilon)\varphi'+\frac{\varphi''}{2\varphi'}(2w'_{\alpha-\varepsilon}+2\varepsilon w^2_{\alpha+\varepsilon}+\varepsilon\varphi). \tag{10}$$

Theorem 1. *Let $w_\alpha=w(z,\alpha,\varepsilon)$ be a solution of Eq.* (1) *with fixed values of α and $\varepsilon^2=1$, and under condition* (8). *Then the function $w_{\alpha-\varepsilon}=w(z,\alpha-\varepsilon)$ defined in* (7b) *is a solution of Eq.* (10).

Theorem 2. *Let $w_{\alpha-\varepsilon}=w(z,\alpha-\varepsilon)$ be a solution of Eq.* (10) *with fixed values of α and $\varepsilon^2=1$, and under condition* (8). *Then the function $w_\alpha=w(z,\alpha,\varepsilon)$ defined in* (7a) *is a solution of Eq.* (1).

It is easy to see that Eq. (10) can be obtained from Eq. (1) by replacing $\varepsilon\to-\varepsilon$, $\alpha\to\alpha-\varepsilon$ and visa versa. This also holds for formulas (7a), (7b). Thus, formulas (7a), (7b) define the direct and inverse Bäcklund transformations for Eq. (1).

Setting $\varphi(z)=z$, we obtain the second Painlevé equation from Eq. (1)

$$w''_\alpha=2w^3_\alpha+zw_\alpha+\alpha. \tag{11}$$

Formulas (7a) and (7b) then take the form

$$w_\alpha=-w_{\alpha-\varepsilon}-\varepsilon\frac{2\alpha-\varepsilon}{2w'_{\alpha-\varepsilon}+2\varepsilon w^2_{\alpha-\varepsilon}+\varepsilon z}, \tag{12}$$

$$w_{\alpha-\varepsilon}=-w_\alpha+\varepsilon\frac{2\alpha-\varepsilon}{2w'_\alpha-2\varepsilon w^2_\alpha-\varepsilon z}. \tag{13}$$

In the case $\varepsilon=1$, transformations (12), (13) for Eq. (11) were obtained in [14].

It can be easily verified that all solution of the Riccati equation

$$2w'_\alpha = 2\varepsilon w_\alpha^2 + \varepsilon\varphi \tag{14}$$

are solutions of Eq. (1) with $2\alpha = \varepsilon$.

Example 1. Equation (14) with $\varepsilon = -1$, $\varphi = -2(z^2 + a)$ (with an arbitrary a), and $w_\alpha = y_a$ takes the form

$$y'_a + y_a^2 = z^2 + a \tag{15}$$

and has a particular solution $y_1 = z$ at $a = 1$. Therefore, the general solution of Eq. (15) with $a = 1$ is given by

$$y_1 = z + \frac{e^{-z^2}}{C + \int e^{-z^2}\,dz},$$

where C is an arbitrary constant.

We consider the equation

$$y'_{a+2} + y_{a+2}^2 = z^2 + a + 2. \tag{16}$$

It can be easily verified that if $y_a = y(z, a)$ is a solution of Eq. (15), then

$$y_{a+2} = y(z, a+2) = z + \frac{a+1}{z + y_a}, \qquad a \neq -1, \tag{17}$$

is a solution of Eq. (16).

We also note that if $y_a = y(z, a)$ is a solution of Eq. (15), then the function $\tilde{y}_a = -iy(iz, -a)$, $i^2 + 1 = 0$, is also a solution of Eq. (15). This property and relation (17) imply the integrability of Eq. (15) in quadratures at $a = 2k + 1$, $k \in \mathbb{Z}$.

Relation (17) can be represented as $(y_{a+2} - z)(y_a + z) = a + 1$, and can be regarded as a discrete analogue of Eq. (15).

We note that Eq. (1) is not a Painlevé-type equation in the case $\varphi''(z) \not\equiv 0$.

3. Analysis of Eq. (2)

Equation (2) can be written as a system of first-order equations

$$\begin{aligned} \varphi w' &= \varepsilon\varphi + (1 - \varepsilon\beta)\varphi' w - \varepsilon y w^2, \\ \varphi y' &= -\varepsilon\varphi - (1 - \varepsilon\beta)\varphi' y + \varepsilon y^2 w. \end{aligned} \tag{18}$$

Eliminating the unknown function w from (18), we obtain an equation for y,

$$y'' = \frac{y'^2}{y} - \frac{\varphi'}{\varphi}y' - \frac{1}{y} + \frac{1}{\varphi}\left(y^2 + (\beta - 2\varepsilon)\varphi'\right) - \frac{\varepsilon - \beta}{\varepsilon} \cdot \frac{\varphi''}{\varphi}y. \tag{19}$$

Equation (19) is obtained from (2) using the transformation $y = w$, $\varepsilon \to -\varepsilon$, $\beta \to \beta - 2\varepsilon$ and vise versa. Thus, the formulas

$$y = \frac{-\varepsilon\varphi w' + (\varepsilon - \beta)\varphi' w + \varphi}{w^2}, \tag{20}$$

$$w = \frac{\varepsilon\varphi y' + (\varepsilon - \beta)\varphi' y + \varphi}{y^2} \tag{21}$$

define the direct and inverse Bäcklund transformations for Eq. (2).

We consider two cases.

1. $\varphi = c = \text{const} \neq 0$. Equation (2) then takes the form

$$ww'' - w'^2 - c^{-1}w^3 + 1 = 0$$

and has the first integral

$$w'^2 - 2c^{-1}w^3 - 1 = Hw^2, \tag{22}$$

where H is an arbitrary constant. Equation (22) can be integrated in terms of elliptic functions [15].

2. $\varphi = z$. In this case, Eq. (2) can be reduced to

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(w^2 + \beta) - \frac{1}{w} \tag{23}$$

and is a particular case of the third Painlevé equation

$$w'' = \frac{w'^2}{w} - \frac{w'}{z} + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w} \tag{24}$$

with the parameter values α, β, $\gamma = 0$, and $\delta = -1$.

We can verify that Eq. (2) with $\varphi = az + b$, $a \neq 0$ is also reducible to Eq. (23) using a scale transformation of the unknown function and the independent variable.

Formulas (20), (21) at $\varphi = z$ were obtained in [16].

Theorem 3. *Equation* (2) *with either* $\varphi = c = \text{const} \neq 0$ *or* $\varphi = az + b$ *(*$a \neq 0$*) is a Painlevé type equation.*

Comparing Eq. (2) with the list of equations in [17], we conclude that it is not a P-type equation if $\varphi''(z) \not\equiv 0$.

4. Analysis of Eq. (3)

The system of equations

$$\varphi w' - \varepsilon\varphi w^2 - (\alpha\varepsilon - 1)w - \sigma\varphi = \frac{\varepsilon\sigma(\sigma\beta + \alpha\varepsilon - 2)w}{yw - \varepsilon\sigma}, \tag{25}$$

$$\varphi y' + \varepsilon\varphi y^2 + (\alpha\varepsilon - 1)y + \sigma\varphi = \frac{-\varepsilon\sigma(\sigma\beta + \alpha\varepsilon - 2)y}{yw - \varepsilon\sigma} \tag{26}$$

under the condition $\sigma\beta + \alpha\varepsilon - 2 \neq 0$ is equivalent to Eq. (3). Eliminating the function w from (25) and (26) under the condition $\sigma\beta + \alpha\varepsilon - 2 \neq 0$, we obtain an equation for y,

$$\varphi yy'' = \varphi y'^2 - \varphi' yy' + (\alpha - \varphi'\varepsilon - \varepsilon)y^3 + (\beta - \varphi'\sigma - \sigma)y + \varphi y^4 - \varphi. \tag{27}$$

Equation (27) can be obtained from (3) using the transformation $w = y$, $\varepsilon \to -\varepsilon$, $\alpha \to \alpha - 2\varepsilon$, $\sigma \to -\sigma$, and $\beta \to \beta - 2\sigma$ and vise versa. Thus, the following theorems hold.

Theorem 4. *Let* $w = w(z, \alpha, \beta, \varepsilon, \sigma)$ *be a solution of Eq.* (3) *with fixed values of the parameters* α, β, $\varepsilon^2 = 1$, *and* $\sigma^2 = 1$. *Then, under the condition* $\sigma\beta + \alpha\varepsilon - 2 \neq 0$, *the function*

$$y = \frac{\varepsilon\sigma}{w} + \frac{\varepsilon\sigma(\sigma\beta + \alpha\varepsilon - 2)}{\varphi w' - \varepsilon\varphi w^2 - (\alpha\varepsilon - 1)w - \sigma\varphi} \tag{28}$$

is a solution of Eq. (27).

Theorem 5. *Let $y = y(z, \alpha, \beta, \varepsilon, \sigma)$ be a solution of Eq.* (27) *with fixed values of the parameters α, β, $\varepsilon^2 = 1$, $\sigma^2 = 1$ such that $\sigma\beta + \alpha\varepsilon - 2 \neq 0$. Then the function*

$$w = \frac{\varepsilon\sigma}{y} - \frac{\varepsilon\sigma(\sigma\beta + \alpha\varepsilon - 2)}{\varphi y' + \varepsilon\varphi y^2 + (\alpha\varepsilon - 1)y + \sigma\varphi} \tag{29}$$

is a solution of Eq. (3).

Thus, relations (28), (29) define the direct and inverse Bäcklund transformations of Eq. (3).

We consider two cases.

1. $\varphi = c = \mathrm{const} \neq 0$. In this case, Eq. (3) takes the form

$$cww'' = cw'^2 + (\alpha - \varepsilon)w^3 + (\beta - \sigma)w + cw^4 - c. \tag{30}$$

This equation is of Painlevé type and can be integrated in terms of elliptic functions [15]. Equation (27) becomes

$$cyy'' = cy'^2 + (\alpha - \varepsilon)y^3 + (\beta - \sigma)y + cy^4 - c \tag{31}$$

and coincides with Eq. (30) up to notation.

Thus, under the conditions of Theorem 4, formula (28) with $\varphi = c = \mathrm{const} \neq 0$ defines a Bäcklund autotransformation for Eq. (30).

2. $\varphi = z$. Equation (3) then takes the form

$$zww'' = zw'^2 - ww' + \alpha w^3 + \beta w + zw^4 - z. \tag{32}$$

Equation (32) is a particular case of the third Painlevé equation (24) with the parameters α, β, $\gamma = 1$, and $\delta = -1$. In this case, Eq. (27) can be represented as

$$zyy'' = zy'^2 - yy' + (\alpha - 2\varepsilon)y^3 + (\beta - 2\sigma)y + zy^4 - z. \tag{33}$$

As mentioned above, Eq. (33) can be obtained from (32) using the transformation $w = y$, $\alpha \to \alpha - 2\varepsilon$, and $\beta \to \beta - 2\sigma$, while Eq. (32) can be obtained from (33) using the transformation $y = w$, $\varepsilon \to -\varepsilon$, $\alpha \to \alpha - 2\varepsilon$, $\sigma \to -\sigma$, and $\beta \to \beta - 2\sigma$. Thus, under the conditions of Theorem 4, the new solution of Eq. (24) with the parameters $(\alpha - 2\varepsilon, \beta - 2\sigma, 1, -1)$, $\varepsilon^2 = \sigma^2 = 1$ can be obtained from solution (24) with the parameters $(\alpha, \beta, 1, -1)$ using transformation (28) (where $\varphi = z$).

If $\varphi = az + b$, $a \neq 0$, Eq. (3) reduces to (24) with $\gamma = 1$ and $\delta = -1$ by a scale transformation of the unknown function and the independent variable. System of equations (25), (26) in the case $\varphi = z$ was obtained in [18].

Theorem 6. *Equation* (3) *in the case $\varphi = c = \mathrm{const} \neq 0$ or $\varphi = az + b$ $(a \neq 0)$ is a Painlevé-type equation.*

The comparison of Eq. (3) with the list of equations in [17] shows that it is not a P-type equation if $\varphi''(z) \not\equiv 0$.

5. Analysis of Eq. (4)

We consider the system of differential equations

$$\begin{aligned} w' &= q + 2\varepsilon\varphi w + \varepsilon w^2 + 2\varepsilon wu, \\ u' &= p - 2\varepsilon\varphi u - \varepsilon u^2 - 2\varepsilon wu, \end{aligned} \tag{34}$$

which is equivalent to Eq. (4) with respect to w. Eliminating the unknown function w from (34), we obtain an equation for u,

$$2uu'' = u'^2 + 3u^4 + 8\varphi u^3 + 4\left(\varphi^2 + \varepsilon\left(\varphi' + q + \frac{p}{2}\right)\right)u^2 - p^2. \tag{35}$$

Theorem 7. *Let $w = w(z, p, q, \varepsilon)$ be a solution of Eq.* (4) *with fixed values of p, q, and $\varepsilon^2 = 1$. Then the function*

$$u = (w' - q - 2\varepsilon\varphi w - \varepsilon w^2)(2\varepsilon w)^{-1} \tag{36}$$

is a solution of Eq. (35).

Theorem 8. *Let $u = u(z, p, q, \varepsilon)$ be a solution of Eq.* (35) *with fixed values of p, q, and $\varepsilon^2 = 1$. Then the function*

$$w = -(u' - p + 2\varepsilon\varphi u + \varepsilon u^2)(2\varepsilon u)^{-1} \tag{37}$$

is a solution of Eq. (4).

It is easy to see that Eq. (35) can be obtained from (4) using the transformations $w = u$, $\varepsilon \to -\varepsilon$, $p \to q$, and $q \to p$. Thus, relations (36), (37) define the (direct and inverse) Bäcklund transformations for Eq. (4).

We consider two cases.

1. $\varphi(z) = z$. Equation (4) is the fourth Painlevé equation [17]

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta, \tag{38}$$

whose general solution has no movable singularities. Relations (36) and (37) at $\varphi = z$ are given in [14], [19].

If $\varphi(z) = az + b$, where a and b are constants and $a(a-1) \neq 0$, then, using the scale transformations $w = \lambda y$ and $az + b = \mu\tau$, Eq. (4) can be reduced to Eq. (38) with the parameters α and β depending on a [20].

2. $\varphi(z) = c = \text{const}$. In this case, Eqs. (4), (35) take the respective form

$$2ww'' = w'^2 + 3w^4 + 8c^2w^3 + 4\left(c^2 + \varepsilon\left(p + \frac{q}{2}\right)\right)w^2 - q^2, \tag{39}$$

$$2uu'' = u'^2 + 3u^4 + 8c^2u^3 + 4\left(c^2 - \varepsilon\left(q + \frac{p}{2}\right)\right)u^2 - p^2. \tag{40}$$

Equation (40) can be obtained from (39) using the transformation $u \to w$, $\varepsilon \to -\varepsilon$, $q \to p$, and $p \to q$. Equations (39) and (40) can be integrated in terms of elliptic functions [15].

Thus, the formulas

$$u = \frac{w' - q - 2\varepsilon cw - \varepsilon w^2}{2\varepsilon w}, \tag{41}$$

$$w = \frac{-(u' - p + 2\varepsilon cu + \varepsilon u^2)}{2\varepsilon u} \tag{42}$$

define the (direct and inverse) Bäcklund transformations for Eq. (39), which is integrable in terms of elliptic functions.

The comparison of Eq. (4) with the list of equations in [17] shows that it is not of P-type if $\varphi''(z) \not\equiv 0$. It is easy to verify that all solutions of the Riccati equation

$$w' = q + 2\varepsilon\varphi w + \varepsilon w^2 \tag{43}$$

are also solutions of Eq. (4) with

$$\beta = -2(1+\alpha\varepsilon)^2. \tag{44}$$

By virtue of (44), Eq. (43) implies that (at $\varphi = z$) all solutions of the equation

$$w' = w^2 + 2zw - 2(1+\alpha) \tag{45}$$

are also solutions of Eq. (38) if $\beta + 2(1+\alpha)^2 = 0$, and all solutions of the equation

$$w' = -w^2 - 2zw + 2(\alpha - 1) \tag{46}$$

are also solutions of Eq. (38) if $\beta + 2(\alpha-1)^2 = 0$.

By the substitution $w = -2(\alpha+1)v^{-1}$ and $(\alpha \neq -1)$ [21], Eq. (45) can be transformed to the equation

$$v' = -v^2 - 2zv + 2(\alpha+1). \tag{47}$$

The comparison of (46) and (47) shows that a solution $w = w_\alpha$ $(\alpha \neq -1)$ of Eq. (45) generates a solution $w_{\alpha+2} = -2(\alpha+1)w_\alpha^{-1}$ of Eq. (46) with $\alpha_1 = \alpha + 2$, and vise versa. Setting $w = -y_a - z$ and $2\alpha + 1 = a$ in Eq. (45), we obtain Eq. (15) for y_a.

6. Analysis of systems (5), (6)

6.1. Solutions of system (5) satisfy one of the two conditions: either

$$[w' + (b-1)\varphi']y + [y' - b\varphi']w = 0, \tag{48}$$

or

$$[w' + (b-1)\varphi']y - [y' - b\varphi']w = 0. \tag{49}$$

System (5) under condition (48) is equivalent with respect to y $(y' - b\varphi' \neq 0)$ to the equation

$$2yy'' = y'^2 + 4y^3 - 2\varphi y^2 + 2b\varphi'' y - b^2\varphi'^2, \tag{50}$$

and is equivalent with respect to w $(w' + (b-1)\varphi' \neq 0)$ to the equation

$$2ww'' = w'^2 + 4w^3 - 2\varphi w^2 - 2(b-1)\varphi'' w - (b-1)\varphi'^2. \tag{51}$$

Theorem 9. *Let $y_b = y(z,b) \not\equiv 0$ be a solution of Eq.* (50) *with a fixed value of the parameter b. Then the function w defined by relation* (5b) *is a solution of Eq.* (51).

It is easy to see that Eq. (51) can be obtained from (50) by the substitution $y = w$, $b \to 1 - b$.

Theorem 10. *Let $w_{b-1} = w(z, b-1) \not\equiv 0$ be a solution of Eq.* (51) *at a fixed value of the parameter b. Then the function y defined in* (5a) *is a solution of Eq.* (50).

System (5) under condition (49) is equivalent with respect to y ($y' - b\varphi' \neq 0$) to the equation

$$2yy'' = 3y'^2 - 4b\varphi' y' + 2\varphi y^2 + 2b\varphi'' y + b^2\varphi'^2, \tag{52}$$

and is equivalent with respect to w ($w' + (b-1)\varphi' \neq 0$) to the equation

$$2ww'' = 3w'^2 + 4(b-1)\varphi' w' + 2\varphi w^2 - 2(b-1)\varphi'' w + (1-b)^2\varphi'^2. \tag{53}$$

Theorem 11. *Let $y_b = y(z, b) \not\equiv 0$ be a solution of Eq.* (52) *at a fixed value of the parameter b. Then the function w defined in* (5b) *is a solution of Eq.* (53).

Theorem 12. *Let $w_{b-1} = w(z, b-1) \not\equiv 0$ be a solution of Eq.* (53) *at a fixed value of the parameter b. Then the function y defined in* (5a) *is a solution of Eq.* (52).

Equations (53) can be obtained from (52) by the substitution $y = w$, $b \to 1 - b$.

Thus, formulas (5a) and (5b) define the (direct and inverse) Bäcklund transformations for Eq. (50) on one hand and the (direct and inverse) Bäcklund transformations for Eq. (52) on the other hand.

Let $\varphi(z) = c = \text{const}$. Then Eqs. (50) and (52) take the respective form

$$2yy'' = y'^2 + 4y^3 - 2cy^2, \tag{54}$$

$$2yy'' = 3y'^2 + 2cy^2. \tag{55}$$

Equation (52) has the first integral

$$y'^2 - 2y^3 + 2cy^2 = Hy, \tag{56}$$

where H is an arbitrary constant. Equation (56) can be integrated in terms of elliptic functions [15].

By the substitution

$$y = p^{-2}(z), \tag{57}$$

Eq. (55) can be reduced to the linear equation $p'' = -(c/2)p$. Thus, we have proved the following theorem.

Theorem 13. *Equations.* (50) *and* (52) *with $\varphi(z) = c = \text{const}$ are P-type equations.*

We note that Eq. (50) with $\varphi(z) = z$ is equation XXXIV from the list in [17].

Theorem 14. *Equation* (52) *with either $b = 0$ or $\varphi(z) = z$ is a Painlevé-type equation.*

Proof. If we set $b = 0$ in Eq. (52), then it can be reduced to the Airy equation $p'' = -(\varphi/2)p$ by transformation (57).

Equation (52) with $\varphi(z) = z$ takes the form

$$2yy'' = 3y'^2 - 4by' + 2zy^2 + b^2. \tag{58}$$

Let $b \neq 0$. By the substitution $y \to by^{-1}$, we transform Eq. (58) to the equation

$$2yy'' = y'^2 - 4y^2y' - y^4 - 2zy^2. \tag{59}$$

Along with Eq. (59), we consider the more general equation

$$2vv'' = v'^2 - 4v^2v' - v^4 + 2F(z)v^2 - \delta, \tag{60}$$

where $F(z)$ is an arbitrary analytic function and δ is a parameter. Equation (60) coincides with Eq. (59) with $F(z) = -z$ and $\delta = 0$ up to notation. It is shown in [22] (where the transformation from [17] is used, see p. 454 in [17]) that Eq. (60) is an equation of P-type. Namely, the general solution of Eq. (60) is a rational function of the integration constants. The Theorem is proved.

Remark 1. Equation (60) with $\delta = 1$ is canonical equation XXVII from the list in [17]. The Bäcklund transformation for Eq. (60) in the case $F(z) = 2(z^2+\alpha)$, $\delta = -2\beta$ (where α and β are arbitrary parameters) was obtained in [23].

Corollary 1. *Equation* (50) *with* $\varphi''(z) \not\equiv 0$ *and Eq.* (52) *with* $b \neq 0$ *and* $\varphi''(z) \not\equiv 0$ *are not Painlevé-type equations.*

In the case $\varphi(z) = z$, system (5) was given in [22].

6.2. In considering system (6) we disregard the case

$$M(z)(-2z + M(z)) \not\equiv 0, \tag{61}$$

$$N(z)(-2z + N(z)) \not\equiv 0 \tag{62}$$

with

$$\alpha + 2\varepsilon = \beta = 0 \tag{63}$$

because the equations of system (6) under conditions (61)–(63) degenerate to $y + w = 0$.

Eliminating the unknown function y from system (6), we conclude that the function w satisfies the following family of differential equations:

- either Eq. (32), i.e., the third Painlevé equation (24) in the case $\gamma = -\delta = 1$,
- or the equation

$$P(z, w, w', w'', \varepsilon, \alpha, \beta) = 0, \tag{64}$$

  where

$$\begin{aligned} P(z, w, w', w'', \varepsilon, \alpha, \beta) = {} & z^4 w^8 + 2\alpha z^3 w^7 + 2z^2(\alpha(\alpha+6)+1)w^6 + \\ & + 2z(\beta z^2 + 6\varepsilon + \alpha(\alpha^2 + 5\alpha\varepsilon + 10))w^5 + \\ & + (-2z^4 - 4\beta\varepsilon z^2 + \alpha^4 + 22\alpha^2 + 8\alpha(\alpha^2+3)\varepsilon + 9)w^4 - \\ & - 2z(\alpha z^2 + \beta(\alpha^2 + 5\alpha\varepsilon + 4))w^3 + 2z^2(\beta^2 + \alpha\varepsilon + 1)w^2 - 2\beta z^3 w + z^4 + \\ & + z(z^3 w'^4 + 2wz^2(2wz + \alpha)\varepsilon w'^3 + \\ & + 2z(3z^2 w^4 + (\alpha - 4\varepsilon)zw^3 - (2\alpha\varepsilon+1)w^3 + \beta zw - z^2)w'^2 + \\ & + 2w(2\varepsilon z^3 w^5 + (2\varepsilon - 4)z^2 w^4 - (7\alpha z + z(\alpha^2+4)\varepsilon)w^3 + \\ & + (4\beta\varepsilon z^2 + \alpha^2 + 5\alpha\varepsilon + 4)w^2 - z(2\varepsilon z^2 + \beta - \alpha\beta\varepsilon)w - \\ & - z^2(\alpha\varepsilon + 2)w'' w - \alpha\varepsilon z^2)w' - 2zw^2(z(\alpha + 2\varepsilon)w^2 + \\ & + (\alpha^2 + 3\alpha\varepsilon + 2)w - \beta z)w''), \end{aligned}$$

- or the equation

$$Q(z, w, w', \varepsilon, \alpha, \beta) = 0, \tag{65}$$

  where

$$\begin{aligned} Q(z, w, w', \varepsilon, \alpha, \beta) = {} & (\alpha\varepsilon + 2)\{[zw' + \varepsilon z w^2 + (\alpha\varepsilon+1)w]^2 + z^2\} - \\ & - 2\beta z[zw' + \varepsilon z w^2 + (\alpha\varepsilon + 1)w]. \end{aligned}$$

Eliminating the unknown function w from system (6), we conclude that the function y satisfies the following family of differential equations:

- either
$$zyy'' = zy'^2 - yy' + (\alpha + 4\varepsilon)y^3 + \beta y + zy^4 - z, \tag{66}$$

- or
$$P(z, y, y', y'', -\varepsilon, \alpha + 4\varepsilon, \beta) = 0, \tag{67}$$

- or
$$Q(z, y, y', -\varepsilon, \alpha + 4\varepsilon, \beta) = 0. \tag{68}$$

It is easy to verify that relation (6a) can be obtained from (6b) using the scheme $y \to w$, $w \to y$, $\varepsilon \to -\varepsilon$, $\alpha \to \alpha + 4\varepsilon$, and $\beta = \beta$, and vise versa. By virtue of this, following the same scheme, we can obtain Eq. (66) from Eq. (32), Eq. (67) from Eq. (64), and Eq. (68) from Eq. (65).

The presence of the term $z^4w'^4$ and of one of the coefficients $-z^3(2\varepsilon + 2)ww'$ at w'' in Eq. (64) does not allow representing it as
$$w'' = L(z, w)w'^2 + S(z, w)w' + T(z, w), \tag{69}$$
where L, S, and T are rational functions of w with the coefficients analytic in z. According to [17] (see p. 437 in [17]), the necessary condition for the absence of movable critical points in the general solution of the equation
$$w'' = R(z, w, w'), \tag{70}$$
(where R is a rational function of w and w' with coefficients analytic in z), i.e., the validity of the Painlevé property, is the representation of (70) in form (69). Thus, Eq. (64) is not of Painlevé type. This can be easily verified, for instance, at $\alpha = -2\varepsilon$ and $\beta \neq 0$. Namely, at these values of the parameters, Eq. (64) does not satisfy the Painlevé test [24] and (according to [25]) does not have entire transcendental solutions (because of the presence of a single dominating term z^4w^8) and polynomial solutions other than $w = 0$. Thus, the following theorems hold.

Theorem 15. *Let $w = w(z, \alpha, \beta)$ be a solution of Eq.* (32) *with fixed values of the parameters α, β, and $\varepsilon^2 = 1$ such that $M(z)(-2z + M(z)) \not\equiv 0$ and $|\alpha + 2\varepsilon| + |\beta| \neq 0$. Then the function y defined in* (6a) *is a solution of Eq.* (64).

Theorem 16. *Let $y = y(z, \alpha, \beta, \varepsilon)$ be a solution of Eq.* (64) *with fixed values of the parameters α, β, and $\varepsilon^2 = 1$ such that $N(z)(-2z + N(z)) \not\equiv 0$ and $|\alpha + 2\varepsilon| + |\beta| \neq 0$. Then the function w defined in* (6b) *is a solution of Eq.* (32).

Consequently, under the conditions of Theorems 15 and 16, relations (6a) and (6b) with fixed values of α, β, and $\varepsilon^2 = 1$ establish a one-to-one correspondence (i.e., define the direct and inverse Bäcklund transformations) between solutions of the third Painlevé equation (24) with the respective parameters $(\alpha, \beta, 1, -1)$ and $(\alpha + 4\varepsilon, \beta, 1, -1)$. On the other hand, formulas (6a), (6b) establish the one-to-one correspondence between solutions of Eqs. (64) and (67) at fixed values of α, β, and $\varepsilon^2 = 1$.

Because the left-hand side of Eq. (65) with $\sigma\beta = \alpha\varepsilon + 2 \neq 0$ and $\sigma^2 = 1$ is a full square, this equation is equivalent to the equation
$$zw' + \varepsilon zw^2 + (\alpha\varepsilon + 1)w - \sigma z = 0, \tag{71}$$
all of whose solutions are also solutions of Eq. (24) with $\sigma\beta - \alpha\varepsilon - 2 = 0$ [14].

System (6) was given in [26].

7. Conclusions

In this paper, we have studied some analytic properties of solutions of second-order nonlinear differential equations of a special form with an arbitrary analytic function. We obtain the (direct and inverse) Bäcklund transformations for each of presented equations. For the presented equations (except one), we prove the existence of one-parameter families of solutions generated by solutions of the Riccati equations with an arbitrary analytic function. The considered equations (which are not equations of Painlevé type in general) under certain constraints on the analytic function can be reduced, in particular, to the second, third (the cases $\gamma = 0$, $\alpha = -\delta = 1$, and $\gamma = -\delta = 1$), or fourth Painlevé equations.

We have discussed the properties of the Bäcklund transformation of the second-order nonlinear differential equations generated by two systems of two first-order nonlinear equations with quadratic nonlinearities in the derivatives of unknown functions.

An important question is associated with the finiteness or infiniteness of the transformation groups admitted by the considered equations. In our opinion, answering requires performing the following studies.

- Find additional symmetries for solutions (if any) of the presented equations with an arbitrary function φ (such symmetries do exist for Painlevé equations) or other Bäcklund transformations, as is the case for Eq. (24) with $\gamma = 1$ and $\delta = -1$ or for Eq. (38) (see [23]).
- Study the behavior of movable singularities of solutions of the presented equations;
- Find (if possible) exact solutions of each equation and try to replicate them using the Bäcklund transformations.

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