ON THE PROBLEM OF INCONSISTENCY IN MANY-VALUED LOGICS

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It is considered a three-valued logical system with a subject area containing three values: 0, 1, 0.5. A three-valued formula is introduced — an analogue of the well-known statement in the liar's paradox. Assuming the axiom about the uniqueness of the truth value of the formula (that is, 0, 1 and 0.5), we obtain a paradox. A brief discussion of the result and conclusions is provided.

Introduction

It is known that the inconsistency of a logical theory makes it possible to derive any formula in it, which makes such a theory little acceptable for practice. One way out is to use para-consistent logic (s), for example [1–5]. In such logic, the system may be inconsistent, but not all formulas in it, including the false one, are deducible.

In essence, para-consistent logic is a meta-logic, that is, its formulas evaluate the formulas of subject theories, for example, true $(2 + 2 = 4)$. The mathematical power of para-consistent logic (s) seems weaker compared to the calculus about which they draw conclusions, but this reasoning requires quite a deep consideration. Another way out is to use many-valued logical calculi, for example, the logic of J. Lukasiewicz or fuzzy logic [6,7].

Thus, in the three-valued logic of J. Lukasiewicz, the formula $f \wedge \neg f$ has the value 0.5 (undefined), which is not 0 (false), as in standard propositional logic. At first glance, it can be assumed that this solves the problem of paradox (es). However, as we will show below, this is not the case. Thus, we can assume that the problem of paradoxes remains valid for any many-valued logical systems, including infinite-valued fuzzy logic.

I. Liar's paradox and its generalization

Let us give some preliminary definitions.

Definition. A contradiction in logic is a situation when, given an assumption (hypothesis) about the meaning of a particular formula, as a result of logically correct reasoning (conclusions), we come to the conclusion that the assumption was incorrect.

Definition. A formula of logical calculus is called contradictory if it cannot be assigned either a false or a true meaning without causing a contradiction.

Definition. A paradox in logic is a situation when, under an assumption (hypothesis) about the true meaning of a particular formula, as a result of logically correct reasoning (conclusions), we come to the conclusion that this formula must be false, and on the contrary, with an assumption (hypothesis) about the false meaning of the same formula, as a result of logically correct reasoning (conclusions), we come to the conclusion that this formula must be true.

Paradoxes in general do not mean that a particular theory is incorrect, but their presence can lead to incorrect conclusions. They serve as an important tool for analyzing and enhancing theories, stressing the necessity to clarify and revise logical system and its philosophical or formal premises.

The well-known liar's paradox consists in the statement that a formula that asserts its own falsity can be neither true nor false, that is, the statement: this formula is false is an example of a paradox. We will use the fact that each formula of logical calculus can be assigned a unique number due to the well-known diagonalization lemma of K. Gedel [2].Let the statement be *false* (this formula) with Gedel number z. Then *false* (z) is a statement about the falsity of a formula with number z, which refers to itself, and therefore can be neither true nor false.

Let's consider the three-valued version: *val*(x,y), where *val*(x,y) evaluates the truth of the formula with number x, and y is the measure (value) of truth indicated for the formula with number x. For example, *val*(1=1, 0.5) evaluates the truth measure of the statement that the truth measure of the formula 1=1 is equal to 0.5. Since 1=1 in reality, the estimate 0.5 is incorrect in the strict sense, so it is fair to assume that *val*(1=1, 0.5)=0. Such reasoning is related to the formal model used, however. On the other hand, the formula *val*(1=0.5, 0.5) can be evaluated, for example, as 1.

We accept the following **axiom** as a premise: a formula that has more than one truth value is inconsistent, that is, the truth value of a formula can only be one of three: 0, 0.5, 1. Refusal of this **axiom** gives a different perspective and goes beyond the scope of our discussions. In connection to this **axiom**, each formula which has two or more truth values is considered contradictory.

Consider a formal model in which only three numbers occur: 0, 0.5 and 1.

Let's define fuzzy measures for equalities: 0=0, 1=1, 0.5=0.5 all have a measure of 1. 0=0.5, 1=0.5 have a truth measure of 0.5. 1=0 has a truth degree of 0. Let the formula *val*(x,0.5) evaluate the validity of the statement that the formula with number x has a truth degree of 0.5. Let this formula itself, which we denote by W, have the number w.

Let's find out what the value of *val*(w,0.5) is. Let *val*(w, 0.5)=0.5. But 0.5=0.5, that is, the formula W with number w has a truth value of 1, which gives a contradiction due to the **axiom**. Let *val*(w=0.5)=1 (0). But 0.5 is not equal to 1 (0), which gives an uncertainty value of the type 1=0.5 or 0=0.5 in the formal system under consideration, that is, *val*(w=0.5) should in this case change the answer to 0.5. And this is a true statement with a score of 1. Again, one gets a contradiction, and we have a paradox.

This reasoning also uses K. Gedel's diagonalization lemma. Here is its wording:

For any formula *h*(x) in language L that contains a single free variable x, there is a formula *diag*(h) in L such that:

1. *diag*(h) is equivalent to the formula *h*(g), where g is the number of the formula *diag*(h).
2. The formula *diag*(h) speaks about itself, i.e. it states that a formula numbered g has the property asserted by *h*.

In our case, the only free variable in the formula *val*(w,0.5) is w.

II. Discussion of the result

The reasons for logical paradoxes are different. Here are some of them [8]: unacceptable applicability of formulas to themselves, an attempt to expand a completed set, a change in reality (other reasons and their combinations are possible). The case considered in the article relates to unacceptable self-applicability.We did not set out to consider how to prevent this kind of paradox. In the general case, many-valued logics are specifically focused on preventing paradoxes. In our case, the axiom used can be considered as an obstacle to correct reasoning, since the obtained result does not remove from the agenda the question of the inconsistency of a particular formal system and the presence of paradoxical formulas in it, since such formulas do not generally allow decisions to be made in a deterministic way. They are looking for a way out in ambiguity and para-consistency.The presented material can substantiate doubts about the freedom of many-valued logics from inconsistency. In this case, the axiom we used is significant. It does not allow two or more different conclusions about the truth of formulas to be done. This is important, for example, in technical systems where control is implemented along one trajectory, and not simultaneously along several others.If we take the use of meta-theories as the basis for confronting inconsistency (paradoxical character), then questions arise about the resolving power of these meta-theories. These questions are relevant and need to be studied.

III. Conclusion

Many-valued logics are supposed to be free from paradoxes because they use the value 0.5 for this kind of situation. We have shown that the three-valued logic of J. Lukasiewicz contains an example of a paradoxical formula. A similar argument can be formulated for logics with more than 3 truth values. This last issue state some research direction, especially with regards to fuzzy logic. For the last, for instance, it is known that modus ponens inference rule may not be valid in every case as produces incorrect conclusion. That is why para-consistent fuzzy logics may take the front-line positions.

IV. References

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