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关于Navier-Stokes方程微分差分问题的可解性

ON THE SOLVABILITY OF A DIFFERENTIAL-DIFFERENCE PROBLEM FOR THE NAVIER-STOKES EQUATIONS

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摘要。研究了矩形管道（三维情况）中 Navier-Stokes 方程组的可解性。研究在步长为 τ 的时间层上进行。
关键词：泊松方程，椭圆算子，Banach 空间，边界算子 Cα $(\overline{\Omega})$，外法向量。
Abstract. *The solvability of the Navier-Stokes equations system in a rectangular pipe (three-dimensional case) is investigated. The study is performed on time layers with a step* τ.
Keywords: *Poisson equation, elliptic operator, Banach space, boundary operator* $C^{\alpha}(\overline{\Omega})$*, outer normal vector.*

The paper investigates the existence and uniqueness of a solution to a boundary value problem for the Navier-Stokes equations in a rectangular pipe with smoothed corners. First, we smooth out all the dihedral and trihedral angles and obtain the region shown in Figure 1 (first smoothing). The vertices are also marked on it. $A, B, C, D, A_1, B_1, C_1, D_1$ of the original rectangular pipe, which are numbered in the order specified above by digits (in the figure, the digits are shown in parentheses near the corresponding vertices).

Rectangle AA_1B_1B is the entrance to the original pipe (before smoothing the corners), rectangle CC_1D_1D is its exit. Continuing smoothing, we smooth the entrance and exit in such a way that the smoothed surface, for values , x_1, satisfying the inequalities: $0 \le x_1 < \delta$ and $L - \delta < x_1 \le L$ (see Figure 1), does not contain either flat or rectilinear parts and is a convex smooth surface, being a connected open set on the boundary of the convex body (second smoothing). This part of the surface is not solid. The solid part of the surface lies in the interval $\delta \le x_1 \le L - \delta$. To clarify, we note that after the second smoothing, the plane touches the smoothed surface at the point $x^{O}(0, \frac{H_2}{2}, \frac{H_3}{2})$, and the plane $x_1 = L -$ touches the point $x^{L}(L, \frac{H_2}{2}, \frac{H_3}{2})$.

Let us adopt the following notations:
$x=(x_1,x_2,x_3)$, $0\le x_1\le L, 0\le x_2\le H_2, 0\le x_3\le H_3$, $0\le t\le T$, $\Omega=(0,L)\times(0,H_2)\times(0,H_3)$, edges:

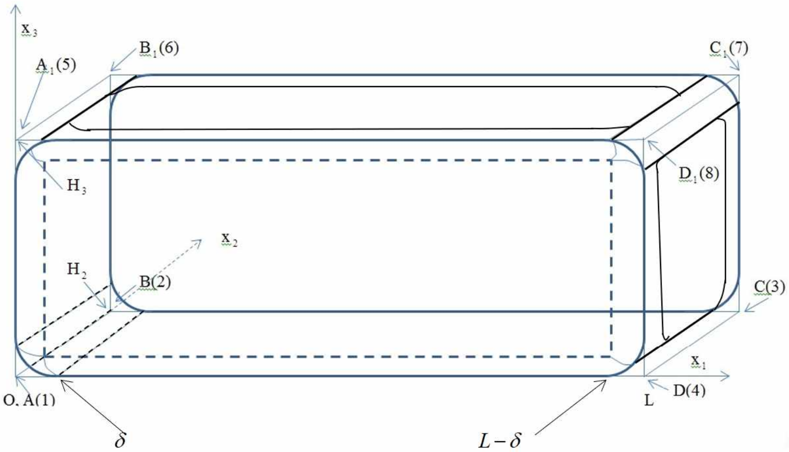

Figure 1.

$S_1=[0\le x_1\le L, 0\le x_2\le H_2, x_3=0]$ – lower, $S_2=[0\le x_1\le L, 0\le x_2\le H_2, x_3=H_3]$ – top, $S_3=[0\le x_1\le L, x_2=0, 0\le x_3\le H_3]$ – front, $S_4=[0\le x_1\le L, x_2=H_2, 0\le x_3\le H_3]$ – back, $S_5=[x_1=0, 0\le x_2\le H_2, 0\le x_3\le H_3]$ – left, $S_6=[x_1=L, 0\le x_2\le H_2, 0\le x_3\le H_3]$ – right (these are the edges of the original region), $S=\bigcup_{k=1}^{6}S_k$ – boundary of the region Ω, $S_T=S\times[0,T]$, $\Omega_T=\Omega\times[0,T]$.

Let us designate $\tilde{S}$ – the surface obtained from the surface S as a result of the second smoothing, $\tilde{\Omega}$ – area bounded by a surface $\tilde{S}$, 00$\overline{\tilde{\Omega}}=\tilde{\Omega}\cup\tilde{S}, \tilde{\Omega}_T=\tilde{\Omega}\times[0,T]$, $\tilde{S}_T=\tilde{S}\times[0,T]$ $\overline{\tilde{\Omega}}_T=\overline{\tilde{\Omega}}\times[0,T]$. The solid part of the surface (it lies in the gap $\delta\le x_1\le L-\delta$) let us designate $\tilde{\tilde{S}}$.

Let's consider the problem (density $\rho=1$, on a hard surface $u_i|_{\tilde{\tilde{S}}}=0$, $i=1,2,3$):

$$\frac{\partial u_i}{\partial t}=\nu\sum_{k=1}^{3}\frac{\partial^2 u_i}{\partial x_k^2}-\sum_{k=1}^{3}u_k\frac{\partial u_i}{\partial x_k}-\frac{\partial p}{\partial x_i},\quad i=1,2,3;\qquad (x,t)\in\tilde{\Omega}_T, \tag{1}$$

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0,\quad (x,t)\in\overline{\tilde{\Omega}}_T, \tag{2}$$

$$u_1|_{t=0}=\bar{b}_1(x)\ x\in\overline{\tilde{\Omega}},\quad \bar{b}_1|_{\tilde{\tilde{S}}}=0,\quad u_1|_{\tilde{S}_T}=\tilde{\psi}_1(s,t)\ (s,t)\in\tilde{S}_T. \tag{3}$$

We proceed to setting the initial conditions for the velocity components u_2 and u_3. The set of actions for finding the functions $\bar{u}_2$, $\bar{u}_3$, we denote as point 1).

1). Assuming for now that $\frac{\partial u_3}{\partial x_3}=0$, we solve the equation $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0$ for the velocity u_2. For any fixed value x_3, $0 \le x_3 \le H_3$, we solve this equation in the same way as it is solved in [1]. The resulting solution, which we denote by u_2^0, as shown in [1], satisfies both the equation being solved and the no-slip conditions on solid sections of the boundary. Having obtained the solution u_2^0, we consider the equation (in the region $\widetilde{S}$)

$$\nu\sum_{k=1}^{2}\frac{\partial^2 u_2}{\partial x_k^2}-u_1\frac{\partial u_2}{\partial x_1}-u_2^0\frac{\partial u_2}{\partial x_2}=0\cdot$$

This equation, according to the well-known Schauder theorem (its proof is, for example, in [2]), has a unique smooth solution (in an arbitrary section $\widetilde{S}$ of the domain by a plane $x_3=x_3^{(0)}$, $0<x_3^{(0)}<H_3$), which we denote $\overline{u}_2$. Solution $\overline{u}_2$, for any fixed value x_3, $\le x \le H$ (at $x_3=0$ and $x_3=H_3$ $\overline{u}_2=0$), we continue until the entrance and exit of the entire three-dimensional region with vertices $A, B, C, D, A_1, B_1, C_1, D_1$. The latter is performed in the same way as, for example, in [1]. For the extended function, now defined in the entire rectangular parallelepiped, we retain the previous notation $\overline{u}_2$. Similar to solving the equation $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0$, it was decided above relatively u_2, we solve the equation $\frac{\partial u_1}{\partial x_1}+\frac{\partial \overline{u}_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}=0$ relatively u_3 with known functions u_1 and $\overline{u}_2$. Indeed, I suppose $\frac{\partial u_1}{\partial x_1}+\frac{\partial \overline{u}_2}{\partial x_2}=\widetilde{u}$, we get the equation $\widetilde{u}+\frac{\partial u_3}{\partial x_3}=0$ of type $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}=0$ and we denote its solution as $\overline{u}_3$. Function $\overline{u}_3$ satisfies the equation

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial \overline{u}_2}{\partial x_2}+\frac{\partial \overline{u}_3}{\partial x_3}=0 \tag{4}$$

2). So, in point 1) the functions (transverse velocities) $\overline{u}_2$ and $\overline{u}_3$. In this point 2), by swapping the roles: functions u_2 and u_3, variables x_2 and x_3, i. e. $u_2 \leftrightarrow u_3$, $\frac{\partial u_1}{\partial x_1}+\frac{\partial \overline{u}_2}{\partial x_2}+\frac{\partial \overline{u}_3}{\partial x_3}=0$ $x_2 \leftrightarrow x_3$, starting with solving the equation $\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}=0$ and performing actions completely analogous to the actions of point 1), we find the functions $\overline{\overline{u}}_3$ and $\overline{\overline{u}}_2$, satisfying the equation

$$\frac{\partial u_1}{\partial x_1}+\frac{\partial \overline{\overline{u}}_2}{\partial x_2}+\frac{\partial \overline{\overline{u}}_3}{\partial x_3}=0. \tag{5}$$

Having designated $u_{2,0}=\frac{1}{2}(\overline{u}_2+\overline{\overline{u}}_2)$, $u \quad =- \ \overline{u} \ +\overline{\overline{u}}$, Let's add (4) and (5). As a result, we get

$$2\frac{\partial u_1}{\partial x_1}+2\cdot\frac{1}{2}\frac{\partial(\overline{u}_2+\overline{\overline{u}}_2)}{\partial x_2}+2\cdot\frac{1}{2}\frac{\partial(\overline{u}_3+\overline{\overline{u}}_3)}{\partial x_3}=0 \quad \text{or} \quad \frac{\partial u_1}{\partial x_1}+\frac{\partial u_{2,0}}{\partial x_2}+\frac{\partial u_{3,0}}{\partial x_3}=0.$$

The initial value of the longitudinal velocity was known and given by the equality $u_1|_{t=0}=\overline{b}_1(x)$. Let's put it for uniformity $u_1|_{t=0}=u_{1,0}$, then we get $\frac{\partial u_{1,0}}{\partial x_1}+\frac{\partial u_{2,0}}{\partial x_2}+\frac{\partial u_{3,0}}{\partial x_3}=0\cdot$

So, we have initial conditions for all equations (1), and they satisfy the continuity equation (2). Let us introduce the notation

$$A_i^{(0)} = \nu \sum_{k=1}^{3} \frac{\partial^2 u_{i,0}}{\partial x_k^2} - \sum_{k=1}^{3} u_{k,0} \frac{\partial u_{i,0}}{\partial x_k}, \quad i = 1,2,3$$

(at $t = 0$ we believe $\frac{\partial u_i}{\partial t} = 0$) and agreement: further, for pairs of identical indices, summation is implied with a change in the identical index from the number to, in particular, if necessary, write down the sum $\sum_{i=1}^{3} b_i(x) \frac{\partial p}{\partial x_i}$ You can apply both the left and right sides of the following equation: $\sum_{i=1}^{3} b_i(x) \frac{\partial p}{\partial x_i} = b_i(x) \frac{\partial p}{\partial x_i}$. Let's consider Poisson's equation

$$\sum_{i=1}^{3} \frac{\partial^2 p}{\partial x_i^2} = \sum_{i=1}^{3} \frac{\partial A_i^{(0)}}{\partial x_i}, \qquad x \in \tilde{\Omega}$$

with a condition on the border $\tilde{S}$: $b_i(x) \frac{\partial p}{\partial x_i} + b(x) p = \varphi(s)$, where $x = s \in \tilde{S}$. Functions $b_i(x)$, $b(x)$ and $\varphi(s)$ we define below. Introducing denoting: for the Laplace operator $\sum_{i=1}^{3} \frac{\partial^2 p}{\partial x_i^2}$ – Δp , for boundary operator $b_i(x) \frac{\partial p}{\partial x_i} + b(x) p$ – Bp, for function $\sum_{i=1}^{3} \frac{\partial A_i^{(0)}}{\partial x_i}$ – $f(x)$, we get the task

$$\Delta \;=\; (\;), \quad Bp|_{\tilde{S}} = \varphi(s). \tag{6}$$

It is known that if an elliptic operator has the form $Lu \equiv a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + a_i(x) \frac{\partial u}{\partial x_i} + a(x) u$, then for the task

$$Lu - \lambda u = f(x), \quad Bu|_{\tilde{S}} = \varphi(s), \tag{7}$$

where λ – complex parameter, the following theorem is true.

Theorem 1. *Let the boundary* $\tilde{S}$ *surface* $\tilde{\Omega}$ *be a surface of class* $C^{2+\alpha}$, *coefficients* a_{ij}, a_i, a *of the operator* L *belong* $C^{\alpha}(\overline{\tilde{\Omega}})$ *and the ellipticity condition is satisfied*

$$a_{ij}(x) \xi_i \xi_j \geq \nu \xi^2, \quad \nu = const > 0.$$

Let the coefficients $b_i(x)$ *and* $b(x)$ *boundary operator* $Bu \equiv b_i(x) \frac{\partial u}{\partial x_i} + b(x) u$ *there are elements* $C^{1+\alpha}(\tilde{S})$,

$$\left. \sum_{i=1}^{n} b_i(x) \cos(\overline{n}, x_i) \right|_{\tilde{S}} \geq \nu_0, \quad \nu_0 = const > 0 \tag{8}$$

and let it be $f(x) \in C^{\alpha}(\overline{\tilde{\Omega}})$, $\varphi(s) \in C^{1+\alpha}(\tilde{S})$. *When problem (7) is uniquely solvable in* $C^{2+\alpha}(\overline{\tilde{\Omega}})$ *for any* f *u* φ *from the specified classes for all* λ, *except for no more than a countable number of values* λ_1, λ_2, ..., *constituting the spectrum of problem (7). The inhomogeneous problem (7) is not solvable for all* f *and* φ (see. [4], chapter III, § 3).

Note 1. From the proof of Theorem 1 given in [4], in particular, it follows that the problem is uniquely solvable $Lu = f$, $Bu|_{\tilde{S}} = \varphi(s)$, i. e. tasks (7) at $\lambda = 0$

. But the point is that this occurs when the inequalities are satisfied $a(x) \le 0$ and $b(x) > 0$. In general, any case $a(x)$ from $C^{\alpha}(\overline{\widetilde{\Omega}})$ and $b(x)$ from $C^{1+\alpha}(\widetilde{S})$ this is not true if the value $\lambda = 0$ belongs to the spectrum specified in Theorem 1.

Let in the problem $\Delta p = f(x)$, to be inequality $Bp|_{\widetilde{S}} = \varphi(s)$ $b(x) > 0$ and on the border $\widetilde{S}$ function $b(x)$ is bounded above by some sufficiently small positive constant β, i.e. the inequality is true $b(x) \le \beta$, $\beta > 0$. Then the task is

$$\Delta p = f(x), \quad Bp|_{\widetilde{S}} = \varphi(s),$$

where $Bp = b_i(x)\dfrac{\partial p}{\partial x_i} + b(x)p$, under the conditions of Theorem 1, it is uniquely solvable.

We will achieve the fulfillment of condition (8) by setting $b_1(x) = \frac{1}{2} + \cos(\overline{n}, x_1)$, $b_i(x) = \eta_i(x) + \cos(\overline{n}, x_i)$, $i = 2,3$, where $\cos(\overline{n}, x_i)$, $i = 1,2,3$ – direction cosines of a unit vector $\overline{n} = \overline{n}(x)$ external normal to $\widetilde{S}$ at the point x, and $\eta_i(x)$ is defined as follows: $\eta_i(x) = -1$, at $0 \le x_i \le \frac{1}{3}H_i$; $-1 \le \eta_i(x) \le 1$, at $\frac{1}{3}H_i \le x_i \le \frac{2}{3}H_i$; $\eta_i(x) = 1$, at $\frac{2}{3}H_i \le x_i \le H_i$; $i = 2,3$, at the same time $x \in \overline{\widetilde{\Omega}}$, function $\eta_i(x)$ fairly smooth, changes from to monotonously and $\eta_i(x)$ at $x_i = \frac{1}{2}H_i$ equals zero: $\eta_i(x)\big|_{x_i = 0.5H_i} = 0$.

Note: when the point $x \in \widetilde{S}$ and $x_i = \frac{1}{2}H_i$ $(i = 2,3)$, $x_1 = 0$ or $x_1 = L$. This means that $b_i(x) = \eta_i(x) + \cos(\overline{n}, x_i)$, $i = 2,3$, vanishes only at points on the surface $\widetilde{S}$ with coordinates $(0, \frac{H_2}{2}, \frac{H_3}{2})$ и $(L, \frac{H_2}{2}, \frac{H_3}{2})$.

Problem (6) takes the form

$$\Delta p = f(x), \quad Bp|_{\widetilde{S}} = \varphi(s), \tag{9}$$

where $f(x) = \sum_{i=1}^{3} \dfrac{\partial A_i^{(0)}}{\partial x_i}$, coefficients $b_i(x)$ and $b(x)$ indicated above, $\varphi(x) = b_i(x)A_i^{(0)}$. We see that the boundary condition has the form $b_i(x)\dfrac{\partial p}{\partial x_i} + b(x)p = b_i(x)A_i^{(0)}$, which is not consistent with equations (1), each of which, $b_i(x) \ne 0$ can be written as $b_i(x)\dfrac{\partial p}{\partial x_i} = b_i(x)A_i^{(0)}$ (here the convention of writing the sum is not applied, but three equalities are written). Summing the last three equations, we obtain $b_i(x)\dfrac{\partial p}{\partial x_i} = b_i(x)A_i^{(0)}$ (now the convention of recording the amount is applied). The following considerations will help to get rid of the indicated discrepancy (they are contained in [4], Chapter X, § 1).

In a limited area $\widetilde{\Omega}$ The following types of problems are considered:

$$L(u) \equiv a_{ij}(x, u, u_x)\frac{\partial^2 u}{\partial x_i \partial x_j} + a(x, u, u_x) = 0, \tag{10}$$

$$L^{(\widetilde{S})}(u) \equiv \left[b(x, u, u_x) + b_i(x, u)u_{x_i} + b_0(x, u) \right]_{\widetilde{S}} = 0, \tag{11}$$

under the assumption that equation (10) is uniformly elliptic and for arbitrarily fixed u and p and at each point $x \in \widetilde{S}$ vector $\widetilde{l}(x,u,p)$ with components $b_{p_i}(x,u,p)+b_i(x,u)$ does not lie in tangent to $\widetilde{S}$ planes. More precisely, it is believed that

$$\left[b_{p_i}(x,u,p)+b_i(x,u)\right]\cos(\overline{n},x_i) \geq \nu_1(|u|,|p|), \quad \nu_1 > 0 . \tag{12}$$

The question of the solvability of problems (10), (11) is reduced to the question of the existence of fixed points for transformations with good properties, and some transformation is considered $u = \Phi(v)$. It is noted that the Leray-Schauder criterion for the existence of fixed points cannot be applied to the transformation under consideration, and it is immediately noted that this is possible in the case when $b(x,u,u_x) \equiv 0$. Recalling problem (9), we see that in our case the last identity holds . And yet we will dwell on the transformation , since it is necessary to get rid of the term in the boundary operator. The consideration leads to the proof of the theorem on the solvability of abstract equations in Banach spaces and the subsequent clarification of the requirements for $\Phi(v)$.

Theorem 2. *Let* X *and* Y *– be two Banach spaces,* I *– be a segment of* $[0,1]$ *, a* x, y *and* τ *– elements* X, Y *and* I *respectively. Let's suppose* Φ *– continuous mapping of the direct product* $X \times I$ *в* Y *, having a derivative* $\Phi_x(x,\tau)$ *, continuous with* (x,τ) *respect to in the operator topology* $L\{X \to Y\}$*, and satisfying the following conditions:*

1) For any solution x *of the equation*

$$\Phi(x,\tau) = 0, \tag{13}$$

answering to an arbitrary τ *from* I*, operator* $\Phi_x(x,\tau)$ *has a limited inverse* $\Phi_x^{-1}(x,\tau)$: $Y \to X$.

2) The set of all solutions of equation (13) that correspond to all $\tau \in I$, *compact in space* X.

3) For some fixed τ *from* I *there is only one solution* x *equations* (13).

Then for each $\tau \in I$ *equation (13) is uniquely solvable in* X.

To reduce the solution of problem (10), (11) to Theorem 2, two Banach spaces are introduced: as X is taken $C^{2+\alpha}(\overline{\widetilde{\Omega}})$, and as a space of pairs of elements $y = \{f, \varphi\}$, where $f(x) \in C^{\alpha}(\overline{\widetilde{\Omega}})$, $\varphi(s) \in C^{1+\alpha}(\widetilde{S})$, with the norm $\|y\|_Y = |f|_{\widetilde{\Omega}}^{(\alpha)} + |\varphi|_{\widetilde{S}}^{(1+\alpha)}$.

Problem (10), (11) is included in the family of problems that depend on the parameter $\tau \in [0,1]$:

$$\left.\begin{aligned} L_\tau(u) &\equiv \tau L(u) + (1-\tau)L_0(u) = 0, x \in \widetilde{\Omega}, \\ L_\tau^{(\widetilde{S})}(u) &\equiv \tau L^{(\widetilde{S})}(u) + (1-\tau)L_0^{(\widetilde{S})}(u) = 0, x \in \widetilde{S}, \end{aligned}\right\} \tag{14}$$

where L_0 и $L_0^{(\widetilde{S})}$ – differential operators of the same type as L and $L^{(\widetilde{S})}$ accordingly, and with the value $\tau = 0$ of problem (14) is uniquely solvable in $C^{2+\alpha}(\overline{\widetilde{\Omega}})$. It is clear that taking as an operator $L_0^{(\widetilde{S})}$ operator $b_i(x)\dfrac{\partial u}{\partial x_i} + b(x)u$, we will achieve solvability of (14) for the value $\tau = 0$ and satisfy condition 3) of Theorem 2. Without further considering the contents of § 1, Chapter X of [4], we simply note that it implies the existence of a unique solution to problem (9) with the boundary operator $b_i(x)\frac{\partial p}{\partial x_i}$. Indicating this solution p_0, we begin to solve the system (1) – (3) with the values $t > 0$.

To find functions u_i, $i = 1,2,3$, and pressure p at values $t > 0$ we will resort to Rothe's method, which essentially reduces the proofs of existence theorems for solutions of initial-boundary value problems for parabolic equations to boundary value problems for elliptic equations (see [1], [3]).

We will dissect the cylinder Ω_T planes $t_m \quad m\tau$, $m = 0, 1, \dots, M$, $\tau = \frac{T}{M}$, and we denote $\tilde{\Omega}_m$ section $\tilde{\Omega}_T$ plane $t_m = m\tau$, $\tilde{S}_m$ – its border, $\overline{\tilde{\Omega}}_m = \tilde{\Omega}_m \cup \tilde{S}_m$. At each section $\tilde{\Omega}_m$ Let's define the functions that we will denote $u_{1,m}$, $u_{2,m}$, $u_{3,m}$, p_m, $m = \overline{0,M}$. The solution was found above for the value $t = 0$, i. e. functions $u_{1,0}$, $u_{2,0}$, $u_{3,0}$, p_0. To find the values $u_{1,m}$, $u_{2,m}$, $u_{3,m}$, p_m at $m = 1,2,\dots,M$ we introduce the difference derivative and replace the derivatives in equations (1) – (3) $\frac{\partial u_i}{\partial t}$ difference derivatives $u_{i\bar{t}} = \frac{1}{\tau}(u_{i,m} - u_{i,m-1})$. Since now the derivatives $\frac{\partial u_i}{\partial t} \neq 0$, then the expressions for will change accordingly $A_i^{(0)} = \nu\sum_{k=1}^{3}\frac{\partial^2 u_{i,0}}{\partial x_k^2} - \sum_{k=1}^{3} u_{k,0}\frac{\partial u_{i,0}}{\partial x_k}$, $i = 1,2,3$. The movement along time sections occurs in the same way as in [1]. Let us arrive at the theorem.

Theorem 3. *Let the following conditions be satisfied:* $\tilde{S} \in C^{3+\alpha}$, $\overline{b}_1(x) \in C^{3+\alpha}(\overline{\tilde{\Omega}})$, $\tilde{\psi}_1 \in C^{3+\alpha}(\tilde{S}_T)$, $f \in C^{\alpha}(\overline{\tilde{\Omega}})$, $\varphi \in C^{1+\alpha}(\tilde{S})$. *Then problem (1) – (3), in which the derivatives* $\frac{\partial u_i}{\partial t}$ *replaced by difference derivatives, for any* $t = t_m = m\tau$, $m = \overline{0,M}$, *and small enough has a unique solution, and* $u_{i,m} \in C^{3+\alpha}(\overline{\tilde{\Omega}}_m)$, $p_m \in C^{2+\alpha}(\overline{\tilde{\Omega}}_m)$.

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