DECOMPOSITION OF BASIC GRAPHS IN FRACTIONAL LINEAR PROGRAMMING PROBLEMS WITH LINEAR CONSTRAINTS

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Efficient computational technologies of decomposing basis graphs in linear-fractional multiflow programming problems with a nested constraint structure are considered. An example of basis graphs decomposition for solving large sparse linear systems for mathematical model of one linear-fractional programming problem are given.

Introduction

Decomposition of the sparse linear underdetermined system with incidence matrix of the graph are considered. Systems of this type appear in nonhomogeneous flow programming problems.

We construct the algorithms of decomposition of the sparse underdetermined linear systems by separating the variables according to the sets of arcs of spanning trees, cyclic arcs and no basis arcs. Effective algorithms and decomposition technologies have been developed for solving sparse systems of linear algebraic equations with a multigraph incidence matrix in the linear-fractional multiflow programming problems with a nested constraint structure are considered. An implementation of the sparse systems solutions in the computer algebra system Wolfram Mathematica with using the technologies of root trees are considered. The example of decomposition for basis graphs of the sparse systems of linear algebraic equations is given.

I. Mathematical model of linear-fractional optimization problem with linear constraints

For the multinetwork $G = (I,U)$ we consider the following linear-fractional programming problems with linear constraints

$$f(x) = \frac{p(x)}{q(x)} = \frac{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} p^k_{ij} x^k_{ij} + \beta}{\sum\limits_{(i,j)\in U} \sum\limits_{k\in K(i,j)} q^k_{ij} x^k_{ij} + \gamma} \longrightarrow \max, \quad (1)$$

$$\sum_{j\in I^+_i(U^k)} x^k_{ij} - \sum_{j\in I^-_i(U^k)} x^k_{ji} = a^k_i, i \in I^k, k \in K;$$

$$\sum_{k\in K_0(i,j)} x^k_{ij} \le d^0_{ij}, (i,j) \in U_0; \quad (2)$$

$$\sum_{(i,j)\in U} \sum_{k\in K(i,j)} \lambda^{kp}_{ij} x^k_{ij} = \alpha_p, p = \overline{1,l};$$

$$x^k_{ij} \ge 0, k \in K_0(i,j), (i,j) \in U_0; \quad (3)$$

$$0 \le x^k_{ij} \le d^k_{ij}, \in K_1(i,j), (i,j) \in U,\ x^k_{ij} \ge 0,$$

$$k \in K(i,j) \setminus K_1(i,j), (i,j) \in U \setminus U_0;$$

$$I^+_i(U^k) = \{j \in I^k : (i,j)^k \in U^k\};$$

$$I^-_i(U^k) = \{j \in I^k : (j,i)^k \in U^k\}. \quad (4)$$

Here K ($|K| < \infty$) is a set of different products (types of flow) transported through the multinetwork G. Without loss of generality, let's put $K = \{1, \ldots, |K|\}$.

Let us denote the connected network corresponding to a certain type k of flow with $S^k = (I^k, U^k)$, where I^k is the set of nodes and U^k is the set of arcs which are available for the flow of type k, $k \in K$. Also, we define for each node $i \in I$ the set of types of flows $K(i) = \{k \in K : i \in I^k\}$ and for each multiarc $(i,j) \in U$ the set $K(i,j) = \{k \in K : (i,j)^k \in U^k\}$. We assume that the denominator $q(x)$ of the objective function (1) does not change sign on a set of multiflows X, $x \in X$.

The work is devoted to methods, algorithms and technologies decomposition of basis multigraphs for constructing the optimal solutions of linear-fractional programming problem (1)–(4). We use modern innovative technologies of sparse matrix analysis [2], algorithmic graph theory and theoretical computer science.

II. Example of basis graphs decomposition

For a multinetwork $G = (I,U)$, $I = \{1,2,3,4,5\}$, $U = \{(1,2),(2,3), (4,1),(4,2), (4,3),(5,1),(5,2), (5,4)\}$ consider a sparse underdetermined system of linear algebraic equations (5) – (6). Multinetwork G presented as a combination of networks S^k (see Fig. 1): $S^k = (I^k, U^k)$, $k \in K = \{1,2,3\}$,

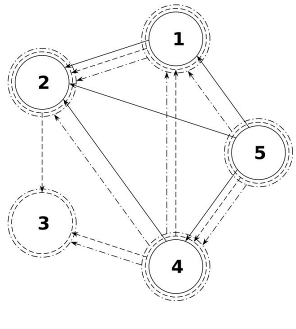

Figure 1 – Multinetwork $G = (I,U)$, Combination of networks S^k, $k \in K = \{1,2,3\}$

$$U^1 = \{(1,2)^1,(4,2)^1,(5,1)^1,(5,2)^1,(5,4)^1\},$$
$$U^2 = \{(1,2)^2, (2,3)^2, (4,1)^2, (4,3)^2, (5,4)^2\},$$

$$U^3 = \{(1,2)^3,\ (4,1)^3,\ (4,2)^3,\ (5,1)^3,\ (5,4)^3\}.$$

$$x^1_{1,2} - x^1_{5,1} = 5, \quad -x^1_{1,2} - x^1_{4,2} - x^1_{5,2} = -13$$
$$x^1_{4,2} - x^1_{5,4} = -1, \quad x^1_{5,1} + x^1_{5,2} + x^1_{5,4} = 9$$

$$x^2_{1,2} - x^2_{4,1} = -1, \quad x^2_{2,3} - x^2_{1,2} = 7$$
$$-x^2_{2,3} - x^2_{4,3} = -13, \quad x^2_{4,1} + x^2_{4,3} - x^2_{5,4} = -1$$
$$x^2_{5,4} = 8$$

$$x^3_{1,2} - x^3_{4,1} - x^3_{5,1} = -4, \quad -x^3_{1,2} - x^3_{4,2} = -10$$
$$-x^3_{4,3} = -6, \quad -x^3_{4,1} + x^3_{4,2} + x^3_{4,3} - x^3_{5,4} = 12$$
$$x^3_{5,1} + x^3_{5,4} = 8 \tag{5}$$

$$x^1_{1,2} + 4x^2_{1,2} + 7x^3_{1,2} + 4x^2_{2,3} + 5x^2_{4,1} +$$
$$+4x^3_{4,1} + 3x^1_{4,2} + 6x^3_{4,2} + 6x^2_{4,3} + 2x^3_{4,3} + 9x^1_{5,1} +$$
$$+3x^3_{5,1} + 10x^1_{5,2} + 4x^1_{5,4} + 2x^2_{5,4} + 9x^3_{5,4} = 328$$

$$10x^1_{1,2} + 6x^2_{1,2} + 5x^3_{1,2} + 2x^2_{4,1} + 4x^3_{4,1} + 9x^1_{4,2} +$$
$$+10x^2_{4,3} + 4x^3_{4,3} + 4x^1_{5,1} + 2x^3_{5,1} + 4x^1_{5,2} + 7x^1_{5,4} +$$
$$+7x^2_{5,4} + 10x^3_{5,4} = 412$$

$$5x^1_{1,2} + 8x^2_{1,2} + x^3_{1,2} + 7x^2_{2,3} + 9x^2_{4,1} + 5x^3_{4,1} +$$
$$+2x^3_{4,2} + 6x^2_{4,3} + 5x^3_{4,3} + 4x^1_{5,1} + +x^3_{5,1} + 7x^1_{5,2} +$$
$$2x^1_{5,4} + 8x^2_{5,4} + 5x^3_{5,4} = 359 \tag{6}$$

Basis graphs(support) $U^k_T \bigcup U^k_C$, $k \in K = \{1,2,3\}$ for the multinetwork $G = (I,U)$ for the system (5) – (6) [1] is represented on figures 2 – 4, where $U^1_T = \{(1,2)^1,\ (4,2)^1,\ (5,4)^1\}$, $U^2_T = \{(1,2)^2,(2,3)^2,(4,3)^2,(5,4)^2\}$, $U^3_T = \{(1,2)^3,(4,2)^3,(4,3)^3,(5,4)^3$ – sets of arcs of spanning trees U^1_T, U^2_T, U^3_T of the graphs $S^1 = (I^1,U^1)$, $S^2 = (I^2,U^2)$, $S^3 = (I^3,U^3)$ respectively (marked with bold lines), $U_C = U^1_C \bigcup U^2_C \bigcup U^3_C$ – set of cyclic arcs, $U^1_C = \{(5,1)^1,(5,2)^1\}$, $U^2_C = \{(4,1)^2\}$, $U^3_C = \emptyset$.

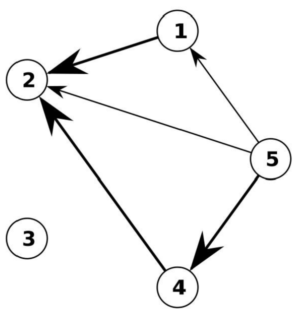

Figure 2 – Basis graph $U^1_T \bigcup U^1_C$ of the system (5)–(6) for $k = 1$. The arcs of basis graph ($k = 1$) for the system (5) are marked in bold

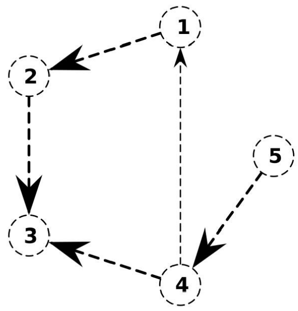

Figure 3 – Basis graph $U^2_T \bigcup U^2_C$ of the system (5)–(6) for $k = 2$. The arcs of basis graph ($k = 2$) for the system (5) are marked in bold

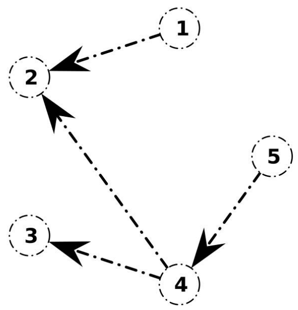

Figure 4 – Basis graph $U^3_T \bigcup U^3_C$ of the system (5)–(6) for $k = 3$. The arcs of basis graph ($k = 3$) for the system (5) are marked in bold

We are build a general solution to a sparse underdetermined system (5) – (6) relative to the basis multigraph (support) (see Fig. 2 – 4) $U^k_T \bigcup U^k_C$, $k \in K = \{1,2,3\}$ of the network $S = (I,U)$ for the system (5) – (6).

Based on the decomposition of basis multiflows we find the flows for the arcs of the set U^k_C, $k \in K = \{1,2,3\}$:

$$x^1_{5,2} \to \frac{1}{34}\left(-449 + 123y^3_{4,1} - 85y^3_{5,1}\right),$$
$$x^1_{5,1} \to \frac{1}{34}\left(3311 - 889y^3_{4,1} + 697y^3_{5,1}\right),$$
$$x^2_{4,1} \to -\frac{504}{17} + \frac{152y^3_{4,1}}{17} - 7y^3_{5,1}.$$

Then we compute the flows for the arcs of the set U^k_T, $k \in K = \{1,2,3\}$.

References

1. Pilipchuk L. A. Linear-fractional extremal inhomogeneous problems network flow programming. – Minsk : BSU, 2013. – 235 p. (in Russian).
2. Pilipchuk L. A. Sparse Linear Systems and Their Applications. – Minsk : BSU, 2013. – 235 p.