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Spin 1 Particle With Anomalous Magnetic Moment in External Uniform Electric Field, Solutions With Cylindrical Symmetry

Alina Ivashkevich[1] | Viktor Red'kov[1] | Alexander Chichurin[2]

[1]Department of Fundamental Interactions and Astrophysics, B.I. Stepanov Institute of Physics, Minsk, Belarus | [2]Department of Mathematical Modeling, The John Paul II Catholic University of Lublin, Lublin, Poland

Correspondence: Alexander Chichurin (achichurin@gmail.com)

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ABSTRACT

A generalized 10-dimensional Duffin–Kemmer–Petiau equation for spin 1 particle with anomalous magnetic moment is examined in cylindrical coordinates (t, r, ϕ, z) in the presence of the external uniform electric field oriented along the axis z. On solutions, we diagonalize operators of the energy and third projection of the total angular momentum. First, we derive the system of 10 equations in partial derivatives for functions $F_i(r, z) = G_i(r)H_i(z)$ $(i = \overline{1, 10})$. The use of the method based on the projective operators permits us to express 10 variables $G_i(r)$ through only three different functions $f_1(r)$, $f_2(r)$, $f_3(r)$, which are solved in Bessel functions. After that, we derive the system of 10 first-order differential equations for functions $H_i(z)$. This system reduces to one independent equation for a separate function and to the system of two linked equations for two remaining primary functions. This system after diagonalization of the mixing matrix gives two separated equations for new variables. All three equations for basic functions are solved in terms of the confluent hypergeometric functions. Thus, the complete system of solutions with cylindrical symmetry for the vector particle with anomalous magnetic moment in the presence of the external uniform electric field is found.

1 | Introduction

The quantum mechanical problem for particles in the external magnetic and electric fields is the classical one for quantum physics, for instance, see [1–4]. The particles of low spin values, $s = 0, 1/2$, were studied in the first place. In the present paper, we turn to a spin $s = 1$ particle. The study of such a particle has a long history; for example, see in [5–20].

Below, we will adhere the general technics developed in [21–25]. A generalized 10-dimensional Duffin–Kemmer–Petiau equation firstly was proposed by Shamaly and Capri [26, 27], they had developed the theory for a vector particle with anomalous magnetic moment. This idea of generalization in order to take into account additional electromagnetic characteristics of the particles has been developed in many of works; in particular, for particle with electric quadruple moment, in the presence of external fields, uniform magnetic or electric, and Coulomb field; see in [28–36].

Now, a spin 1 particle with the anomalous magnetic moment will be examined in the presence of the external uniform electric field. The main wave equation is specified in cylindrical coordinates (t, r, ϕ, z) and corresponding tetrad.[1] Solutions with cylindrical symmetry are searched. On solutions, we diagonalize the operators of the energy and the third projection of the total angular momentum. For Duffin–Kemmer–Petiau matrices, we apply the cyclic presentation.

First, we derive the system of 10 first-order differential equations in partial derivatives for functions $F_j(r,z) = F_j(r)F_j(z)$, $(j = \overline{1,10})$. The use of the method by Fedorov–Gronskiy [37] permits us to express all 10 functions $F_j(r)$ through only three different ones $f_1(r), f_2(r), f_3(r)$.

After that, we derive the system of 10 differential equations for functions $F_j(z)$ dependent on the coordinate z. This system is resolved by means of the method which generalized the known method applied in solving the similar problem in Cartesian coordinates [32–35]. In this way, we derive the system of three 2nd-order differential equations for three primary functions which is divided into one independent equation for a separate function and the system of two linked equations for two functions. The last system after diaginalizing the mixing matrix reduces to two separated equations for new functions. All three equations are solved in terms of the confluent hypergeometric functions. When separating the variables within the matrix approach, and in practical realization of the method by Fedorov–Gronskiy. Numerical study of the obtained analytical solutions is performed.

Thus, the complete system of solutions for the vector particle with anomalous magnetic moment in the presence of the external uniform electric field has been constructed.

2 | Separation of the Variables

The wave equation for a spin 1 particle with anomalous magnetic moment [21] has the form (assuming the use of the tetrad formalism [2])

$$\left\{ \beta^c \left[(e^{\beta}_{(c)} \partial_\beta + \frac{1}{2} j^{ab} \gamma_{abc}(x)) \; - ieA_c \right] + \lambda \frac{1}{2} F_{\alpha\beta}(x) j^{\alpha\beta}(x) P - M \right\} \Psi = 0 \,, \quad P = \begin{vmatrix} I_4 & 0 \\ 0 & 0 \end{vmatrix}; \qquad (1)$$

where the symbol P stands for the projective operator separating the vector component in the complete wave function. In cylindrical coordinates,

$$x^\alpha = (t, r, \phi, z), \quad dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2,$$

and in the presence of the uniform electric field along the axis z, $A_t = -Ez$, $F_{tz} = F_{03} = E$, Equation (1) takes the form (for brevity, we simplify the notation, $eE \Rightarrow E$)

$$\left[\beta^0 \left(\frac{\partial}{\partial t} + iEz \right) + \beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{\partial_\phi + J^{12}}{r} + \beta^3 \frac{\partial}{\partial z} + \Gamma J^{03} P - M \right] \Psi = 0 \qquad (2)$$

where the physical dimensions of the quantities are

$$[M] = 1/L, \quad \Gamma = i\lambda E, \quad [\Gamma] = 1/L, \quad [E] = 1/L^2;$$

Γ is the imaginary parameter referring to the anomalous magnetic moment of the particle.

Let us search solutions in the form (we will apply the block structure of the wave function and the matrices)

$$\Psi = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} h_0(r,z) \\ h_i(r,z) \\ E_i(r,z) \\ B_i(r,z) \end{vmatrix} = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} H_1(r,z) \\ H_2(r,z) \end{vmatrix}, \quad \beta^a = \begin{vmatrix} 0 & L^a \\ K^a & 0 \end{vmatrix} \qquad (3)$$

Correspondingly, the system of equations reads

$$\begin{aligned} &\left[L^0(-i\epsilon + iEz) + L^1 \frac{\partial}{\partial r} + L^2 \frac{im + j_2^{12}}{r} + L^3 \frac{\partial}{\partial z} \right] \\ &H_2 + \Gamma j_1^{03} H_1 = M H_1 \\ &\left[K^0(-i\epsilon + iEz) + K^1 \frac{\partial}{\partial r} + K^2 \frac{im + j_1^{12}}{r} + K^3 \frac{\partial}{\partial z} \right] H_1 = M H_2. \end{aligned} \qquad (4)$$

First, we assume the use of the Cartesian basis for Duffin–Kemmer–Petiau matrices

$$L^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}, \quad L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{vmatrix},$$

$$L^2 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{vmatrix}, \quad L^3 = \begin{vmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix},$$

$$K^0 = \begin{vmatrix} 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad K^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{vmatrix},$$

$$K^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad K^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

The needed generators in Cartesian basis are given by the formulas

$$j_1^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad j_1^{03} = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

$$j_2^{ab} H_2 = j_1^{ab} H_2 + H_2 \tilde{j}_1^{ab}, \quad H_2 = \{E_i, B_i\},$$

$$J_{(2)}^{12} = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}, \quad J_{(2)}^{03} = \begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

For the following, it will be more convenient to apply the cyclic basis. It is defined by the requirement that the generator j_1^{12} for vector field $H_1 = (\Psi_l)$ be diagonal. The needed transformation is given as follows: $\overline{H}_1 = U H_1$, where

$$U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad U^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{vmatrix} \qquad (5)$$

Correspondingly, the needed generators for vector and tensor transform to the cyclic basis in accordance with the rules, $\overline{j}_1^{ab} = U j_1^{ab} U^{-1}$, $\overline{j}_2^{ab} = \overline{j}_1^{ab} \otimes I + I \otimes \overline{j}_1^{ab}$. In this way, we get

$$\overline{j}_1^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{vmatrix}, \quad \overline{j}_2^{12} = i \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix},$$

$$\overline{j}_1^{03} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \overline{j}_2^{03} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Also, we should transform to cyclic basis the matrices β^a:

$$\overline{H}_1 = U H_1, \quad U = C_1, \quad \overline{H}_2 = (U \otimes U) H_2 = C_2 H_2,$$

$$\begin{vmatrix} 0 & \overline{L}^a \\ \overline{K}^a & 0 \end{vmatrix} = \begin{vmatrix} 0 & C_1 L^a C_2^{-1} \\ C_2 K^a C_1^{-1} & 0 \end{vmatrix}.$$

Taking in mind the matrices

$$C_2 = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$C_2^{-1} = \begin{vmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -i & 0 \end{vmatrix},$$

we get

$$\overline{L}^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix},$$

$$\overline{L}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\overline{L}^2 = \begin{vmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \overline{L}^3 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

$$\overline{K}^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\overline{K}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix}, \quad \overline{K}^2 = \begin{vmatrix} -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\overline{K}^{3} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}.$$

Therefore, in cyclic basis, we have the system

$$\begin{aligned} &\Big[-i(\epsilon - Ez)\overline{L}^{0} + \overline{L}^{1}\frac{\partial}{\partial r} + \frac{1}{r}\overline{L}^{2}(im + \overline{j}_2^{12}) + \frac{\partial}{\partial z}\overline{L}^{3}\Big] \\ &\overline{H}_2 + \Gamma \overline{j}_1^{03}\overline{H}_1 = M\overline{H}_1, \\ &\Big[-i(\epsilon - Ez)\overline{K}^{0} + \overline{K}^{1}\frac{\partial}{\partial r} + \frac{1}{r}\overline{K}^{2}(im + \overline{j}_1^{12}) + \frac{\partial}{\partial z}\overline{K}^{3}\Big]\overline{H}_1 = M\overline{H}_2. \end{aligned} \tag{6}$$

Below, for brevity, we will omit the bar symbol over the variables.

After the needed calculation, we find the system of 10 partial differential equations in the variables r and z:

$$\begin{aligned} &\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{(m-1)}{r}\Big)E_1 - \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{(m+1)}{r}\Big) \\ &E_3 - \frac{\partial}{\partial z}E_2 + \Gamma h_2 = Mh_0, \\ &\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)B_2 + i(\epsilon - Ez)E_1 - \frac{\partial}{\partial z}B_3 = Mh_1, \\ &-\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{(m+1)}{r}\Big)B_1 - \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{(m-1)}{r}\Big) \\ &B_3 + i(\epsilon - Ez)E_2 + \Gamma h_0 = Mh_2, \\ &\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)B_2 + i(\epsilon - Ez)E_3 + \frac{\partial}{\partial z}B_1 = Mh_3, \\ &\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)h_0 + i(Ez - \epsilon)h_1 = ME_1, \\ &+ i(Ez - \epsilon)h_2 - \frac{\partial}{\partial z}h_0 = ME_2, \\ &-\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)h_0 + i(Ez - \epsilon)h_3 = ME_3, \\ &-\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)h_2 + \frac{\partial}{\partial z}h_3 = MB_1, \\ &\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{(m-1)}{r}\Big)h_1 + \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{(m+1)}{r}\Big)h_3 = MB_2, \\ &-\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)h_2 - \frac{\partial}{\partial z}h_1 = MB_3. \end{aligned}$$

With the use of the notations

$$\begin{aligned} &Ez - \epsilon = W, \quad a_m = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big), \quad b_m = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big), \\ &a_{m+1} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m+1}{r}\Big), \quad b_{m-1} = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m-1}{r}\Big), \end{aligned} \tag{7}$$

we can present these equations shorter (the prime stands for the derivative in z)

$$\begin{aligned} &b_{m-1}E_1 - a_{m+1}E_3 - E_2' + \Gamma h_2 = Mh_0, \ a_m B_2 + i(-W)E_1 - B_3' = Mh_1, \\ &-a_{m+1}B_1 - b_{m-1}B_3 - iWE_2 + \Gamma h_0 = Mh_2, \ b_m B_2 - iWE_3 + B_1' = Mh_3; \\ &a_m h_0 + iWh_1 = ME_1, +iWh_2 - h_0' = ME_2, \ -b_m h_0 + iWh_3 = ME_3, \\ &-b_m h_2 + h_3' = MB_1, \ b_{m-1}h_1 + a_{m+1}h_3 = MB_2, \ -a_m h_2 - h_1' = MB_3. \end{aligned} \tag{8}$$

For studying the dependence of the functions on the variable r, we will apply the method by Fedorov–Gronskiy [37]. To this end, let us introduce the matrix of the third projection of the spin, $Y = -i\overline{J}^{12}$; we readily verify that it obeys the minimal equation $Y(Y-1)(Y+1) = 0$. This minimal equation permits us to introduce three projective operators

$$\begin{aligned} &P_1 = P_{-1} = \frac{1}{2}Y(Y-1), \quad P_2 = P_{+1} = \frac{1}{2}Y(Y+1) \\ &P_3 = P_0 = 1 - Y^2 \end{aligned} \tag{9}$$

with the needed properties

$$P_0^2 = P_0, \quad P_{+1}^2 = P_{+1}, \quad P_{-1}^2 = P_{-1}, \quad P_0 + P_{+1} + P_{-1} = 1.$$

So the complete wave function can be decomposed into the sum of three parts

$$\Psi = \Psi_0 + \Psi_{+1} + \Psi_{-1}, \quad \Psi_\sigma = P_\sigma \Psi, \quad \sigma = 0, +1, -1.$$

Explicitly, these operators read

$$P_1 = \begin{vmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1 \end{vmatrix},$$

$$P_3 = \begin{vmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0 \end{vmatrix}.$$

Extending the method by Fedorov–Gronskiy [37], we assume that each projective constituent is defined by only one function of the variable r:

$$\Psi_1(r,z) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ h_3(z) \\ 0 \\ 0 \\ E_3(z) \\ B_1(z) \\ 0 \\ 0 \end{vmatrix} f_1(r), \quad \Psi_2(r) = \begin{vmatrix} 0 \\ h_1(z) \\ 0 \\ 0 \\ E_1(z) \\ 0 \\ 0 \\ 0 \\ 0 \\ B_3(z) \end{vmatrix} f_2(r) \tag{10}$$

$$\Psi_3(r,z) = \begin{vmatrix} h_0(z) \\ 0 \\ h_2(z) \\ 0 \\ 0 \\ E_2(z) \\ 0 \\ 0 \\ B_2(z) \\ 0 \end{vmatrix} f_3(r)$$

Thus, we are to apply the following substitutions:

$$\begin{gathered} h_3(r,z) = f_1(r)h_3(z), \quad h_1(r,z) = f_2(r)h_1(z), \quad h_0(r,z) = f_3(r)h_0(z), \\ E_3(r,z) = f_1(r)E_3(z), \quad E_1(r,z) = f_2(r)E_1(z), \quad h_2(r,z) = f_3(r)h_2(z), \\ B_1(r,z) = f_1(r)B_1(z), \quad B_3(r,z) = f_2(r)B_3(z), \quad E_2(r,z) = f_3(r)E_2(z), \\ B_2(r,z) = f_3(r)B_2(z). \end{gathered} \tag{11}$$

Taking these into account, from (8), we obtain the equations (they are collected into three groups; besides, we impose some differential constraints on the functions)

$$P_1,$$

$$\begin{aligned} b_m f_3 B_2(z) - iW f_1 E_3(z) + Z f_1 B_1(z) = M f_1 h_3(z) &\Rightarrow b_m f_3 = C_1 f_1, \\ -b_m f_3 h_0(z) + iW f_1(r) h_3(z) = M f_1(r) E_3(z) &\Rightarrow b_m f_3 = C_1 f_1, \\ -b_m f_3 h_2(z) + Z f_1 h_3(z) = M f_1 B_1(z) &\Rightarrow b_m f_3 = C_1 f_1; \end{aligned}$$

$$P_2,$$

$$\begin{aligned} a_m f_3 B_2(z) - iW f_2 E_1(z) - Z f_2 B_3(z) = M f_2 h_1(z) &\Rightarrow a_m f_3 = C_2 f_2 \\ a_m f_3 h_0(z) + iW f_2 h_1(z) = M f_2 E_1(z) &\Rightarrow a_m f_3) = C_2 f_2 \\ -a_m f_3 h_2(z) - Z f_2) h_1(z) = M f_2 B_3(z) &\Rightarrow a_m f_3 = C_2 f_2; \end{aligned}$$

$$P_3,$$

$$\begin{aligned} & b_{m-1} f_2 E_1(z) - a_{m+1} f_1 E_3(z) - Z f_3 E_2(z) + \Gamma f_3 h_2(z) = M f_3 h_0(z) \\ & \quad \Rightarrow b_{m-1} f_2 = C_3 f_3, \quad a_{m+1} f_1(r) = C_4 f_3, \\ & -a_{m+1} f_1(r) B_1(z) - b_{m-1} f_2 B_3(z) - iW f_3 E_2(z) \\ & \quad + \Gamma f_3 h_0(z) = M f_3 h_2(z) \\ & \quad \Rightarrow b_{m-1} f_2 = C_3 f_3, \quad a_{m+1} f_1 = C_4 f_3, \\ & \quad + iW f_3 h_2(z) - Z f_3 h_0(z) = M f_3 E_2(z), \\ & b_{m-1} f_2(r) h_1(z) + a_{m+1} f_1 h_3(z) = M f_3 B_2(z) \\ & \quad \Rightarrow b_{m-1} f_2 = C_3 f_3, \quad a_{m+1} f_1 = C_4 f_3. \end{aligned}$$

After reducing the total multipliers in all equations, we arrive at the differential system in the variable z:

$$\begin{aligned} & C_1 B_2 - iW E_3 + B_1' = M h_3, \; -C_1 h_0 \\ & \quad + iW h_3 = M E_3, \; -C_1 h_2 + h_3' = M B_1; \\ & C_2 B_2 - iW E_1 - B_3' = M h_1, \; C_2 h_0 \\ & \quad + iW h_1 = M E_1, \; -C_2 h_2 - h_1' = M B_3; \\ & C_3 E_1 - C_4 E_3 - E_2' + \Gamma h_2 = M h_0, \\ & \quad - C_4 B_1 - C_3 B_3 - iW E_2 + \Gamma h_0 = M h_2, \\ & iW h_2 - h_0' = M E_2, \quad C_3 h_1 + C_4 h_3 = M B_2. \end{aligned} \tag{12}$$

Collect the constrains together

$$\begin{aligned} b_m f_3 = C_1 f_1, \quad a_{m+1} f_1 = C_4 f_3, \quad \text{let} \quad C_4 = C_1; \\ a_m f_3 = C_2 f_2, \quad b_{m-1} f_2 = C_3 f_3, \quad \text{let} \quad C_3 = C_2. \end{aligned} \tag{13}$$

From (13), we readily derive the 2nd-order equations for separate functions

$$\begin{aligned} \left(b_m a_{m+1} - C_1^2\right) f_1(r) = 0, \quad \left(a_{m+1} b_m - C_1^2\right) f_3(r) = 0, \\ \left(a_m b_{m-1} - C_2^2\right) f_2(r) = 0, \quad \left(b_{m-1} a_m - C_2^2\right) f_3(r) = 0. \end{aligned} \tag{14}$$

Allowing for the identities

$$\begin{aligned} b_m a_{m+1} &= \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2}\right) \\ a_{m+1} b_m &= \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) \\ a_m b_{m-1} &= \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2}\right) \\ b_{m-1} a_m &= \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right) \end{aligned}$$

we rewrite the above equations differently

$$\begin{gathered} \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - 2C_1^2\right) f_1 = 0 \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - 2C_1^2\right) f_3 = 0 \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - 2C_2^2\right) f_2 = 0 \\ \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - 2C_2^2\right) f_3 = 0; \end{gathered}$$

so that $C_1^2 = C_2^2 = C$. In the variable $x = i\sqrt{2C}\, r$, they take on the Bessel form

$$\begin{gathered} \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(m+1)^2}{x^2}\right) f_1 = 0, \quad f_1(x) = J_{m+1}(x); \\ \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(m-1)^2}{x^2}\right) f_2 = 0, \quad f_2(x) = J_{m-1}(x); \\ \left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{m^2}{x^2}\right) f_3 = 0, \quad f_3(x) = J_m(x). \end{gathered} \tag{15}$$

3 | Solving the System in z Variable

Now, we turn to the system (12):

$$
\begin{aligned}
&CB_2 - iWE_3 + B_1' = Mh_3,\ -Ch_0 + iWh_3 = ME_3,\ -Ch_2 + h_3' = MB_1,\\
&CB_2 - iWE_1 - B_3' = Mh_1,\ Ch_0 + iWh_1 = ME_1,\ -Ch_2 - h_1' = MB_3,\\
&CE_1 - CE_3 - E_2' + \Gamma h_2 = Mh_0,\ -CB_1 - CB_3 - iWE_2 + \Gamma h_0 = Mh_2,\\
&iWh_2 - h_0' = ME_2,\ Ch_1 + Ch_3 = MB_2.
\end{aligned} \tag{16}
$$

Let us compare the system (16) with the system derived in [32] when solving the similar problem in Cartesian coordinates

$$
\begin{aligned}
&- iaE_1 - ibE_2 + \Gamma h_3 - E_3' = Mh_0,\\
&CE_1 - CE_3 + \Gamma h_2 - E_2' = Mh_0,\\
&- ibB_3 - iWE_1 + B_2' = Mh_1,\\
&CB_2 - iWE_1 - B_3' = Mh_1,\\
&iaB_3 - iWE_2 - B_1' = Mh_2,\\
&CB_2 - iWE_3 + B_1' = Mh_3,\\
&ibB_1 - iaB_2 - iWE_3 + \Gamma h_0 = Mh_3,\\
&- CB_1 - CB_3 - iWE_2 + \Gamma h_0 = Mh_2,\\
&- iah_0 + i(Ez - \epsilon)h_1 = ME_1,\\
&Ch_0 + iWh_1 = ME_1,\\
&- ibh_0 + i(Ez - \epsilon)h_2 = ME_2,\\
&- Ch_0 + iWh_3 = ME_3,\\
&\quad iWh_3 - h_0' = ME_3,\\
&+ iWh_2 - h_0' = ME_2,\\
&ibh_3 - h_2' = MB_1,\\
&- Ch_2 + h_3' = MB_1,\\
&- iah_3 + h_1' = MB_2,\\
&- Ch_2 - h_1' = MB_3,\\
&- ibh_1 + iah_2 = MB_3,\\
&Ch_1 + Ch_3 = MB_2.
\end{aligned}
$$

We can see that transition from the left-side system to the right-side one is reached by the formal changes

$$
\begin{aligned}
&h_0 \Rightarrow h_0, \quad h_1 \Rightarrow h_1, \quad h_2 \Rightarrow h_3, \quad h_3 \Rightarrow h_2,\\
&E_1 \Rightarrow E_1, \quad E_2 \Rightarrow E_3, \quad E_3 \Rightarrow E_2,\\
&B_1 \Rightarrow B_1, \quad B_2 \Rightarrow B_3, \quad B_3 \Rightarrow B_2.
\end{aligned}
$$

Therefore, in the new system, we can try the same method as in [32]. So we will consider the variables h_1, h_3, E_2, B_2 as primary ones. First, we resolve the subsystem of six equations

$$
\begin{aligned}
&CE_1 - CE_3 + \Gamma h_2 - E_2' = Mh_0,\\
&- CB_1 - CB_3 - iWE_2 + \Gamma h_0 = Mh_2,\\
&Ch_0 + iWh_1 = ME_1, \quad -Ch_0 + iWh_3 = ME_3,\\
&- Ch_2 + h_3' = MB_1, \quad -Ch_2 - h_1' = MB_3
\end{aligned}
$$

as algebraic with respect to the functions $h_0, h_2, E_1, E_3, B_1, B_3$; this results in

$$
\begin{aligned}
h_0 = \frac{1}{D}\{&M\left[\left(2C^2 - M^2\right)E_2' + \Gamma C\left(h_1' - h_3'\right)\right]\\
&-iW\left[C\left(2C^2 - M^2\right)\left(h_1 - h_3\right) + \Gamma M^2 E_2\right]\},
\end{aligned}
$$

$$
\begin{aligned}
h_2 = -\frac{1}{D}\{&iMW\left[\left(M^2 - 2C^2\right)E_2 + \Gamma C\left(h_3 - h_1\right)\right]\\
&+C\left(2C^2 - M^2\right)\left(h_1' - h_3'\right) + \Gamma M^2 E_2'\},
\end{aligned}
$$

$$
\begin{aligned}
E_1 = \frac{1}{MD}\{&iW\left[C^2\left(2C^2 - M^2\right)h_3\right.\\
&\left.+h_1\left(2C^4 - 3C^2M^2 + M^4 - \Gamma^2M^2\right) - \Gamma CM^2E_2\right]\\
&+CM\left[\left(2C^2 - M^2\right)E_2' + \Gamma C\left(h_1' - h_3'\right)\right]\},
\end{aligned}
$$

$$
\begin{aligned}
E_3 = \frac{1}{MD}\{&iW\left[C^2\left(2C^2 - M^2\right)h_1\right.\\
&\left.+h_3\left(2C^4 - 3C^2M^2 + M^4 - \Gamma^2M^2\right) + \Gamma CM^2E_2\right]\\
&-CM\left[\left(2C^2 - M^2\right)E_2' + \Gamma C\left(h_1' - h_3'\right)\right]\},
\end{aligned}
$$

$$
\begin{aligned}
B_1 = \frac{1}{MD}\{&\left(2C^4 - 3C^2M^2 + M^4 - \Gamma^2M^2\right)h_3'\\
&+C\left[iMW\left[\left(M^2 - 2C^2\right)E_2 + \Gamma C\left(h_3 - h_1\right)\right]\right.\\
&\left.+C\left(2C^2 - M^2\right)h_1' + \Gamma M^2E_2'\right]\},
\end{aligned}
$$

$$
\begin{aligned}
B_3 = \frac{1}{MD}\{&-iCMW\left[\left(2C^2 - M^2\right)E_2 + \Gamma C\left(h_1 - h_3\right)\right]\\
&+C^2\left(M^2 - 2C^2\right)h_3' + \left(-2C^4 + 3C^2M^2 - M^4 + \Gamma^2M^2\right)\\
&h_1' + \Gamma CM^2E_2'\},
\end{aligned}
$$

and then substitute these expressions into the remaining four equations

$$
\begin{aligned}
&CB_2 - iWE_1 - B_3' = Mh_1, \quad CB_2 - iWE_3 + B_1' = Mh_3,\\
&+iWh_2 - h_0' = ME_2, \quad Ch_1 + Ch_3 = MB_2.
\end{aligned}
$$

In this way, we obtain four 2nd-order equations:

1.

$$
\begin{aligned}
&CB_2 + \frac{-\Gamma CMW^2 + iC\left(2C^2 - M^2\right)W'}{\left(M^2 - 2C^2\right)^2 - \Gamma^2M^2}\\
&E_2 - \frac{\Gamma CM}{\left(M^2 - 2C^2\right)^2 - \Gamma^2M^2}E_2''\\
&+ \frac{C^2\left(\left(2C^2 - M^2\right)W^2 - i\Gamma MW'\right)}{M\left(M^2 - 2C^2\right)^2 - \Gamma^2M^3}h_3\\
&+ \frac{C^2\left(2C^2 - M^2\right)}{M\left(M^2 - 2C^2\right)^2 - \Gamma^2M^3}h_3''
\end{aligned}
$$

$$+\frac{\left(-\left(M^3-2C^2M\right)^2+i\Gamma C^2MW'+W^2\left(2C^4-3C^2M^2+M^4-\Gamma^2M^2\right)+\Gamma^2M^4\right)}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}h_1$$

$$+\frac{\left(2C^4-3C^2M^2+M^4-\Gamma^2M^2\right)}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}h_1''=0,$$

2.

$$CB_2+\frac{\left(\Gamma CMW^2-iC\left(2C^2-M^2\right)W'\right)}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}$$

$$E_2+\frac{\Gamma CM}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}E_2''$$

$$+\frac{C^2\left(\left(2C^2-M^2\right)W^2-i\Gamma MW'\right)}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}$$

$$h_1+\frac{\left(2C^4-C^2M^2\right)}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}h_1''$$

$$+\frac{\left(-\left(M^3-2C^2M\right)^2+i\Gamma C^2MW'+W^2\left(2C^4-3C^2M^2+M^4-\Gamma^2M^2\right)+\Gamma^2M^4\right)}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}h_3$$

$$+\frac{2C^4-3C^2M^2+M^4-\Gamma^2M^2}{M\left(M^2-2C^2\right)^2-\Gamma^2M^3}h_3''=0,$$

3.

$$\frac{M\left(\left(M^2-2C^2\right)W^2-\left(M^2-2C^2\right)^2+\Gamma^2M^2+i\Gamma MW'\right)}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}E_2+\frac{M^3-2C^2M}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}E_2''$$

$$+\frac{iC\left(\left(2C^2-M^2\right)W'+i\Gamma MW^2\right)}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}h_1-\frac{\Gamma CM}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}h_1''$$

$$+\frac{C\left(\Gamma MW^2+i\left(M^2-2C^2\right)W'\right)}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}h_3+\frac{\Gamma CM}{\left(M^2-2C^2\right)^2-\Gamma^2M^2}h_3''=0,$$

4.

$$-MB_2+Ch_1+Ch_3=0\quad\Rightarrow\quad\frac{M}{C}B_2=(h_1+h_3);$$

the last equation provides us with the linear relationship between three variables, so in fact, we have only three equations for three independent variables.

Let us sum and subtract Equations (1) and (2). The summing gives

$$2CMB_2+(W^2-M^2)(h_1+h_3)+(h_1''+h_3'')=0,$$

whence, allowing for Equation (4), we obtain

$$2CMB_2+\frac{M}{C}(W^2-M^2)B_2+\frac{M}{C}B_2''=0,$$

thus, we have a separate equation for the function B_2:

$$\left[\frac{d^2}{dz^2}+(W^2-M^2)+2C^2\right]B_2=0,\quad B_2=\frac{C}{M}(h_1+h_3)\qquad(17)$$

Subtraction gives

$$\left(-2C^2M+M^3-\Gamma^2M\right)h_1''+M\left(\Gamma^2+2C^2-M^2\right)h_3''$$

$$+\left(MW^2\left(-\Gamma^2-2C^2+M^2\right)-M\left(M^2-2C^2\right)^2\right.$$
$$\left.+2i\Gamma C^2W'+\Gamma^2M^3\right)h_1$$

$$-\Big(MW^2\big(-\Gamma^2-2C^2+M^2\big)-M\big(M^2-2C^2\big)^2$$
$$+2i\Gamma C^2W'+\Gamma^2M^3\Big)h_3$$

$$+\big(-2\Gamma CMW^2+2iC\big(2C^2-M^2\big)W'\big)E_2-2\Gamma CME_2''=0,$$

or (let us apply more convenient notations $h_1(z)-h_3(z)=G(z),\ E_2(z)=F(z)$)

$$\begin{aligned}&\big(-M\Gamma^2-2\,MC^2+M^3\big)G''\\&+\Big[MW^2\big(-\Gamma^2-2C^2+M^2\big)-M\big(M^2-2C^2\big)^2\\&+2i\Gamma C^2W'+\Gamma^2M^3\Big]G\\&-2\Gamma CMF''(z)+\big[-2\Gamma CMW^2+2iC\big(2C^2-M^2\big)W'\big]F=0\end{aligned} \quad (18)$$

Let us write down Equation (3)

$$\begin{aligned}&M\Big[\big(M^2-2C^2\big)W^2-\big(M^2-2C^2\big)^2+\Gamma^2M^2+i\Gamma MW'\Big]\\&F+M(M^2-2C^2)F''\\&+\big[iC(2C^2-M^2)W'-C\Gamma MW^2\big]h_1-\Gamma CMh_1''\\&+\big[iC(M^2-2C^2)W'+C\Gamma MW^2\big]h_3+\Gamma CMh_3''=0,\end{aligned}$$

it can be presented differently

$$\begin{aligned}&M\Big[\big(M^2-2C^2\big)W^2-\big(M^2-2C^2\big)^2+\Gamma^2M^2+i\Gamma MW'\Big]\\&F+M(M^2-2C^2)F''\\&+iC(2C^2-M^2)W'(h_1+h_3)-C\Gamma MW^2G-\Gamma CMG''=0\end{aligned}$$

or

$$\begin{aligned}&M\Big[\big(M^2-2C^2\big)W^2-\big(M^2-2C^2\big)^2+\Gamma^2M^2+i\Gamma MW'\Big]F\\&+M(M^2-2C^2)F''+i(2C^2-M^2)W'MB_2\\&-C\Gamma M\big[G''+W^2G\big]=0\end{aligned} \quad (19)$$

Thus, we have three equations for three functions:

$$\begin{aligned}&\left[\frac{d^2}{dz^2}+(W^2-M^2)+2C^2\right]B_2=0,\\&\big(-M\Gamma^2-2\,MC^2+M^3\big)G''+[MW^2\big(-\Gamma^2-2C^2+M^2\big)\\&-M\big(M^2-2C^2\big)^2+2i\Gamma C^2W'+\Gamma^2M^3]G\\&-2\Gamma CMF''+\big[-2\Gamma CMW^2+2iC\big(2C^2-M^2\big)W'\big]F=0,\\&i(2C^2-M^2)W'B_2-C\Gamma\big[G''+W^2G\big]\\&+(M^2-2C^2)F''+\Big[\big(M^2-2C^2\big)W^2-\big(M^2-2C^2\big)^2+\Gamma^2M^2\\&+i\Gamma MW'\Big]F+i(2C^2-M^2)W'B_2-C\Gamma\big[G''+W^2G\big]=0.\end{aligned}$$

Let us fix the parameter C, by setting $(M^2-2C^2)=0$. In this way, we eliminate the variable B_2 from the second equation. After that, we can search for solutions of the system as follows:

$$(I)\qquad B_2(z)\neq 0,\quad F(z)=0,\quad G(z)=0;$$

$$(II)\qquad B_2(z)=0\quad F(z)\neq 0,\quad G(z)\neq 0.$$

In the case II, we have two linked equations for two variables (recall that $W'=E$)

$$\begin{aligned}&M\big(-\Gamma^2+M^2-2C^2\big)G''+[MW^2\big(-\Gamma^2-2C^2+M^2\big)\\&-M\big(M^2-2C^2\big)^2+2i\Gamma EC^2+\Gamma^2M^3]G\\&-2\Gamma CMF''+\big[-2\Gamma CMW^2+2iC\big(2C^2-M^2\big)E\big]F=0,\\&-C\Gamma\big[G''+W^2G\big]+(M^2-2C^2)F''\\&+\Big[\big(M^2-2C^2\big)W^2-\big(M^2-2C^2\big)^2+\Gamma^2M^2+i\Gamma ME\Big]F=0.\end{aligned}$$

It may be presented in symbolical form

$$P_1G''+R_1(z)G+Q_1F''+S_1(z)F=0,$$
$$P_2G''+R_2(z)G+Q_2F''+S_2(z)F=0.$$

Let us multiply the first equation by α, the second — by β and sum the results. We will consider two possibilities.

The first one is

$$\alpha P_1+\beta P_2=1,\quad \alpha Q_1+\beta Q_2=0,$$
$$G''+(\alpha R_1+\beta R_2)G+0+(\alpha S_1+\beta S_2)F=0,$$

where

$$\alpha=\frac{M^2-2C^2}{M\big(M^2-2C^2\big)^2-\Gamma^2M^3},\qquad \beta=\frac{2\Gamma C}{\big(M^2-2C^2\big)^2-\Gamma^2M^2},$$

$$G''+\left[W^2+\frac{(2C^2-M^2)\Big(M\big(M^2-2C^2\big)^2-2i\Gamma C^2E-\Gamma^2M^3\Big)}{M\big(M^2-2C^2\big)^2-\Gamma^2M^3}\right]G$$

$$-\frac{2C(\Gamma M+iE)}{M}\;F=0.$$

The second one is

$$\alpha P_1+\beta P_2=0,\quad \alpha Q_1+\beta Q_2=1,$$
$$(\alpha R_1+\beta R_2)G+F''+(\alpha S_1+\beta S_2)F=0,$$

where

$$\alpha=\frac{\Gamma C}{M\big(M^2-2C^2\big)^2-\Gamma^2M^3},\qquad \beta=\frac{-\Gamma^2-2C^2+M^2}{\big(M^2-2C^2\big)^2-\Gamma^2M^2},$$

$$\frac{\Gamma C\Big(-M\big(M^2-2C^2\big)^2+2i\Gamma C^2E+\Gamma^2M^3\Big)}{M\big(M^2-2C^2\big)^2-\Gamma^2M^3}G$$

$$+F''+\left(\Gamma^2+2C^2-M^2+\frac{i\Gamma E}{M}+W^2\right)F=0.$$

Thus, we have derived two equations

$$G''+\left[W^2+\frac{(2C^2-M^2)\Big(M\big(M^2-2C^2\big)^2-2i\Gamma C^2E-\Gamma^2M^3\Big)}{M\big(M^2-2C^2\big)^2-\Gamma^2M^3}\right]G$$

$$-\frac{2C(\Gamma M+iE)}{M}F=0,$$

$$F'' + \left[W^2 + \Gamma^2 + 2C^2 - M^2 + \frac{i\Gamma E}{M}\right]F$$

$$+\Gamma C \frac{-M\left(M^2-2C^2\right)^2 + 2i\Gamma C^2 E + \Gamma^2 M^3}{M\left(M^2-2C^2\right)^2 - \Gamma^2 M^3} G = 0.$$

Taking in mind the identity $C^2 = M^2$, we simplify them to the form

$$\left(\frac{d^2}{dz^2} + W^2\right)G - \left(\Gamma + \frac{iE}{M}\right)2CF = 0,$$
$$\left(\frac{d^2}{dz^2} + W^2\right)F + \left(\Gamma + \frac{iE}{M}\right)\Gamma F - C\left(\Gamma + \frac{iE}{M}\right)G = 0. \tag{20}$$

This system can be rewritten in matrix notations

$$D_2\begin{vmatrix} G \\ F \end{vmatrix} = \left(\Gamma + \frac{iE}{M}\right)\begin{vmatrix} 0 & 2C \\ C & -\Gamma \end{vmatrix}\begin{vmatrix} G \\ F \end{vmatrix}, \quad D_2\Psi = \left(\Gamma + \frac{iE}{M}\right)A\Psi,$$

the last system should be diagonalized by linear transformation

$$\overline{\Psi} = S\Psi, \quad D_2\Psi = \left(\Gamma + \frac{iE}{M}\right)A\Psi, \quad SA = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} S.$$

Furthermore, we get two linear systems

$$\begin{vmatrix} -\lambda_1 & C \\ 2C & -(\Gamma+\lambda_1) \end{vmatrix}\begin{vmatrix} s_{11} \\ s_{12} \end{vmatrix} = 0, \quad \begin{vmatrix} -\lambda_2 & C \\ 2C & -(\Gamma+\lambda_2) \end{vmatrix}\begin{vmatrix} s_{21} \\ s_{22} \end{vmatrix} = 0.$$

They lead to the roots

$$\lambda_1 = -\frac{\Gamma}{2} - \sqrt{\frac{\Gamma^2}{4} + M^2} = M\left(-x - \sqrt{1+x^2}\right), \quad x = \frac{\Gamma}{2M};$$
$$\lambda_2 = -\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + M^2} = M\left(-x + \sqrt{1+x^2}\right), \quad x = \frac{\Gamma}{2M}; \tag{21}$$

the transformation matrix may be taken as follows:

$$S = \begin{vmatrix} 1 & \lambda_1/C \\ 1 & \lambda_2/C \end{vmatrix} \tag{22}$$

In this way, we obtain two separate equations

$$\left(\frac{d^2}{dz^2} + W^2(z) - (M\Gamma + iE)\frac{\lambda_1}{M}\right)\overline{G} = 0,$$
$$\left(\frac{d^2}{dz^2} + W^2(z) - (M\Gamma + iE)\frac{\lambda_2}{M}\right)\overline{F} = 0; \tag{23}$$

they may be rewritten differently (recall that $x = \Gamma/2M$)

$$\left(\frac{d^2}{dz^2} + W^2(z) + (M\Gamma + iE)\left(x + \sqrt{1+x^2}\right)\right)\overline{G} = 0,$$
$$\left(\frac{d^2}{dz^2} + W^2(z) + (M\Gamma + iE)\left(x - \sqrt{1+x^2}\right)\right)\overline{F} = 0, \tag{24}$$

or shortly

$$\left(\frac{d^2}{dz^2} + (Ez-\epsilon)^2 - \Lambda_1\right)\overline{G} = 0, \quad \left(\frac{d^2}{dz^2} + (Ez-\epsilon)^2 - \Lambda_2\right)\overline{F} = 0 \tag{25}$$

Let us write down the equation for function $B_2(z)$:

$$\left[\frac{d^2}{dz^2} + W^2 - M^2 + 2C^2\right]B_2 = 0$$
$$\Rightarrow \quad \left[\frac{d^2}{dz^2} + (Ez-\epsilon)^2\right]B_2 = 0 \tag{26}$$

All three equations are of the same mathematical structure.

4 | Solving the Differential Equation

The derived equation has the same formal structure as for a scalar relativistic particle in the uniform electric field

$$\left(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - \mu^2\right)\Phi(z) = 0 \tag{27}$$

We transform Equation (27) to the new variable (assuming that $E > 0$)

$$Z = i\frac{(Ez+\epsilon)^2}{E}, \quad \sigma = \frac{\mu^2}{4E} \tag{28}$$

then we get the confluent hypergeometric equation [38]

$$\left(\frac{d^2}{dZ^2} + \frac{1/2}{Z}\frac{d}{dZ} - \frac{1}{4} + \frac{i\sigma}{Z}\right)\Phi(Z) = 0 \tag{29}$$

This equation has two singular points. The point $Z = 0$ is regular, behavior of solutions near this point is given by the formulas $Z \to 0$, $\Phi(Z) = Z^A$, $A = 0, 1/2$. The point $Z = \infty$ is irregular point of the rank 2. Indeed, in the inverse variable $y = Z^{-1}$, we get the equation

$$\left(\frac{d^2}{dy^2} + \frac{3}{2y}\frac{d}{dy} - \frac{1}{4y^4} + \frac{i\sigma}{y^3}\right)\Phi = 0 \tag{30}$$

Asymptotic of solutions at $y \to 0$ should have the structure $\Phi = y^C e^{D/y}$. Furthermore, we arrive at

$$D^2 - \frac{1}{4} = 0, \quad -2CD + 2D - \frac{3}{2}D + i\sigma = 0\ ,$$

whence it follows

$$D_1 = +\frac{1}{2}, \quad C_1 = \frac{1}{4} + i\sigma\ ; \qquad D_2 = -\frac{1}{2}, \quad C_2 = \frac{1}{4} - i\sigma \tag{31}$$

Therefore, in infinity, there are possible two behaviors

$$Z \to \infty, \qquad \Phi = Z^{-C}e^{DZ} = \begin{cases} Z^{-C_1}e^{D_1 Z} = Z^{-1/4-i\sigma}e^{+Z/2}, \\ Z^{-C_2}e^{D_2 Z} = Z^{-1/4+i\sigma}e^{-Z/2}, \end{cases} \tag{32}$$

where (we use the main branch of the logarithmic function)

$$Z = i\frac{(\epsilon+Ez)^2}{E} = iZ_0, \qquad Z_0 > 0\ , \quad e^{\pm Z/2} = e^{\pm iZ_0/2}\ ,$$
$$Z^{-1/4\mp i\sigma} = \left(e^{\ln iZ_0}\right)^{-1/4\mp i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4\mp i\sigma}\ . \tag{33}$$

Let us find solutions in the whole region of the variable Z. To this end, we apply the substitution $\Phi(Z) = Z^A\, e^{BZ}\, f(Z)$, taking in mind the constraints $A = 0\,, 1/2\,,\ B = -1/2\,,$ we get the equation

$$\left(Z\frac{d^2}{dZ^2} + (2A + 1/2 - Z)\frac{d}{dy} - (A + 1/4 - i\sigma)\right)f(Z) = 0\,,$$

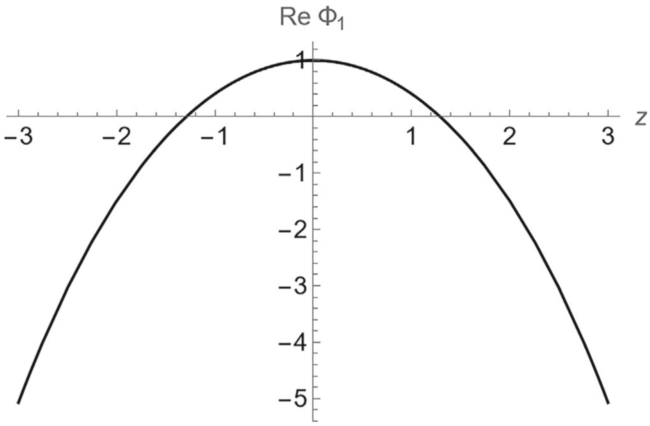

FIGURE 1 | Plot of the real function $\Phi_1(z)$ from (37).

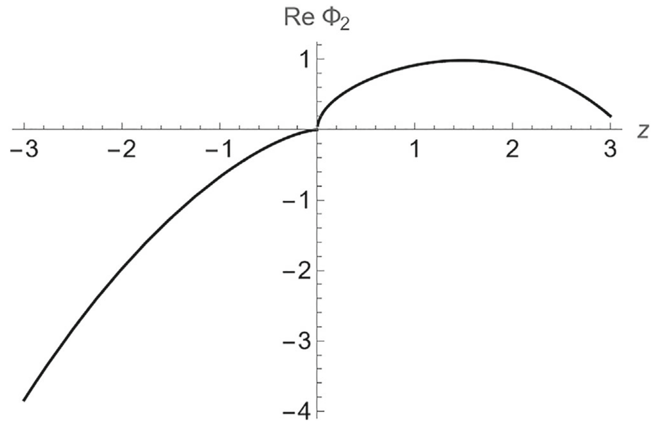

FIGURE 2 | Plot of $Re\ \Phi_2(z)$ from (37).

which can be recognized as the confluent hypergeometric equation with parameters

$$a = A + 1/4 - i\sigma\ , \quad c = 2A + 1/2\ , \quad f(Z) = Z^A\ e^{-Z/2}\ F(a, c; Z)\ .$$

Without loss of generality, we can use the value $A = 0$:

$$A = 0, \quad a = 1/4 - i\sigma\ , \quad c = +1/2\ , \quad \Phi(Z) = e^{-Z/2}\ f(Z) \tag{34}$$

The confluent hypergeometric equation has different sets of linearly independent solutions. First, consider the following ones (see in [38]):

$$\begin{aligned} Y_1(Z) &= F(a, c; Z) = e^Z F(c-a, c; -Z), \\ Y_2(Z) &= Z^{1-c} F(a-c+1, 2-c; Z) = Z^{1-c} e^Z F(1-a, 2-c; -Z)\ . \end{aligned} \tag{35}$$

They lead to the complete functions

$$\begin{aligned} \Phi_1 &= e^{-Z/2} F(a, c; Z) = e^{+Z/2} F(c-a, c; -Z), \\ \Phi_2 &= e^{-Z/2} Z^{1-c} F(a-c+1, 2-c; Z) \\ &= Z^{1-c} e^{+Z/2} F(1-a, 2-c; -Z)\ . \end{aligned} \tag{36}$$

Taking in mind the identities

$$\begin{aligned} & c = \frac{1}{2}, \quad a = \frac{1}{4} - i\sigma, \quad c - a = \frac{1}{4} + i\sigma = a^*, \\ & c = c^* = \frac{1}{2}, \quad Z^* = -Z\ , \\ & a - c + 1 = \frac{3}{4} - i\sigma = (1-a)^*, \quad (2-c) = (2-c)^* = \frac{3}{2}\ , \end{aligned}$$

we can conclude that the solution $\Phi_1(Z)$ is given by the real-valued function (see Figure 1), the second $\Phi_2(Z)$ (see Figures 2 and 3) has a definite symmetry under complex conjugation:

$$\Phi_1(Z) = +[\Phi_1(Z)]^*, \qquad \Phi_2(Z) = i[\Phi_2(Z)]^* \tag{37}$$

This property of the function $\Phi_2(Z)$ may be presented differently when using other normalization (see Figure 4)

$$\overline{\Phi}_2(Z) = \frac{1-i}{\sqrt{2}} \Phi_2(Z) = \left(\frac{1-i}{\sqrt{2}} \Phi_2(Z) \right)^* = (\overline{\Phi}_2(Z))^* \tag{38}$$

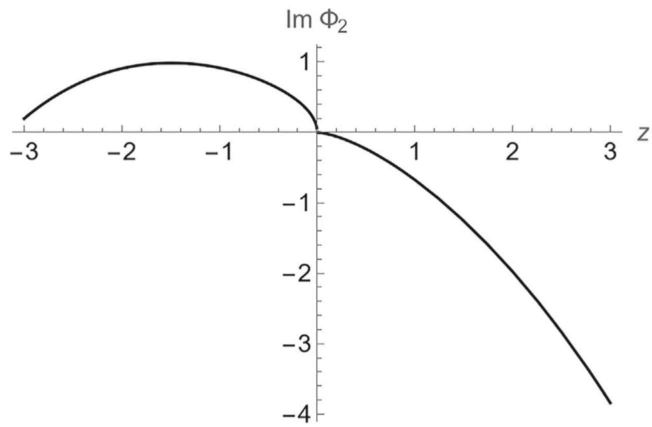

FIGURE 3 | Plot of $Im\ \Phi_2(z)$ from (38).

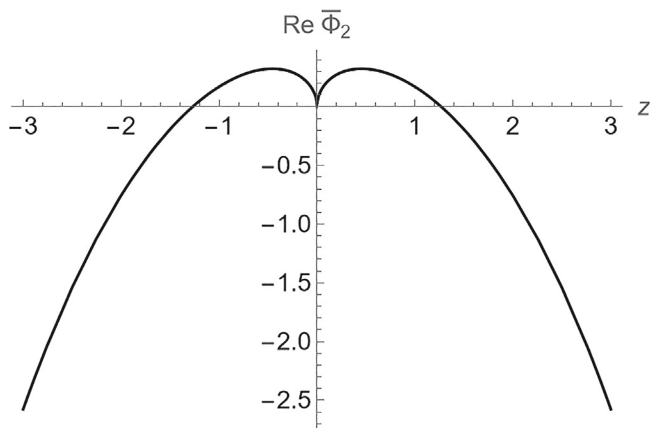

FIGURE 4 | Plot of the real function $\overline{\Phi}_2(z)$ from (38).

At small Z, the above solutions behave

$$\begin{aligned} & Y_1(Z) \approx 1, \qquad Y_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\frac{i}{eE}}\ (\epsilon + eEz)\ ; \\ & \Phi_1(Z) \approx 1, \qquad \Phi_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\frac{i}{eE}}\ \ (\epsilon + eEz)\ . \end{aligned} \tag{39}$$

At large $Z = iZ_0, Z_0 \to +\infty$, we can apply the asymptotic formula (see in [38])

$$F(a,c,Z) = \left(\frac{\Gamma(c)}{\Gamma(c-a)} (-Z)^{-a} + \ldots \right) + \left(\frac{\Gamma(c)}{\Gamma(a)} e^Z Z^{a-c} + \ldots \right) \tag{40}$$

Taking into account identities

$$\begin{aligned} (-Z)^{-a} &= (-iZ_0)^{-1/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4+i\sigma} \\ &= e^{-(-1/4+i\sigma)i\pi/2}\ e^{(-1/4+i\sigma)\ln Z_0}\ , \\ Z^{a-c} &= (iZ_0)^{-1/4-i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4-i\sigma} \\ &= e^{+(-1/4-i\sigma)i\pi/2}\ e^{(-1/4-i\sigma)\ln Z_0}\ , \end{aligned}$$

$$\frac{\Gamma(c)}{\Gamma(c-a)} = \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}, \qquad \frac{\Gamma(c)}{\Gamma(a)} = \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)}$$

we find the following behavior of the solutions:

$$\begin{aligned} Y_1(Z) &= F(a,c,Z) \\ &= e^{iZ_0/2} \left\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}\ e^{-(-1/4+i\sigma)i\pi/2}\ e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} \right. \\ &\quad \left. + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2}\ e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\} \end{aligned} \tag{41}$$

Whence after transition to the variable $\Phi_1(Z)$, we get

$$\begin{aligned} \Phi_1(Z) = &\left\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\sigma)}\ e^{-(-1/4+i\sigma)i\pi/2}\ e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} \right. \\ &\left. + \frac{\Gamma(1/2)}{\Gamma(1/4-i\sigma)} e^{+(-1/4-i\sigma)i\pi/2}\ e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\}, \end{aligned} \tag{42}$$

where we can see the sum of two conjugate terms. Similarly, we can examine behavior in infinity of the second solution $F(a-c+1, 2-c; Z)$:

$$F(a-c+1,2-c,Z) = \frac{\Gamma(2-c)}{\Gamma(1-a)} (-Z)^{-a+c-1} + \frac{\Gamma(2-c)}{\Gamma(a-c+1)} e^Z Z^{a-1}\ . \tag{43}$$

Whence taking into account the identities

$$\begin{aligned} (-Z)^{-a+c-1} &= (-iZ_0)^{-3/4+i\sigma} = \left(e^{\ln Z_0 - i\pi/2}\right)^{-3/4+i\sigma} \\ &= e^{-(-3/4+i\sigma)i\pi/2}\ e^{(-3/4+i\sigma)\ln Z_0}\ , \end{aligned}$$

$$Z^{a-1} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-3/4-i\sigma} = e^{+(-3/4-i\sigma)i\pi/2}\ e^{(-3/4-i\sigma)\ln Z_0}\ ,$$

$$\frac{\Gamma(2-c)}{\Gamma(1-a)} = \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}, \qquad \frac{\Gamma(2-c)}{\Gamma(a-c+1)} = \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)},$$

we find the following behavior in infinity:

$$\begin{aligned} &F(a-c+1,2-c,Z) = e^{iZ_0/2} \\ &\times \left\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}\ e^{-(-3/4+i\sigma)i\pi/2}\ e^{(-3/4+i\sigma)\ln Z_0} e^{-iZ_0/2} \right. \\ &\left. + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2}\ e^{(-3/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\}. \end{aligned} \tag{44}$$

Whence for the function $\Phi_2(Z)$, we derive (allowing for $\sqrt{Z} = e^{(1/2)(\ln Z_0 + i\pi/2)}$)

$$\begin{aligned} \Phi_2(Z) &= \sqrt{Z} F(a-c+1, 2-c, Z) = e^{i\pi/4} \\ &\times \left\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\sigma)}\ e^{-(-3/4+i\sigma)i\pi/2}\ e^{(-1/4+i\sigma)\ln Z_0} e^{-iZ_0/2} \right. \\ &\left. + \frac{\Gamma(3/2)}{\Gamma(3/4-i\sigma)} e^{+(-3/4-i\sigma)i\pi/2}\ e^{(-1/4-i\sigma)\ln Z_0} e^{+iZ_0/2} \right\}. \end{aligned} \tag{45}$$

It is possible to construct two solutions which at infinity behave as conjugate functions. To this end, we should use other pair of independent solutions (see in [38])

$$Y_5(Z) = \Psi(a,c;Z), \qquad Y_7(Z) = e^Z \Psi(c-a,c;-Z) \tag{46}$$

Two pairs $\{Y_5, Y_7\}$ and $\{Y_1, Y_2\}$ are related by Kummer formulas (see in [38])

$$\begin{aligned} Y_5 &= \frac{\Gamma(1-c)}{\Gamma(a-c+1)} Y_1 + \frac{\Gamma(c-1)}{\Gamma(a)} Y_2 \\ Y_7 &= \frac{\Gamma(1-c)}{\Gamma(1-a)} Y_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)} e^{i\pi c} Y_2 \end{aligned} \tag{47}$$

Whence we can derive asymptotic relations in the region $|Z| \to \infty$

$$\begin{aligned} Y_5(Z) &= Z^{-a} = (iZ_0)^{-1/4+i\sigma} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\sigma}\ , \\ Y_7(Z) &= e^Z \Psi(c-a,c;-Z) = e^Z (-iZ_0)^{a-c} \\ &= e^{iZ_0} (-iZ_0)^{-1/4-i\sigma} = e^{iZ_0} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\sigma}\ . \end{aligned} \tag{48}$$

The last formulas after translating them to variables $\Phi(Z)$ take on the form (see Figures 5 and 6)

$$\begin{aligned} \Phi_5(Z) &= e^{-Z/2}\ Y_5(Z) = e^{-iZ_0/2} \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\sigma}\ , \\ \Phi_7(Z) &= e^{-Z/2}\ Y_7(Z) = e^{+iZ_0/2} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\sigma}\ . \end{aligned} \tag{49}$$

These functions are conjugate to each other, they are presented in the combinations (42) and (45).

5 | Conclusions and Open Problems

In this paper, the quantum-mechanical equation for a spin 1 particle with anomalous magnetic moment is solved exactly in cylindrical coordinates (t, r, φ, z), the presence of the external uniform electric field was taken into account. In fact, the problem was reduced to the system on 10 first-order equations in partial derivatives over the variables (r, z). In resolving this system of equations, deciding role belongs to the application of method by Fedorov–Gronskiy [37] extended to the system of equations in partial derivative.

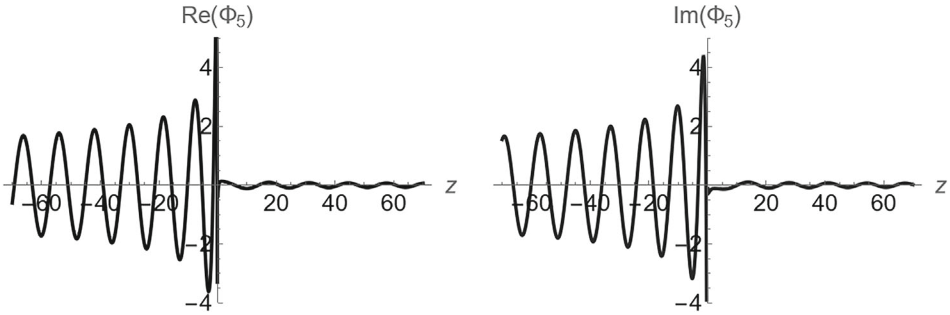

FIGURE 5 | Plots of functions $Re\ \Phi_5(z)$ and $Im\ \Phi_5(z)$ from (49), $\sigma = 1$.

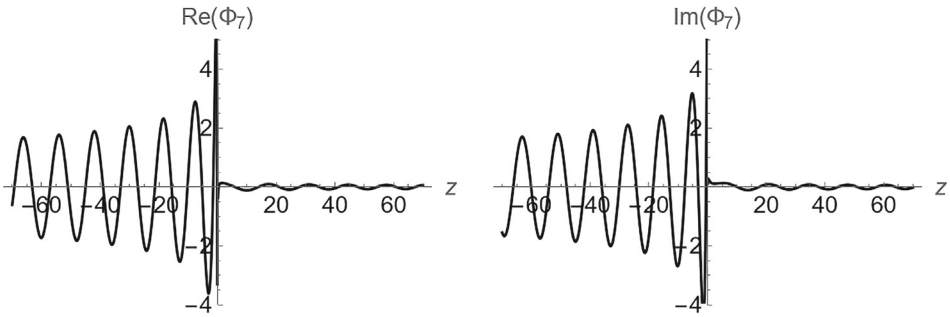

FIGURE 6 | Plots of functions $Re\ \Phi_7(z)$ and $Im\ \Phi_7(z)$ from (49), $\sigma = 1$.

The proposed method is rather general, it may be applied for studying different problems with cylindrical symmetry; for instance, for fields with spin 3/2 and 2. Besides, the obtained exact solutions may be used for experimental measuring the anomalous magnetic moment of the spin 1 particle, for instance of the vector bosons.

6 | Conclusions

In this paper, the quantum-mechanical equation for a spin 1 particle with anomalous magnetic moment is solved exactly in cylindrical coordinates (t, r, φ, z), the presence of the external uniform electric field was taken into account. In fact, the problem was reduced to the system on 10 first-order equations in partial derivatives over the variables (r, z). In resolving this system of equations, deciding role belongs to the application of method by Fedorov–Gronskiy [37] extended to the system of equations in partial derivative.

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Author Contributions

The authors equally contributed to this work in the conceptualization of the study, the analysis, and in the writing of the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest.

Endnotes

[1] Similar problem with Cartesian symmetry was examined in [32–35].

References

1. V. Barrera-Figueroa, V. S. Rabinovich, and S. A. C. Loredo-Ramrez, “On the Calculation of the Discrete Spectra of One-Dimensional Dirac Operators,” *Mathematical Methods in the Applied Sciences* 45, no. 16 (2022): 10218–10246.

2. A. V. Chichurin, E. M. Ovsiyuk, and V. M. Redkov, *Problems in Quantum Mechanics and Field Theory With Mathematical Modelling* (CRC Press, 2024).

3. M. Melgaard and G. Rozenblum, “Eigenvalue Asymptotics for Weakly Perturbed Dirac and Schrdinger Operators With Constant Magnetic Fields of Full Rank,” *Communications in Partial Differential Equations* 28, no. 34 (2003): 697–736.

4. R. T. de Aldecoa, “Asymptotics Near ± of the Spectral Shift Function for Dirac Operators With Non-Constant Magnetic Fields,” *Communications in Partial Differential Equations* 36 (2011): 10–41.

5. A. Proca, “Sur les Equations Fondamentales des Particules Élémentaires,” *Comptes Rendus de l’Académie des Sciences Paris* 202 (1936): 1490–1492.

6. A. Proca, “Théorie Non Relativiste des Particules a Spin Entier,” *Journal de Physique et Le Radium* 9 (1938): 61–66.

7. R. Y. Duffin, “On the Characteristic Matrices of Covariant Systems,” *Physics Review* 54, no. 12 (1938): 1114–1114.

8. N. Kemmer, “The Particle Aspect of Meson Theory,” *Proceedings. Royal Society of London* A73 (1939): 91–116.

9. H. C. Corben and J. Schwinger, “The Electromagnetic Properties of Mesotrons,” *Physics Review* 58 (1940): 953.

10. E. Schrödinger, “Pentads, Tetrads, and Triads of Meson Matrices,” *Proceedings of the Royal Irish Academy. Section A* 48 (1943): 135–146.

11. E. Schrödinger, “Systematics of Meson Matrices,” *Proceedings of the Royal Irish Academy. Section A* 49 (1943): 29–42.

12. N. Kemmer, “The Algebra of Meson Matrices,” *Proceedings. Cambridge Philological Society* 39 (1943): 189–196.

13. I. Fujiwara, “On the Duffin-Kemmer algebra,” *Progress in Theoretical Physics* 10, no. 6 (1953): 589–616.

14. N. Kemmer, “On the Theory of Particles of Spin 1,” *Helvetica Physica Acta* 33, no. 8 (1960): 829–838.

15. A. Da Silveira, “Kemmer Wave Equation in Riemann Space,” *Journal of Mathematical Physics* 1, no. 6 (1960): 489–491.

16. G. Casanova, “Particules Neutre de Spin 1,” *Comptes rendus de l'Académie des Sciences Paris. A* 268 (1969): 673–676.

17. J. A. Young and S. A. Bludman, “Electromagnetic Properties of a Charged Vector Meson,” *Physics Review* 131, no. 5 (1963): 2326–2334.

18. S. Gupta, “Wave Equation of Charged Particle of Spin 1, the Kemmer Equation,” *Indian Journal of Physics* 43, no. 2 (1969): 92–105.

19. T. Goldman, W. Tsai, and A. Yildiz, “Consistency of Spin One Theory,” *Physical Review D* 5, no. 8 (1972): 1926–1930.

20. E. A. Lord, “Six-Dimensional Formulation of Meson Equations,” *International Journal of Theoretical Physics* 5, no. 4-6 (1972): 339–348.

21. A. A. Bogush, V. V. Kisel, N. G. Tokarevskaya, and V. M. Red'kov, “Duffin – Kemmer – Petiau Formalism Reexamined: Nonrelativistic Approximation for Spin 0 and Spin 1 Particles in the Riemannian Space-Time,” *Annales de la Fondation Louis de Broglie* 32, no. 2-3 (2007): 355–381.

22. V. M. Red'kov, *Fields in Riemannian Space and the Lorentz Group (Russian)* (Belarusian Science, 2009).

23. E. M. Ovsiyuk, V. V. Kisel, and V. M. Red'kov, *Maxwell Electrodynamics and Boson Fields in Spaces of Constant Curvature* (Nova Science Publishers Inc, 2014).

24. V. A. Pletukhov, V. M. Red'kov, and V. I. Strazhev, *Relativistic Wave Equations and Intrinsic Degrees of Freedom (Russian)* (Belarusian Science, 2015).

25. V. V. Kisel, E. M. Ovsiyuk, O. V. Veko, Y. A. Voynova, V. Balan, and V. M. Red'kov, *Elementary Particles With Internal Structure in External Fields*, vol. I, II (Nova Science Publishers Inc., 2018).

26. A. Shamaly and A. Z. Capri, “First-Order Wave Equations for Integral Spin,” *Nuovo Cimento* B2, no. 2 (1971): 235–253.

27. A. Shamaly and A. Z. Capri, “Unified Theories for Massive Spin 1 Fields,” *Canadian Journal of Physics* 51, no. 14 (1973): 1467–1470.

28. V. V. Kisel, “Electrical Polarizability of Particles With Spin 1 in the Theory of Relativistic Wave Equations (Russian),” *Proceedings of the National Academy of Sciences of Belarus Physics and Mathematics Series* 18, no. 3 (1982): 73–78.

29. V. V. Kisel, E. M. Ovsiyuk, and V. M. Red'kov, “On the Wave Functions and Energy Spectrum for a Spin 1 particle in External Coulomb Field,” *Nonlinear Phenomena and Complex Systems* 13, no. 4 (2010): 352–367.

30. V. V. Kisel, V. M. Red'kov, and E. M. Ovsiyuk, “Wave Functions and Energy Spectrum for Spin 1 Particle in Coulomb Field (in Russian),” *Doklady Natsionalnoi Akademii Nauk Belarusi* 55, no. 1 (2011): 50–55.

31. E. M. Ovsiyuk, O. V. Veko, Y. A. Voynova, A. D. Koral'kov, V. V. Kisel, and V. M. Red'kov, “On Describing Bound States for a Spin 1 Particle in the external Coulomb Field,” *Balkan Society of Geometers Proceedings* 25 (2018): 59–78.

32. E. M. Ovsiyuk, Y. A. Voynova, V. V. Kisel, V. Balan, and V. M. Red'kov, “Spin 1 Particle With AnomalousMagnetic Moment in the External Uniform Electric field,” in *Quaternions: Theory and Applications*, ed. S. Griffin (Nova Science Publ. Inc., 2017), 47–84.

33. E. M. Ovsiyuk, Y. A. Voynova, V. V. Kisel, V. Balan, and V. M. Red'kov, “Techniques of Projectiv Operators Used to Construct Solutions for a Spin 1 Particle With Anomalous Magnetic Moment in the External Magnetic Field,” in *Quaternions: Theory and Applications*, ed. S. Griffin (Nova Science Publ. Inc., 2017), 11–46.

34. V. M. Kisel, Y. A. Voynova, E. M. Ovsiyuk, V. Balan, and V. M. Red'kov, “Spin 1 Particle With Anomalous Magnetic Moment in the external Uniform Magnetic Field,” *Nonlinear Phenomena and Complex Systems* 20, no. 1 (2017): 21–39.

35. E. M. Ovsiyuk, Y. A. Voynova, V. V. Kisel, V. Balan, and V. M. Red'kov, “Spin 1 Particle With Anomalous Magnetic Moment in the External Uniform Electric Field,” *Nonlinear Phenomena and Complex Systems* 21, no. 1 (2018): 1–20.

36. V. V. Kisel, E. M. Ovsiyuk, Y. A. Voynova, and V. M. Red'kov, “Quantum Mechanics of Spin 1 particle with quadrupole Moment in External Uniform Magnetic Field (Russian),” *Problems of Physics, Mathematics, and Thechnics* 32, no. 3 (2017): 18–27.

37. V. K. Gronskiy and F. I. Fedorov, “Magnetic Properties of a Particle With Spin 3/2,” *Doklady of the National Academy of Sciences of Belarus* 4, no. 7 (1960): 278–283.

38. H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, vol. 1 (McGraw-Hill, 1953).