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Spin 1 Particle with Two Additional Electromagnetic Characteristics in Presence of the Both Magnetic and Electric Fields

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In the paper, we study a generalized Duffin–Kemmer equation for spin 1 particle with two characteristics, anomalous magnetic moment and polarizability in presence of external uniform magnetic and electric fields. After separating the variables, we get the system of ten first order partial differential equations for 10 functions $f_i(r,z)$. To describe the r-dependence of 10 functions $f_A(r,z), A=1,...,10$, we apply the method by Fedorov – Gronskiy; so the complete 10-component wave function is decomposed into the sum of three projective constituents, dependence of each component on the polar coordinate r is determined by only one corresponding function, $F_i(r), i=1,2,3$; these three basic functions are constructed in terms of the confluent hypergeometric functions, at this there arises the quantization rule due to the presence of magnetic field. After that we derive a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved by using the elimination method and with the help of special linear combining of the involved functions. As the result, we find three separated second order differential equations, their solutions are constructed in the terms of the confluent hypergeometric functions. Thus, in this paper, the three types of solutions for a vector particle with two additional electromagnetic characteristics in presence of external uniform magnetic and electric fields are found.

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1. Introduction

In [2], Within the general method by Gel'fand – Yaglom [1], for a spin 1 particle it was constructed a relativistic generalized system of the first order equations for a particle with two additional characteristics, anomalous magnetic moment and polarizability (a number of relevant papers see in [3]—[14]).

First, the model was developed for a free particle, and the system of spinor equations was obtained; then it was transformed to tensor form. In tensor form, the presence of external electromagnetic fields was taken into account. After eliminating the accessory variables of the complete wave function, it was derived the generalized Proca system of 10 equations, it contains two additional interaction terms which are interpreted as related to the anomalous magnetic moment and polarizability. In [16], this equation was solved in presence of the uniform magnetic field; in [17], this equation solved in the presence of electric field (also see [18]).

In the present paper, we will consider the situation when the both fields, magnetic and

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electric, are presented. After separating the variables, we get the system of ten first order partial differential equations for 10 functions $f_i(r,z)$. To describe the r-dependence of 10 functions $f_A(r,z), A=1,...,10$, we apply the method by Fedorov – Gronskiy [15]; in this approach, the complete 10-component wave function is decomposed into the sum of three projective constituents, dependence of each component on the polar coordinate is determined by only one function, $F_i(r), i=1,2,3$; they are constructed in terms of the confluent hypergeometric functions. At this there arises the quantization rule due to the presence of magnetic field.

After that we derive a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved, as the result, we find three independent solutions.

2. Matrix equation in Minkowski space

We start [2] with the following tensor equations (let $D_a = \partial_a + ieA_a$)

$$D^b \Phi_{ab} + e\mu F_{ab}\Phi^b + e\sigma D_a(F^{cd}\Phi_{cd}) - M\Phi_a = 0, \quad D_a\Phi_b - D_b\Phi_a - M\Phi_{ab} = 0; \tag{1}$$

they may be compared with the ordinary Proca system

$$D^b\Phi_{ab} - M\Phi_a = 0, \quad D_a\Phi_b - D_b\Phi_a - M\Phi_{ab} = 0. \tag{2}$$

In (1), we can see two additional interaction terms, proportional to parameters μ (anomalous magnetic moment) and σ (polarizability); in the paper [16], it was proved that that both parameters μ, σ are imaginary: $\mu \Longrightarrow i\mu, \sigma \Longrightarrow i\sigma$, we will take this into account later on. Below we apply the 10-dimensional column:

$$\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) = (H_1; H_2).$$

Let us recall the matrix form of the Proca system when $\mu = 0, \sigma = 0$. The first equation gives $K^a D_a H_2 - MH_1 = 0$, where

$$K^0 = \left| \begin{array}{cccccc} . & . & . & . & . & . \\ -1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & -1 & . & . & . \end{array} \right|, K^1 = \left| \begin{array}{cccccc} -1 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & -1 & . \end{array} \right|,$$

$$K^2 = \left| \begin{array}{cccccc} . & -1 & . & . & . & . \\ . & . & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & +1 & . & . \end{array} \right|, K^3 = \left| \begin{array}{cccccc} . & . & -1 & . & . & . \\ . & . & . & . & +1 & . \\ . & . & . & -1 & . & . \\ . & . & . & . & . & . \end{array} \right|.$$

The second equation in (2) leads to $D_a L^a H_1 - MH_2 = 0$,where

$$L^0 = \left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|, L^1 = \left| \begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right|, L^2 = \left| \begin{array}{cccc} . & . & . & . \\ -1 & . & . & . \\ . & . & . & . \\ . & . & . & 1 \\ . & . & . & . \\ . & -1 & . & . \end{array} \right|, L^3 = \left| \begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ -1 & 0 & 0 & . \\ . & . & -1 & . \\ . & 1 & . & . \\ . & . & . & . \end{array} \right|.$$

Thus, the system of equations for the ordinary spin 1 particle is presented in the block form as

$$K^a D_a H_2 - MH_1 = 0, \quad L^a D_a H_1 - MH_2 = 0.$$

Let us detail the first additional term in (1) (taking in mind identities: $F_{01}\Phi^1 = -E^1\Phi_1$, $F_{12}\Phi^2 = B^3\Phi_2$, and so on)

$$e\mu F_{ab}\Phi^b = e\mu \left| \begin{array}{cccc} 0 & -E^1 & -E^2 & -E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{array} \right| \left| \begin{array}{c} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{array} \right|,$$

whence allowing for the structure of six Lorentzian generators for vector field

$$j^{23}=S_1=\begin{vmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{vmatrix},j^{31}=S_2=\begin{vmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{vmatrix},j^{12}=S_3=\begin{vmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{vmatrix},$$

$$j^{01}=T_1=\begin{vmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{vmatrix},j^{02}=T_2=\begin{vmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{vmatrix},j^{03}=T_3=\begin{vmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{vmatrix},$$

we obtain the more short presentation for this term

$$e\mu F_{\alpha\beta}\Phi^{\beta}\quad\Longrightarrow\quad -e\mu\Big(\mathbf{S}\,\mathbf{B}+\mathbf{T}\,\mathbf{E}\Big)H_1.$$

The second additional term in (1) is

$$e\sigma D_a(F^{cd}\Phi_{cd})=2e\sigma D_a\begin{vmatrix}E^1&0&0&0&0&0\\0&E^2&0&0&0&0\\0&0&E^3&0&0&0\\0&0&0&-B^1&0&0\\0&0&0&0&-B^2&0\\0&0&0&0&0&-B^3\end{vmatrix}\begin{vmatrix}\Phi_{01}\\\Phi_{02}\\\Phi_{03}\\\Phi_{23}\\\Phi_{31}\\\Phi_{12}\end{vmatrix}.$$

So we have a generalized Duffin – Kemmer – Petiau equation in the block form

$$\begin{gathered}\Big(K^aD_aH_2\Big)_c-e\mu\Big[(\mathbf{S}\,\mathbf{B}+\mathbf{T}\,\mathbf{E})H_1\Big]_c+e\sigma D_c(F^{kl}\Phi_{kl})-M(H_1)_c=0,\\\Big(L^aD_aH_1\Big)_{[kl]}-M(H_2)_{[kl]}=0.\end{gathered}\tag{3}$$

3. Extension to curved space-time models

In Riemannian space, we start with the

$$D_\beta\Phi_\alpha^{\ \beta}+e\mu F_{\alpha\beta}\Phi^\beta+e\sigma\hat{\partial}_\alpha(F^{\rho\delta}\Phi_{\rho\delta})-M\Phi_\alpha=0,\quad D_\alpha\Phi_\beta-D_\beta\Phi_\alpha-M\Phi_{\alpha\beta}=0;$$

below two different derivative symbols will be used: $D_\alpha=\nabla_\alpha+ieA_\alpha,\ \hat{\partial}_\alpha=\partial_\alpha+ieA_\alpha$. Let us transform these equations to tetrad form (apply the notation $e^{\beta}_{(b)}(\partial_\beta+ieA_\beta)=\hat{\partial}_{(b)}$) Using the Ricci rotation coefficients, we present the above equations as follows

$$\hat{\partial}_{(b)}\Phi_c^{\ b}-\gamma_{acb}\Phi^{ab}+e^{\beta}_{(b);\beta}\Phi_c^{\ b}+e\mu F_{cb}\Phi^b+e\sigma(\partial_{(c)}F^{ab})\Phi_{ab}+e\sigma F^{ab}\hat{\partial}_{(c)}\Phi_{ab}-M\Phi_c=0,$$

$$\hat{\partial}_{(c)}\Phi_d-\hat{\partial}_{(d)}\Phi_c+\gamma_{bdc}\Phi^b-\gamma_{bcd}\Phi^b-M\Phi_{cd}=0.$$

We should recall the known matrix tetrad form of the ordinary vector particle

$$\Big[\beta^c\Big(e^{\alpha}_{(c)}(x)\frac{\partial}{\partial x^\alpha}+\frac{1}{2}J^{ab}\gamma_{abc}\Big)-M\Big]\Phi=0,\ \ \Phi=\begin{vmatrix}\Phi_a\\\Phi_{ab}\end{vmatrix}=\begin{vmatrix}H_1\\H_2\end{vmatrix};\tag{4}$$

two additional interactions terms are

$$e\mu F_{\alpha\beta}\Phi^\beta=e\mu\Big(\mathbf{S}\,\mathbf{B}-\mathbf{T}\,\mathbf{E}\Big)H_1,\quad e\sigma\hat{\partial}_\alpha(F^{cd}\Phi_{cd})=e\sigma\begin{vmatrix}\hat{\partial}_{(0)}(F^{cd}\Phi_{cd})\\\hat{\partial}_{(1)}(F^{cd}\Phi_{cd})\\\hat{\partial}_{(2)}(F^{cd}\Phi_{cd})\\\hat{\partial}_{(3)}(F^{cd}\Phi_{cd})\end{vmatrix}.$$

So, we have the following generalized system of equations

$$
\begin{gathered}
\Big(K^c D^{(2)}_c H_2\Big)_n - e\mu\Big[(\mathbf{SB}+\mathbf{TE})H_1\Big]_n + e\sigma e^{\alpha}_{(n)}\hat{\partial}_\alpha(F^{kl}\Phi_{kl}) - M(H_1)_n = 0,\\
\Big(L^c D^{(1)}_c H_1\Big)_{[kl]} - (MH_2)_{[kl]} = 0.
\end{gathered} \tag{5}
$$

4. Particle in the uniform magnetic and electric fields

It is convenient to use the cylindrical coordinates $x^\alpha = (t, r, \phi, z)$: the relevant tetrad, Ricci rotation coefficients, the uniform magnetic and electric field are determined as

$$
\begin{gathered}
dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \qquad e^{\alpha}_{(a)} = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & 0 & 0 & 1 \end{vmatrix},\\
\gamma_{122} = \frac{1}{r}, A_0 = -Ez,\; F_{03} = e^{t}_{(0)} e^{z}_{(3)} F_{tz} = E,\; \hat{\partial}_0 \Longrightarrow -i\epsilon - ieEz,\\
A_\phi = -Br^2/2, \quad F_{r\phi} = -Br, \quad F_{12} = e^{r}_{(1)} e^{\phi}_{(2)} F_{r\phi} = -B.
\end{gathered} \tag{6}
$$

Correspondingly, the system of equations takes on the form

$$
\Big[K^0(\partial_0 - ieEz) + K^1\partial_r + K^2\frac{1}{r}\Big(\partial_\phi + \frac{ieBr^2}{2} + j^{12}_2\Big) + K^3\partial_z\Big]H_2 - MH_1
$$

$$
-e\mu B j^{12}_1 H_1 - e\mu E j^{03}_1 H_1 - 2Be\sigma \begin{vmatrix} (\partial_0 - ieEz)\\ \partial_r\\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2})\\ \partial_3 \end{vmatrix} \Phi_{12} + 2Ee\sigma \begin{vmatrix} (\partial_0 - ieEz)\\ \partial_r\\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2})\\ \partial_z \end{vmatrix} \Phi_{03} = 0,
$$

$$
\Big[L^0(\partial_0 - ieEz) + L^1\partial_r + L^2\frac{1}{r}(\partial_\phi + ieBr^2/2 + j^{12}_1) + L^3\partial_z\Big]_1 - MH_2 = 0.
$$

It is more convenient to apply the so called cyclic basis. It is defined by requirement to have a diagonal generator j^{12}_1 for the vector field $H_1 = (\Phi_l)$. The needed transformation $\bar{\Phi} = U\Phi$ is

$$
U = \begin{vmatrix} 1 & 0 & 0 & 0\\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \bar{j}^{12} = \begin{vmatrix} 0 & 0 & 0 & 0\\ 0 & -i & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & +i \end{vmatrix}. \tag{7}
$$

Vector and tensor generators transform according to the rules

$$
\bar{J}^{ab}_1 = U j^{ab} U^{-1}, \quad \bar{J}^{ab}_2 = \bar{j}^{ab} \otimes I + I \otimes \bar{j}^{ab};
$$

$$
J^{12}_{(2)} = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \Longrightarrow \bar{J}^{12}_2 = i \begin{vmatrix} -1 & . & . & . & . & .\\ . & 0 & . & . & . & .\\ . & . & 1 & . & . & .\\ . & . & . & 1 & . & .\\ . & . & . & . & 0 & .\\ . & . & . & . & . & -1 \end{vmatrix}.
$$

We should transform the Duffin – Kemmer matrices β^a to the cyclic basis as well; it is convenient to apply the block presentation: $\bar{H}_1 = C_1 H_1,\ (C_1 = U),\ \bar{H}_2 = (U \otimes U) H_2 = C_2 H_2$; further we obtain

$$\left| \begin{array}{cc} 0 & \bar{K}^a \\ \bar{L}^a & 0 \end{array} \right| = \left| \begin{array}{cc} 0 & C_1 K^a C_2^{-1} \\ C_2 L^a C_1^{-1} & 0 \end{array} \right| .$$

We derive

$$C_1 = U = \left| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{array} \right|, \quad C_2 = \left| \begin{array}{cccccc} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{array} \right| . \tag{8}$$

Further, we readily find the needed blocks:

$$\bar{K}^0 = \left| \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{array} \right|, \bar{K}^1 = \left| \begin{array}{cccccc} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{array} \right|,$$

$$\bar{K}^2 = \left| \begin{array}{cccccc} \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{array} \right|, \ \bar{K}^3 = \left| \begin{array}{cccccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right|,$$

$$\bar{L}^0 = \left| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|, \bar{L}^1 = \left| \begin{array}{cccc} 1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -1/\sqrt{2} & 0 \end{array} \right|,$$

$$\bar{L}^2 = \left| \begin{array}{cccc} -i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i/\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -i/\sqrt{2} & 0 \\ 0 & i/\sqrt{2} & 0 & -i/\sqrt{2} \\ 0 & 0 & i/\sqrt{2} & 0 \end{array} \right|, \bar{L}^3 = \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array} \right|,$$

and also expressions for the needed generators

$$\bar{J}_1^{12} = i \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{array} \right|, \quad \bar{j}_1^{03} = \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right|,$$

Taking this in mind, we can transform the above two equations to the cyclic basis; so we get

$$\Big[\bar{K}^0(\partial_0 - ieEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}\Big(\partial_\phi + \frac{ieBr^2}{2} + \bar{j}_2^{12}\Big) + \bar{K}^3\partial_z\Big]\bar{H}_2 - M\bar{H}_1$$

$$-e\mu B\bar{j}_1^{12}\bar{H}_1 - e\mu E\bar{j}_1^{03}\bar{H}_1 - 2Be\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} (-i\bar{\Phi}_{31})$$

$$+2Ee\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} \bar{\Phi}_{02} = 0,$$

so that

$$\Big[\bar{L}^0(\partial_0 - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2} + \bar{j}_1^{12}) + \bar{L}^3\partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0, \tag{9}$$

$$\Big[\bar{L}^0(\partial_0 - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(\partial_\phi + \bar{j}_1^{12}) + \bar{L}^3\partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0. \tag{10}$$

Now let us perform the separation of the variables, applying the substitution

$$\bar{H}_1 = e^{-i\epsilon t}e^{im\phi}\begin{vmatrix} \bar{\Phi}_0(r,z) \\ \bar{\Phi}_1(r,z) \\ \bar{\Phi}_2(r,z) \\ \bar{\Phi}_3(r,z) \end{vmatrix}, \bar{H}_2 = e^{-i\epsilon t}e^{im\phi}\begin{vmatrix} \bar{\Phi}_{01}(r,z) \\ \bar{\Phi}_{02}(r,z) \\ \bar{\Phi}_{03}(r,z) \\ \bar{\Phi}_{23}(r,z) \\ \bar{\Phi}_{31}(r,z) \\ \bar{\Phi}_{12}(r,z) \end{vmatrix} = e^{-i\epsilon t}e^{im\phi}\begin{vmatrix} \bar{E}_1(r,z) \\ \bar{E}_2(r,z) \\ \bar{E}_3(r,z) \\ \bar{B}_1(r,z) \\ \bar{B}_2(r,z) \\ \bar{B}_3(r,z) \end{vmatrix},$$

in this way we obtain (for brevity let as make the change in notations: $eB \to B, eE \to E$)

$$\Big[\bar{K}^0(-i\epsilon - iEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}\Big(im + \frac{iBr^2}{2} + \bar{j}_2^{12}\Big) + \bar{K}^3\partial_z\Big]\bar{H}_2 - M\bar{H}_1$$

$$-\mu B\bar{j}_1^{12}\bar{H}_1 - 2B\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (-i\epsilon - iEz) \\ \partial_r \\ \frac{1}{r}(im + \frac{iBr^2}{2}) \\ \partial_z \end{vmatrix} (-i\bar{B}_2)$$

$$-\mu E\bar{j}_1^{03}\bar{H}_1 + 2E\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (-i\epsilon - iEz) \\ \partial_r \\ \frac{1}{r}(im + \frac{iBr^2}{2}) \\ \partial_z \end{vmatrix} \bar{E}_2 = 0,$$

$$\Big[\bar{L}^0(-i\epsilon - iEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(im + \frac{iBr^2}{2} + \bar{j}_1^{12}) + \bar{L}^3\partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0;$$

Further we obtain the explicit form of 10 equations (for brevity we will omit the overline symbol). With the use of shortening notations

$$\begin{aligned} a_m &= \frac{\partial}{\partial r} + \frac{Br}{2} + \frac{m}{r}, \quad a_{m+1} = \frac{\partial}{\partial r} + \frac{Br}{2} + \frac{m+1}{r}, \\ b_m &= \frac{\partial}{\partial r} - \frac{Br}{2} - \frac{m}{r}, \quad b_{m-1} = \frac{\partial}{\partial r} - \frac{Br}{2} - \frac{m-1}{r}, \end{aligned} \tag{11}$$

these equations read

$$1 \quad \frac{1}{\sqrt{2}}b_{m-1}E_1 - \frac{1}{\sqrt{2}}a_{m+1}E_3 - \partial_z E_2 + \mu E\Phi_2 - 2iE\sigma(\epsilon + Ez)E_2 + 2B\sigma(Ez+\epsilon)B_2 = M\Phi_0,$$

$$2 \quad i(\epsilon + Ez)E_1 + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)a_m B_2 - \partial_z B_3\sqrt{2}E\sigma a_m E_2 + iB\mu\Phi_1 = M\Phi_1,$$

$$3 \quad + i(\epsilon + Ez)E_2 - \frac{1}{\sqrt{2}}a_{m+1}B_1 - \frac{1}{\sqrt{2}}b_{m-1}B_3 - \mu E\Phi_0 + 2E\sigma\partial_z E_2 + 2iB\sigma\partial_z B_2 = M\Phi_2,$$

$$4 \quad i(\epsilon + Ez)E_3 + \frac{1}{\sqrt{2}}i(2B\sigma - i)b_m B_2 + \partial_z B_1 + \sqrt{2}E\sigma b_m E_2 - iB\mu\Phi_3 = M\Phi_3,$$

$$5 \quad \frac{1}{\sqrt{2}}a_m\Phi_0 - i(\epsilon + Ez)\Phi_1 = ME_1, \quad 6 \quad -\partial_z\Phi_0 - i(\epsilon + Ez)\Phi_2 = ME_2,$$

$$7 \quad -\frac{1}{\sqrt{2}}b_m\Phi_0 - i(\epsilon + Ez)\Phi_3 = ME_3, \quad 8 \quad -\frac{1}{\sqrt{2}}b_m\Phi_2 + \partial_z\Phi_3 = MB_1,$$

$$9 \quad \frac{1}{\sqrt{2}}b_{m-1}\Phi_1 + \frac{1}{\sqrt{2}}a_{m+1}\Phi_3 = MB_2, \quad 10 \quad -\frac{1}{\sqrt{2}}a_m\Phi_2 - \partial_z\Phi_1 = MB_3.$$

5. Projective operators method

To analyze the system of equations, we will use the method of projective operators. To this end, we consider the third spin projection $Y = -i\bar{J}^{12}$, and make sure that it satisfies the minimal equation $Y(Y-1)(Y+1) = 0$. This minimal equation allows us to introduce three projective operators

$$P_0 = 1 - Y^2, \quad P_{+1} = \frac{1}{2}Y(Y+1), \quad P_{-1} = \frac{1}{2}Y(Y-1), \tag{12}$$

with the needed properties $P_0^2 = P_0,\ P_{+1}^2 = P_{+1}, P_{-1}^2 = P_{-1},\ P_0 + P_{+1} + P_{-1} = 1$. Accordingly, the complete wave function can be expanded in the sum of three parts

$$\bar{\Phi} = \bar{\Phi}_0 + \bar{\Phi}_{+1} + \bar{\Phi}_{-1}, \quad \bar{\Phi}_\sigma = P_\sigma\bar{\Phi}, \quad \sigma = 0,\, +1,\, -1. \tag{13}$$

These components have the following dependence on the variable r (in accordance with the Fedorov – Gronsky method, each projective component should be determined by only one function of r):

$$\bar{\Phi}_0 = \left|\begin{array}{c} \bar{\Phi}_0(z) \\ 0 \\ \bar{\Phi}_2(z) \\ 0 \\ 0 \\ \bar{E}_2(z) \\ 0 \\ 0 \\ \bar{B}_2(z) \\ 0 \end{array}\right| F_1(r), \bar{\Phi}_{+1} = \left|\begin{array}{c} 0 \\ 0 \\ 0 \\ \bar{\Phi}_3 \\ 0 \\ 0 \\ \bar{E}_3(z) \\ \bar{B}_1(z) \\ 0 \\ 0 \end{array}\right| F_2(r), \Phi_{-1} = \left|\begin{array}{c} 0 \\ \bar{\Phi}_1(z) \\ 0 \\ 0 \\ \bar{E}_1(z) \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{B}_3(z) \end{array}\right| F_3(r); \tag{14}$$

$$\begin{gathered} 1 \quad \frac{1}{\sqrt{2}}E_1(z)b_{m-1}F_3(r) - \frac{1}{\sqrt{2}}E_3(z)a_{m+1}F_2(r) - \partial_z E_2(z)F_1(r) \\ -\mu E\Phi_2(z)F_1(r) - 2iE\sigma(\epsilon + Ez)E_2(z)F_1(r) + \\ +2B\sigma(Ez+\epsilon)B_2(z)F_1(r) = M\Phi_0(z)F_1(r), \end{gathered}$$

$$\begin{gathered} 2 \quad i(\epsilon + Ez)E_1(z)F_3(r) + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)B_2(z)a_m F_1(r) - \partial_z B_3(z)F_3(r) \\ -\sqrt{2}E\sigma E_2(z)a_m F_1(r) + iB\mu\Phi_1(z)F_3(r) = M\Phi_1(z)F_3(r), \end{gathered}$$

$$3 \quad + i(\epsilon + Ez)E_2(z)F_1(r) - \frac{1}{\sqrt{2}}B_1(z)a_{m+1}F_2(r) - \frac{1}{\sqrt{2}}B_3(z)b_{m-1}F_3(r)$$
$$-\mu E\Phi_0(z)F_1(r) + 2E\sigma\partial_z E_2(z)F_1(r)+2iB\sigma\partial_z B_2(z)F_1(r) = M\Phi_2(z)F_1(r),$$
$$4 \quad i(\epsilon + Ez)E_3(z)F_2(r) + \frac{1}{\sqrt{2}}i(2B\sigma - i)B_2(z)b_m F_1(r) + \partial_z B_1(z)F_2(r)$$
$$+\sqrt{2}E\sigma E_2(z)b_m F_1(r) - iB\mu\Phi_3(z)F_2(r) = M\Phi_3(z)F_2(r),$$
$$5 \quad \frac{1}{\sqrt{2}}\Phi_0(z)a_m F_1(r) - i(\epsilon + Ez)\Phi_1(z)F_3(r) = ME_1(z)F_3(r),$$
$$6 \quad -\partial_z\Phi_0(z)F_1(r) - i(\epsilon + Ez)\Phi_2(z)F_1(r) = ME_2(z)F_1(r),$$
$$7 \quad -\frac{1}{\sqrt{2}}\Phi_0(z)b_m F_1(r) - i(\epsilon + Ez)\Phi_3(z)F_2(r) = ME_3(z)F_2(r),$$
$$8 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)b_m F_1(r) + \partial_z\Phi_3(z)F_2(r) = MB_1(z)F_2(r),$$
$$9 \quad \frac{1}{\sqrt{2}}\Phi_1(z)b_{m-1}F_3(r) + \frac{1}{\sqrt{2}}\Phi_3(z)F_2(r) = MB_2(z)F_1(r),$$
$$10 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)a_m F_1(r) - \partial_z\Phi_1(z)F_3(r) = MB_3(z)F_3(r),$$

In order to obtain equations in the variable z, we should impose the following constraints

$$b_{m-1}F_3 = C_1F_1, \quad a_mF_1 = C_4F_3, \quad a_{m+1}F_2 = C_2F_1, \quad b_mF_1 = C_3F_2, \tag{15}$$

so we get

$$1 \quad \frac{1}{\sqrt{2}}E_1(z)C_1 - \frac{1}{\sqrt{2}}E_3(z)C_2 - \partial_z E_2(z) - \mu E\Phi_2(z) - 2iE\sigma(\epsilon + Ez)E_2(z) + 2B\sigma(Ez + \epsilon)B_2(z) = M\Phi_0(z),$$
$$2 \quad i(\epsilon + Ez)E_1(z) + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)B_2(z)C_4 - \partial_z B_3(z) - \sqrt{2}E\sigma E_2(z)C_4 + iB\mu\Phi_1(z) = M\Phi_1(z),$$
$$3 \quad + i(\epsilon + Ez)E_2(z) - \frac{1}{\sqrt{2}}B_1(z)C_2 - \frac{1}{\sqrt{2}}B_3(z)C_1 - \mu E\Phi_0(z) + 2E\sigma\partial_z E_2(z) + 2iB\sigma\partial_z B_2(z) = M\Phi_2(z),$$
$$4 \quad i(\epsilon + Ez)E_3(z) + \frac{1}{\sqrt{2}}i(2B\sigma - i)B_2(z)C_3 + \partial_z B_1(z) + \sqrt{2}E\sigma E_2(z)C_3 - iB\mu\Phi_3(z) = M\Phi_3(z),$$
$$5 \quad \frac{1}{\sqrt{2}}\Phi_0(z)C_4 - i(\epsilon + Ez)\Phi_1(z) = ME_1(z), \quad 6 \quad -\partial_z\Phi_0(z) - i(\epsilon + Ez)\Phi_2(z) = ME_2(z),$$
$$7 \quad -\frac{1}{\sqrt{2}}\Phi_0(z)C_3 - i(\epsilon + Ez)\Phi_3(z) = ME_3(z), \quad 8 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)C_3 + \partial_z\Phi_3(z) = MB_1(z),$$
$$9 \quad \frac{1}{\sqrt{2}}\Phi_1(z)C_1 + \frac{1}{\sqrt{2}}\Phi_3(z)C_2 = MB_2(z), \quad 10 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)C_4 - \partial_z\Phi_1(z) = MB_3(z),$$

6. Explicit form of three basic functions

From the differential constraints

$$b_{m-1}F_3(r) = C_1F_1(r), \quad a_mF_1(r) = C_4F_3(r),$$
$$a_{m+1}F_2(r) = C_2F_1(r), \quad b_mF_1(r) = C_3F_2(r);$$

It is obvious that the parameters in each pair can be chosen the same: $C_4 = C_1$, $C_3 = C_2$. In this case, the obtain the following constraints

$$b_{m-1}F_3(r) = C_1F_1(r), \quad a_mF_1(r) = C_1F_3(r),$$
$$a_{m+1}F_2(r) = C_2F_1(r), \quad b_mF_1(r) = C_2F_2(r);$$

and the following second order equations

$$(b_{m-1}a_m - C_1^2)F_1 = 0, \quad (a_m b_{m-1} - C_1^2)F_3 = 0,$$
$$(a_{m+1}b_m - C_2^2)F_1 = 0, \quad (b_m a_{m+1} - C_2^2)F_2 = 0.$$

These equations explicitly read

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\Big)F_1 = 0,$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm - B - C_2^2\Big)F_1 = 0,$$

therefore $C_2^2 = C_1^2 - 2B$ and

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\Big)F_3 = 0,$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_2^2\Big)F_2 = 0.$$

Thus, we have only 3 and the constraint $C_2^2 = C_1^2 - 2B$):

$$1, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\Big)F_1 = 0,$$
$$2, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_1^2 + 2B\Big)F_2 = 0,$$
$$3, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\Big)F_3 = 0.$$

Let $B - C_1^2 = X$, then

$$1, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + X\Big)F_1 = 0,$$
$$2, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - B(m-1) + X\Big)F_2 = 0,$$
$$3, \qquad \Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - B(m+1) + X\Big)F_3 = 0.$$

In the variable $x = \frac{Br^2}{2}$ read

$$1, \quad F_1(x) = x^{+\frac{|m|}{2}} e^{-x/2} F_1(x), \quad F_1(x) = \Phi(-n_1, \, |m| + 1, x)$$

$$X = 2B\Big(\frac{|m| + m}{2} + \frac{1}{2} + n_1\Big) > B, \quad n_1 = 0, 1, 2, ...$$

$$2, \quad F_2(x) = x^{+\frac{|m+1|}{2}} e^{-x/2} F_3(x), \quad F_3(x) = \Phi(-n_3, \, |m+1| + 1, x)$$

$$X = 2B\Big(\frac{|m+1| + m - 1}{2} + \frac{1}{2} + n_3\Big) > B, \quad n_3 = 0, 1, 2, ...$$

$$3, \quad F_3(x) = x^{+\frac{|m-1|}{2}} e^{-x/2} F_2(x), \quad F_2(x) = \Phi(-n_2, \, |m-1| + 1, x)$$

$$X = 2B\Big(\frac{|m-1| + m + 1}{2} + \frac{1}{2} + n_2\Big) > B, \quad n_2 = 0, 1, 2, ...$$

The quantity X is the same, below we apply the variant

$$X = 2BN > 0, \quad N = \frac{|m| + m}{2} + \frac{1}{2} + n, \quad n = 0, 1, 2, ... \; ; \tag{16}$$

note that parameter N takes half-integer values. Note the formulas

$$C_4 = C_1 = i\sqrt{X - B}, \quad C_3 = C_2 = i\sqrt{X + B}.$$

7. Solving equations in z-variable

Let us turn to the system in z-variable and take into account relations

$$C_4 = C_1 = i\sqrt{X-B}, \quad C_3 = C_2 = i\sqrt{X+B};$$

in this way, we obtain (dividing equations in two groups):

Subsystem I, it is algebraic with respect to $\bar{\Phi}_0, \bar{\Phi}_2, \bar{E}_1, \bar{E}_3, \bar{B}_1, \bar{B}_3$:

$$1 \quad \frac{1}{\sqrt{2}}E_1(z)i\sqrt{X-B} - \frac{1}{\sqrt{2}}E_3(z)i\sqrt{X+B} - \partial_z E_2(z)$$

$$-\mu E\Phi_2(z) - 2iE\sigma(\epsilon+Ez)E_2(z) + 2B\sigma(Ez+\epsilon)B_2(z) = M\Phi_0(z),$$

$$3 \quad +i(\epsilon+Ez)E_2(z) - \frac{1}{\sqrt{2}}B_1(z)i\sqrt{X+B} - \frac{1}{\sqrt{2}}B_3(z)i\sqrt{X-B}$$

$$-\mu E\Phi_0(z) + 2E\sigma\partial_z E_2(z) + 2iB\sigma\partial_z B_2(z) = M\Phi_2(z),$$

$$5 \quad \frac{1}{\sqrt{2}}\Phi_0(z)i\sqrt{X-B} - i(\epsilon+Ez)\Phi_1(z) = ME_1(z), \quad 7 \quad -\frac{1}{\sqrt{2}}\Phi_0(z)i\sqrt{X+B} - i(\epsilon+Ez)\Phi_3(z) = ME_3(z),$$

$$8 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)i\sqrt{X+B} + \partial_z\Phi_3(z) = MB_1(z), \qquad 10 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)i\sqrt{X-B} - \partial_z\Phi_1(z) = MB_3(z),$$

Subsystem II of 4 equations:

$$2 \quad i(\epsilon+Ez)E_1(z) + \frac{1}{\sqrt{2}}(1-2iB\sigma)B_2(z)i\sqrt{X-B} - \partial_z B_3(z)$$

$$-\sqrt{2}E\sigma E_2(z)i\sqrt{X-B} + iB\mu\Phi_1(z) = M\Phi_1(z),$$

$$4 \quad i(\epsilon+Ez)E_3(z) + \frac{1}{\sqrt{2}}i(2B\sigma-i)B_2(z)i\sqrt{X+B} + \partial_z B_1(z) + \sqrt{2}E\sigma E_2(z)i\sqrt{X+B} - iB\mu\Phi_3(z) = M\Phi_3(z),$$

$$6 \quad -\partial_z\Phi_0(z) - i(\epsilon+Ez)\Phi_2(z) = ME_2(z), \quad 9 \quad \frac{1}{\sqrt{2}}\Phi_1(z)i\sqrt{X-B} + \frac{1}{\sqrt{2}}\Phi_3(z)i\sqrt{X+B} = MB_2(z).$$

The last equation 9 permits to eliminate the variable B_2 from the three ones.

Let us resolve the system I with respect to the variables $\bar{\Phi}_0, \bar{\Phi}_2, \bar{E}_1, \bar{E}_3, \bar{B}_1, \bar{B}_3$; so we obtain

$$\Phi_0 = \frac{1}{2\Big(\Big(M^2+X\Big)^2 - M^2E^2\mu^2\Big)}$$

$$\times\Big[4BM\Big(\Big(M^2+X\Big)(Ez+\epsilon) - i\partial_z ME\mu\Big)\sigma B_2 - 2M\Big(iE(Ez+\epsilon)\Big(M\mu + 2\Big(M^2+X\Big)\sigma\Big)$$

$$+\partial_z\Big(M^2 + 2E^2\mu\sigma M + X\Big)\Big)E_2 + \sqrt{2}\Big(\Big(M^2+X\Big)(Ez+\epsilon) - i\partial_z ME\mu\Big)\Big(\sqrt{X-B}\Phi_1 - \sqrt{B+X}\Phi_3\Big)\Big],$$

$$\Phi_2 = \frac{1}{2\Big(\Big(M^2+X\Big)^2 - M^2E^2\mu^2\Big)}$$

$$\times\Big[i\Big(4BM\Big(\partial_z\Big(M^2+X\Big) + iME(Ez+\epsilon)\mu\Big)\sigma B_2 + 2M\Big(EzM^2 + \epsilon M^2 - i\partial_z E\mu M + EXz + X\epsilon + 2E\Big(ME(Ez+\epsilon)\mu$$

$$-i\partial_z\Big(M^2+X\Big)\Big)\sigma\Big)E_2 + \sqrt{2}\Big(\partial_z\Big(M^2+X\Big) + iME(Ez+\epsilon)\mu\Big)\Big(\sqrt{X-B}\Phi_1 - \sqrt{B+X}\Phi_3\Big)\Big)\Big],$$

$$E_1 = -\frac{1}{2M\Big(M^2 - E\mu M + X\Big)\Big(M^2 + E\mu M + X\Big)}$$

$$\times\Big[iE\sqrt{X-B}\sqrt{B+X}z\Phi_3M^2 + i\sqrt{X-B}\sqrt{B+X}\epsilon\Phi_3M^2$$

$$-2i\sqrt{2}B\sqrt{X-B}\Big(\Big(M^2+X\Big)(Ez+\epsilon) - i\partial_z ME\mu\Big)\sigma B_2M$$

$$+\sqrt{2}\sqrt{X-B}\Big(i\partial_z\Big(M^2+2E^2\mu\sigma M+X\Big)-E(Ez+\epsilon)\Big(M\mu+2\Big(M^2+X\Big)\sigma\Big)\Big)E_2M$$
$$+\partial_z E\sqrt{X-B}\sqrt{B+X}\mu\Phi_3 M+i\Big(-2M^2E^2(Ez+\epsilon)\mu^2$$
$$+i\partial_z ME(X-B)\mu+\Big(M^2+X\Big)\Big(2M^2+B+X\Big)(Ez+\epsilon)\Big)\Phi_1+iEX\sqrt{X-B}\sqrt{B+X}z\Phi_3+iX\sqrt{X-B}\sqrt{B+X}\epsilon\Phi_3\Big],$$

$$E_3=-\frac{1}{2M\Big(M^2-E\mu M+X\Big)\Big(M^2+E\mu M+X\Big)}$$
$$\times\Big[iE\sqrt{X-B}\sqrt{B+X}z\Phi_1M^2+i\sqrt{X-B}\sqrt{B+X}\epsilon\Phi_1M^2$$
$$+2i\sqrt{2}B\sqrt{B+X}\Big(\Big(M^2+X\Big)(Ez+\epsilon)-i\partial_z ME\mu\Big)\sigma B_2M+\sqrt{2}\sqrt{B+X}\Big(E(Ez+\epsilon)\Big(M\mu+2\Big(M^2+X\Big)\sigma\Big)$$
$$-i\partial_z\Big(M^2+2E^2\mu\sigma M+X\Big)\Big)E_2M+\partial_z E\sqrt{X-B}\sqrt{B+X}\mu\Phi_1M+iEX\sqrt{X-B}\sqrt{B+X}z\Phi_1+iX\sqrt{X-B}\sqrt{B+X}\epsilon\Phi_1$$
$$-iB\Big(\Big(M^2+X\Big)(Ez+\epsilon)-i\partial_z ME\mu\Big)\Phi_3+i\Big(-2M^2E^2(Ez+\epsilon)\mu^2+i\partial_z MEX\mu+\Big(M^2+X\Big)\Big(2M^2+X\Big)(Ez+\epsilon)\Big)\Phi_3\Big],$$

$$B_1=\frac{1}{2M\Big(M^2-E\mu M+X\Big)\Big(M^2+E\mu M+X\Big)}$$
$$\times[\partial_z\sqrt{X-B}\sqrt{B+X}\Phi_1M^2+2\sqrt{2}B\sqrt{B+X}\Big(\partial_z\Big(M^2+X\Big)+iME(Ez+\epsilon)\mu\Big)\sigma B_2M$$
$$+\sqrt{2}\sqrt{B+X}\Big(EzM^2+\epsilon M^2-i\partial_z E\mu M+EXz+X\epsilon+2E\Big(ME(Ez+\epsilon)\mu-i\partial_z\Big(M^2+X\Big)\Big)\sigma\Big)E_2M$$
$$+iE^2\sqrt{X-B}\sqrt{B+X}z\mu\Phi_1M+iE\sqrt{X-B}\sqrt{B+X}\epsilon\mu\Phi_1M+\partial_z X\sqrt{X-B}\sqrt{B+X}\Phi_1-B\Big(\partial_z\Big(M^2+X\Big)$$
$$+iME(Ez+\epsilon)\mu\Big)\Phi_3+\Big(-2\partial_z M^2E^2\mu^2-iMEX(Ez+\epsilon)\mu+\partial_z\Big(M^2+X\Big)\Big(2M^2+X\Big)\Big)\Phi_3\Big],$$

$$B_3=-\frac{1}{2M\Big(M^2-E\mu M+X\Big)\Big(M^2+E\mu M+X\Big)}$$
$$\times\Big[\partial_z\sqrt{X-B}\sqrt{B+X}\Phi_3M^2-2\sqrt{2}B\sqrt{X-B}\Big(\partial_z\Big(M^2+X\Big)+iME(Ez+\epsilon)\mu\Big)\sigma B_2M$$
$$-\sqrt{2}\sqrt{X-B}\Big(EzM^2+\epsilon M^2-i\partial_z E\mu M+EXz+X\epsilon+2E\Big(ME(Ez+\epsilon)\mu-i\partial_z\Big(M^2+X\Big)\Big)\sigma\Big)E_2M$$
$$+iE^2\sqrt{X-B}\sqrt{B+X}z\mu\Phi_3M+iE\sqrt{X-B}\sqrt{B+X}\epsilon\mu\Phi_3M$$
$$+\Big(-2\partial_z M^2E^2\mu^2+iME(B-X)(Ez+\epsilon)\mu+\partial_z\Big(M^2+X\Big)\Big(2M^2+B+X\Big)\Big)\Phi_1+\partial_z X\sqrt{X-B}\sqrt{B+X}\Phi_3\Big],$$

Now, substitute these expressions in equations of the group II, this results in
1

$$B_2\Big(-\frac{\sqrt{2}B\partial_z^2\sigma\sqrt{X-B}(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2}$$
$$+\frac{\sqrt{X-B}((2B\sigma+i)(M^2-\mu ME+X)(M^2+\mu ME+X)-2B\sigma(M^2+X)(Ez+\epsilon)^2-2iB\mu M\sigma E^2)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\Big)$$
$$+E_2\Big(\frac{i\partial_z^2E\sqrt{X-B}(2\sigma(M^2+X)+\mu M)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-$$
$$-\frac{iE\sqrt{X-B}(-i(M^2+2\mu M\sigma E^2+X)+2\sigma(M^2-\mu ME+X)(M^2+\mu ME+X)-((Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)))}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\Big)$$
$$+\Phi_3\Big(\frac{\partial_z^2\sqrt{X-B}\sqrt{B+X}(M^2+X)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}+\frac{\sqrt{X-B}\sqrt{B+X}((M^2+X)(Ez+\epsilon)^2+i\mu ME^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\Big)$$
$$+\Phi_1\frac{\Big(\partial_z^2((M^2+X)(B+2M^2+X)-2\mu^2M^2E^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}$$

$$+\frac{1}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}$$

$$\times\Big[2iB\mu M(M^2-\mu ME+X)(M^2+\mu ME+X)+(M^2+X)(B+2M^2+X)(Ez+\epsilon)^2+i\mu ME^2(B-X)+2\mu^2M^4E^2$$

$$-2\mu^2M^2E^2(Ez+\epsilon)^2-2M^2(M^2+X)^2\Big]\Big)=0,$$

2

$$B_2\Big(\frac{\sqrt{2}B\partial_z^2\sigma\sqrt{B+X}(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2}+\frac{\sqrt{B+X}(\frac{2B\sigma(\mu^2M^2E^2-(M^2+X)(M^2-(Ez+\epsilon)^2+X)+i\mu ME^2)}{(M^2+X)^2-\mu^2M^2E^2}+i)}{\sqrt{2}}\Big)$$

$$+E_2\Big(\frac{E\sqrt{B+X}(2i\sigma(M^2-\mu ME+X)(M^2+\mu ME+X)-i(Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)+M^2+2\mu M\sigma E^2+X)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}$$

$$-\frac{i\partial_z^2E\sqrt{B+X}(2\sigma(M^2+X)+\mu M)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\Big)$$

$$+\Phi_1\Big(\frac{\partial_z^2\sqrt{X-B}\sqrt{B+X}(M^2+X)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}+\frac{\sqrt{X-B}\sqrt{B+X}((M^2+X)(Ez+\epsilon)^2+i\mu ME^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\Big)$$

$$+\Phi_3\Big(\frac{\partial_z^2(4-\frac{2(B+X)(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2})}{4M}$$

$$+\frac{1}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}$$

$$\times\Big[2iB\mu^3M^3E^2-i\mu M(B(2(M^2+X)^2+E^2)+E^2X)+(M^2+X)(-(B-X)(Ez+\epsilon)^2-2M^4+2M^2((Ez+\epsilon)^2-X))$$

$$+2\mu^2M^2E^2(M-Ez-\epsilon)(M+Ez+\epsilon)\Big]\Big)=0,$$

3

$$B_2\Big(\frac{2iB\partial_z^2\mu M^2\sigma E}{(M^2+X)^2-\mu^2M^2E^2}-\frac{2BM\sigma E(M^2-i\mu M(Ez+\epsilon)^2+X)}{(M^2+X)^2-\mu^2M^2E^2}\Big)$$

$$+\Phi_1\Big(\frac{i\partial_z^2\mu ME\sqrt{X-B}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-\frac{E\sqrt{X-B}(M^2-i\mu M(Ez+\epsilon)^2+X)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\Big)$$

$$+\Phi_3\Big(\frac{E\sqrt{B+X}(M^2-i\mu M(Ez+\epsilon)^2+X)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-\frac{i\partial_z^2\mu ME\sqrt{B+X}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\Big)$$

$$+E_2\Big(\frac{\partial_z^2M(M^2+2\mu M\sigma E^2+X)}{(M^2+X)^2-\mu^2M^2E^2}$$

$$+\frac{M(-M^4+M^2(E^2(\mu^2+2i\sigma+z^2)+2Ez\epsilon-2X+\epsilon^2)+\mu ME^2(2\sigma(Ez+\epsilon)^2+i)+X(2i\sigma E^2+(Ez+\epsilon)^2-X))}{(M^2+X)^2-\mu^2M^2E^2}\Big)=0,$$

4

$$\frac{i\Phi_1\sqrt{X-B}}{\sqrt{2}}+\frac{i\Phi_3\sqrt{B+X}}{\sqrt{2}}+B_2(-M)=0.$$

With the help of the fourth equation, we can eliminate the variable B_2 from remaining three equations (let us change the notations $\Phi_1=G,\Phi_3=H,E_2=F$); in order to remove fractions, we multiply each equation by

$$2M(M^2+X-ME\mu)(M^2+X+ME\mu)((M^2+X)^2-M^2E^2\mu^2)),$$

so we get the following three equations

1

$$\Big[i\sqrt{2}\partial_z^2ME\sqrt{X-B}(M^2-\mu ME+X)(M^2+\mu ME+X)(2\sigma(M^2+X)+\mu M)$$

$$-i\sqrt{2}ME\sqrt{X-B}(M^2-\mu ME+X)(M^2+\mu ME+X)(-i(M^2+2\mu M\sigma E^2+X)$$

$$+2\sigma(M^2-\mu ME+X)(M^2+\mu ME+X)-((Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)))\Big]F$$

$$+\Big[\partial_z^2(M^2-\mu ME+X)(M^2+\mu ME+X)(2iB\sigma(B-X)(M^2+X)+BM^2+BX+2M^4-2\mu^2M^2E^2+3M^2X+X^2)$$

$$+(M^2-\mu ME+X)(M^2+\mu ME+X)(i(X-B)((2B\sigma+i)(M^2-\mu ME+X)(M^2+\mu ME+X)$$

$$-2B\sigma(M^2+X)(Ez+\epsilon)^2-2iB\mu M\sigma E^2)+2iB\mu M(M^2-\mu ME+X)(M^2+\mu ME+X)$$

$$+(M^2+X)(B+2M^2+X)(Ez+\epsilon)^2+i\mu ME^2(B-X)+2\mu^2M^4E^2-2\mu^2M^2E^2(Ez+\epsilon)^2-2M^2(M^2+X)^2)\Big]G$$

$$+\Big[\partial_z^2(1-2iB\sigma)\sqrt{X-B}\sqrt{B+X}(M^2+X)(M^2-\mu ME+X)(M^2+\mu ME+X)$$

$$+i(2B\sigma+i)\sqrt{X-B}\sqrt{B+X}((M^2+X)^2-\mu^2M^2E^2)(\mu^2(-M^2)E^2+(M^2+X)(M^2-(Ez+\epsilon)^2+X)-i\mu ME^2)\Big]H=0,$$

2

$$\Big[\sqrt{2}ME\sqrt{B+X}(M^2-\mu ME+X)(M^2+\mu ME+X)(2i\sigma(M^2-\mu ME+X)(M^2+\mu ME+X)$$

$$-i(Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)+M^2+2\mu M\sigma E^2+X)$$

$$-i\sqrt{2}\partial_z^2ME\sqrt{B+X}(M^2-\mu ME+X)(M^2+\mu ME+X)(2\sigma(M^2+X)+\mu M)\Big]F$$

$$+\Big[\partial_z^2(1+2iB\sigma)\sqrt{X-B}\sqrt{B+X}(M^2+X)(M^2-\mu ME+X)(M^2+\mu ME+X)-i(2B\sigma-i)\sqrt{X-B}\sqrt{B+X}((M^2+X)^2$$

$$-\mu^2M^2E^2)(\mu^2(-M^2)E^2+(M^2+X)(M^2-(Ez+\epsilon)^2+X)-i\mu ME^2)\Big]G+$$

$$+\Big[\partial_z^2(M^2-\mu ME+X)(M^2+\mu ME+X)(2iB\sigma(B+X)(M^2+X)-BM^2-BX+2M^4-2\mu^2M^2E^2+3M^2X+X^2)$$

$$+(M^2-\mu ME+X)(M^2+\mu ME+X)(2iB\mu^3M^3E^2-i\mu M(B(2(M^2+X)^2+E^2)+E^2X)+$$

$$+i(B+X)(i((M^2+X)^2-\mu^2M^2E^2)-2B\sigma(\mu^2(-M^2)E^2+(M^2+X)(M^2-(Ez+\epsilon)^2+X)-i\mu ME^2))$$

$$+(M^2+X)(-(B-X)(Ez+\epsilon)^2-2M^4+2M^2((Ez+\epsilon)^2-X))+2\mu^2M^2E^2(M-Ez-\epsilon)(M+Ez+\epsilon))\Big]H=0,$$

3

$$\Big[2\partial_z^2M^2(M^2-\mu ME+X)(M^2+\mu ME+X)(M^2+2\mu M\sigma E^2+X)$$

$$+2M^2(M^2-\mu ME+X)(M^2+\mu ME+X)(-M^4+M^2(E^2(\mu^2+2i\sigma+z^2)+2Ez\epsilon-2X+\epsilon^2)$$

$$+\mu ME^2(2\sigma(Ez+\epsilon)^2+i)+X(2i\sigma E^2+(Ez+\epsilon)^2-X))\Big]F$$

$$+\Big[-\sqrt{2}\partial_z^2\mu M^2E(2B\sigma-i)\sqrt{X-B}(M^2-\mu ME+X)(M^2+\mu ME+X)$$

$$-i\sqrt{2}ME(2B\sigma-i)\sqrt{X-B}(M^2-\mu ME+X)(M^2+\mu ME+X)(M^2-i\mu M(Ez+\epsilon)^2+X)\Big]G$$

$$+\Big[\sqrt{2}ME(1-2iB\sigma)\sqrt{B+X}(M^2-\mu ME+X)(M^2+\mu ME+X)(M^2-i\mu M(Ez+\epsilon)^2+X)$$

$$-\sqrt{2}\partial_z^2\mu M^2E(2B\sigma+i)\sqrt{B+X}(M^2-\mu ME+X)(M^2+\mu ME+X)\Big]H=0,$$

Let us write the last system in a symbolical form

$$1 \quad a_1F''+b_1F+c_1G''+d_1G+l_1H''+n_1H=0,$$

$$2 \quad a_2F''+b_2F+c_2G''+d_2G+l_2H''+n_2H=0,$$

$$3 \quad a_3F''+b_3F+c_3G''+d_3G+l_3H''+n_3H=0.$$

We will combine equations as follows

$$1)\cdot\alpha+2)\cdot\beta+3)\cdot\gamma=0;$$

in this way we obtain the following equation

$$(\alpha a_1+\beta a_2+\gamma a_3)F''+(\alpha b_1+\beta b_2+\gamma b_3)F+$$

$$(\alpha c_1+\beta c_2+\gamma c_3)G''+(\alpha d_1+\beta d_2+\gamma d_3)G+$$
$$(\alpha l_1+\beta l_2+\gamma l_3)H''+(\alpha n_1+\beta n_2+\gamma n_3)H=0.$$

Here we will distinguish 3 different cases:

$$I \quad \begin{array}{c} \alpha_1 a_1+\beta_1 a_2+\gamma_1 a_3=1,\\ \alpha_1 c_1+\beta_1 c_2+\gamma_1 c_3=0,\\ \alpha_1 l_1+\beta_1 l_2+\gamma_1 l_3=0; \end{array}$$

$$II \quad \begin{array}{c} \alpha_2 a_1+\beta_2 a_2+\gamma_2 a_3=0,\\ \alpha_2 c_1+\beta_2 c_2+\gamma_2 c_3=1,\\ \alpha_2 l_1+\beta_2 l_2+\gamma_2 l_3=0; \end{array} \qquad III \quad \begin{array}{c} \alpha_3 a_1+\beta_3 a_2+\gamma_3 a_3=0,\\ \alpha_3 c_1+\beta_3 c_2+\gamma_3 c_3=0,\\ \alpha_3 l_1+\beta_3 l_2+\gamma_3 l_3=1; \end{array}$$

the corresponding three solutions have the form

$$I \qquad \begin{array}{l} \alpha_1=\dfrac{\mu E(2B\sigma-i)\sqrt{X-B}}{2\sqrt{2}\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \beta_1=\dfrac{i\mu E(1-2iB\sigma)\sqrt{B+X}}{2\sqrt{2}\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \gamma_1=\dfrac{M^2\big(M^2-\mu^2E^2+X\big)+2iB^2\sigma\big(M^2+X\big)}{2M^2(M^2-\mu ME+X)^2(M^2+\mu ME+X)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)}; \end{array}$$

$$II \qquad \begin{array}{l} \alpha_2=\dfrac{2iB^2\sigma+B(-1+2i\sigma X)+2M\big(M+2\mu\sigma E^2\big)+X}{4\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \beta_2=\dfrac{i(2B\sigma+i)\sqrt{X-B}\sqrt{B+X}}{4\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \gamma_2=-\dfrac{iE\sqrt{X-B}\big(2\sigma\big(M^2+X\big)+\mu M\big)}{2\sqrt{2}M\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)}; \end{array}$$

$$III \qquad \begin{array}{l} \alpha_3=\dfrac{i(-2B\sigma+i)\sqrt{X-B}\sqrt{B+X}}{4\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \beta_3=\dfrac{2iB^2\sigma-2iB\sigma X+B+2M\big(M+2\mu\sigma E^2\big)+X}{4\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)},\\ \gamma_3=\dfrac{iE\sqrt{B+X}\big(2\sigma\big(M^2+X\big)+\mu M\big)}{2\sqrt{2}M\big((M^2+X)^2-\mu^2M^2E^2\big)^2(2iB^2\sigma+M^2+2\mu M\sigma E^2)}. \end{array}$$

So the above equations can be presented as

$$1)\quad F''+(\alpha_1 b_1+\beta_1 b_2+\gamma_1 b_3)F$$
$$+(\alpha_1 d_1+\beta_1 d_2+\gamma_1 d_3)G+(\alpha_1 n_1+\beta_1 n_2+\gamma_1 n_3)H=0,$$
$$2)\quad G''+(\alpha_2 b_1+\beta_2 b_2+\gamma_2 b_3)F$$
$$+(\alpha_2 d_1+\beta_2 d_2+\gamma_2 d_3)G+(\alpha_2 n_1+\beta_2 n_2+\gamma_2 n_3)H=0,$$
$$3)\quad H''+(\alpha_3 b_1+\beta_3 b_2+\gamma_3 b_3)F$$
$$+(\alpha_3 d_1+\beta_3 d_2+\gamma_3 d_3)G+(\alpha_3 n_1+\beta_3 n_2+\gamma_3 n_3)H=0;$$

Let us make the needed change $\sigma\Rightarrow i\sigma, \mu\Rightarrow i\mu$, so we get

$$1)\quad F''+\frac{1}{M^2-2\sigma(B^2+\mu ME^2)}\Big[(-2\sigma(B^2(-M^2+(Ez+\epsilon)^2-X)+ME^2(M+\mu((Ez+\epsilon)^2-X)))$$
$$+4B^2\sigma^2E^2(\mu M+1)-M(M^3+M(\mu^2E^2-(Ez+\epsilon)^2+X)+\mu E^2))\Big]F$$

$$+\frac{E(2B\sigma-1)\sqrt{X-B}(M(-B\mu+\mu M(B\mu+M)+M)-2B^2\sigma(\mu M+1))}{\sqrt{2}(M^3-2M\sigma(B^2+\mu ME^2))}G$$
$$+\frac{E(2B\sigma+1)\sqrt{B+X}(M(B\mu+\mu M(M-B\mu)+M)-2B^2\sigma(\mu M+1))}{\sqrt{2}(M^3-2M\sigma(B^2+\mu ME^2))}H=0,$$

$$2)\quad G''+\frac{1}{2M^2-4\sigma(B^2+\mu ME^2)}\Big[(2B^3\sigma(\mu M-1)+B^2(2\sigma(2M^2+\mu MX-2(Ez+\epsilon)^2+X)$$
$$+\mu M-4\sigma^2E^2(\mu M+1)-1)+B(-(\mu M-1)(2M^2+X)+2\sigma E^2(\mu M(2\mu M-1)+1)+4\sigma^2E^2X(\mu M+1))$$
$$-2\sigma E^2(\mu M(-2M^2+2(Ez+\epsilon)^2-X)+X)+2M^2(-M^2+(Ez+\epsilon)^2-X))\Big]G$$
$$-\frac{ME\sqrt{X-B}(\mu M+1)(4\sigma^2E^2+1)}{\sqrt{2}(M^2-2\sigma(B^2+\mu ME^2))}F$$
$$+\frac{(2B\sigma+1)\sqrt{X-B}\sqrt{B+X}(B(\mu M-1)-2\sigma E^2(\mu M+1))}{4\sigma(B^2+\mu ME^2)-2M^2}H=0,$$

$$3)\quad H''+\frac{1}{2M^2-4\sigma(B^2+\mu ME^2)}\Big[(-2B^3\sigma(\mu M-1)+B^2(2\sigma(2M^2+\mu MX-2(Ez+\epsilon)^2+X)$$
$$+\mu M-4\sigma^2E^2(\mu M+1)-1)+B((\mu M-1)(2M^2+X)+2\sigma E^2(\mu M(1-2\mu M)-1)-4\sigma^2E^2X(\mu M+1))$$
$$-2\sigma E^2(\mu M(-2M^2+2(Ez+\epsilon)^2-X)+X)+2M^2(-M^2+(Ez+\epsilon)^2-X))\Big]H$$
$$+\frac{ME\sqrt{B+X}(\mu M+1)(4\sigma^2E^2+1)}{\sqrt{2}(M^2-2\sigma(B^2+\mu ME^2))}F$$
$$+\frac{(2B\sigma-1)\sqrt{X-B}\sqrt{B+X}(B(\mu M-1)+2\sigma E^2(\mu M+1))}{4\sigma(B^2+\mu ME^2)-2M^2}G=0.$$

It is convenient to apply the short notation $(Ez+\epsilon)=\Sigma$; in this way we obtain

$$1)\qquad F''+\Sigma^2F+$$
$$\frac{-M^2(-2B^2\sigma+W^2(\mu^2+2\sigma)+X)+\mu MW^2(2\sigma(2B^2\sigma+X)-1)+2B^2\sigma(2\sigma W^2+X)-M^4}{M^2-2\sigma(B^2+\mu MW^2)}F+$$
$$+\frac{W(2B\sigma-1)\sqrt{X-B}(M(-B\mu+\mu M(B\mu+M)+M)-2B^2\sigma(\mu M+1))}{\sqrt{2}(M^3-2M\sigma(B^2+\mu MW^2))}G+$$
$$+\frac{W(2B\sigma+1)\sqrt{B+X}(M(B\mu+\mu M(M-B\mu)+M)-2B^2\sigma(\mu M+1))}{\sqrt{2}(M^3-2M\sigma(B^2+\mu MW^2))}H=0,$$

$$2)\qquad G''+\Sigma^2G+$$
$$+\frac{1}{2M^2-4\sigma(B^2+\mu MW^2)}\Big[2B^3\sigma(\mu M-1)+B^2(2\sigma(2M^2+\mu MX+X)+\mu M-4\sigma^2W^2(\mu M+1)-1)+$$
$$+B(-(\mu M-1)(2M^2+X)+2\sigma W^2(\mu M(2\mu M-1)+1)+4\sigma^2W^2X(\mu M+1))+$$
$$+2\sigma W^2(2\mu M^3+X(\mu M-1))-2M^2(M^2+X)\Big]G-$$
$$-\frac{MW\sqrt{X-B}(\mu M+1)(4\sigma^2W^2+1)}{\sqrt{2}(M^2-2\sigma(B^2+\mu MW^2))}F+$$
$$+\frac{(2B\sigma+1)\sqrt{X-B}\sqrt{B+X}(B(\mu M-1)-2\sigma W^2(\mu M+1))}{4\sigma(B^2+\mu MW^2)-2M^2}H=0,$$

$$3)\qquad H''+\Sigma^2H+$$
$$+\frac{1}{2M^2-4\sigma(B^2+\mu MW^2)}\Big[-2B^3\sigma(\mu M-1)+B^2(2\sigma(2M^2+\mu MX+X)+$$

$$+\mu M - 4\sigma^2 W^2(\mu M + 1) - 1) + B((\mu M - 1)(2M^2 + X) + 2\sigma W^2(\mu M(1 - 2\mu M) - 1) -$$
$$-4\sigma^2 W^2 X(\mu M + 1)) + 2\sigma W^2(2\mu M^3 + X(\mu M - 1)) - 2M^2(M^2 + X)\Big] H +$$
$$+ \frac{MW\sqrt{B + X}(\mu M + 1)(4\sigma^2 W^2 + 1)}{\sqrt{2}(M^2 - 2\sigma(B^2 + \mu M W^2))} F +$$
$$+ \frac{(2B\sigma - 1)\sqrt{X - B}\sqrt{B + X}(B(\mu M - 1) + 2\sigma W^2(\mu M + 1))}{4\sigma(B^2 + \mu M W^2) - 2M^2} G = 0,$$

Let us present the system in matrix form

$$\Delta = \frac{d^2}{dz^2} + \Sigma^2(z), \Delta \begin{vmatrix} F \\ G \\ H \end{vmatrix} = A \begin{vmatrix} F \\ G \\ H \end{vmatrix}, \quad \Delta\Psi(z) = A\Psi(z), A = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Now we will find the transformation which diagonalizes the system

$$\bar{\Psi} = S\Psi, \quad \Delta\bar{\Psi}(z) = \bar{A}\bar{\Psi}(z), \quad \bar{A} = SAS^{-1} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}, S = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix},$$

we should find solutions for equation $SA = \bar{A}S$; explicitly it reads

$$\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix};$$

whence follow three similar subsystems

$$\begin{cases} (A_1 - \lambda_1)s_{11} + A_2 s_{12} + A_3 s_{13} = 0 \\ B_1 s_{11} + (B_2 - \lambda_1)s_{12} + B_3 s_{13} = 0 \\ C_1 s_{11} + C_2 s_{12} + (C_3 - \lambda_1)s_{13} = 0 \end{cases},$$

$$\begin{cases} (A_1 - \lambda_2)s_{21} + A_2 s_{22} + A_3 s_{23} = 0 \\ B_1 s_{21} + (B_2 - \lambda_2)s_{22} + B_3 s_{23} = 0 \\ C_1 s_{21} + C_2 s_{22} + (C_3 - \lambda_2)s_{23} = 0 \end{cases},$$

$$\begin{cases} (A_1 - \lambda_3)s_{31} + A_2 s_{32} + A_3 s_{33} = 0 \\ B_1 s_{31} + (B_2 - \lambda_3)s_{32} + B_3 s_{33} = 0 \\ C_1 s_{31} + C_2 s_{32} + (C_3 - \lambda_3)s_{33} = 0 \end{cases},$$

or differently

$$\begin{vmatrix} (A_1 - \lambda) & A_2 & A_3 \\ B_1 & (B_2 - \lambda) & B_3 \\ C_1 & C_2 & (C_3 - \lambda) \end{vmatrix} \begin{vmatrix} s_{i1} \\ s_{i2} \\ s_{i3} \end{vmatrix} = 0, \quad i = 1, 2, 3. \tag{17}$$

From vanishing the determinant

$$\det \begin{vmatrix} (A_1 - \lambda) & A_2 & A_3 \\ B_1 & (B_2 - \lambda) & B_3 \\ C_1 & C_2 & (C_3 - \lambda) \end{vmatrix} = 0$$

we derive the cubic equation

$$\lambda^3 - \lambda^2 (A_1 + B_2 + C_3) + \lambda (A_2 B_1 - A_1 B_2 + A_3 C_1 - A_1 C_3 + B_3 C_2 - B_2 C_3)$$

$$+\lambda^0(A_3B_2C_1 - A_2B_3C_1 - A_3B_1C_2 + A_1B_3C_2 + A_2B_1C_3 - A_1B_2C_3) = 0,$$

or shortly $\lambda^3 + d_2\lambda^2 + d_1\lambda + d = 0$.

Let us write down equations, solutions of which determine the elements of the matrix S:

$$\begin{array}{l} (A_1-\lambda_1)s_{11} + A_2 s_{12} + A_3 = 0 \\ B_1 s_{11} + (B_2-\lambda_1)s_{12} + B_3 = 0 \end{array}, \quad \text{assuming } s_{13} = 1;$$

$$\begin{array}{l} (A_1-\lambda_2)s_{21} + A_2 s_{22} + A_3 = 0 \\ B_1 s_{21} + (B_2-\lambda_2)s_{22} + B_3 = 0 \end{array}, \quad \text{assuming } s_{23} = 1;$$

$$\begin{array}{l} (A_1-\lambda_3)s_{31} + A_2 s_{32} + A_3 = 0 \\ B_1 s_{31} + (B_2-\lambda_3)s_{32} + B_3 = 0 \end{array}, \quad \text{assuming } s_{33} = 1.$$

After performing this transformation, we get 3 separate equations

$$\Big(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - \lambda_1\Big)\bar{F} = 0\,, \quad \Big(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - \lambda_2\Big)\bar{G} = 0\,,$$

$$\Big(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - \lambda_3\Big)\bar{H} = 0\,.$$

These equations have the same structure as for a scalar particle in the uniform electric field

$$\Big(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - \lambda\Big)\Phi(z) = 0\,. \tag{18}$$

We transform eq. (18) to the new variable (assuming that $E > 0$)

$$Z = i\frac{(Ez+\epsilon)^2}{E}, \quad \Lambda = \frac{\lambda}{4E}\,, \tag{19}$$

then we get the confluent hypergeometric equation

$$\Big(\frac{d^2}{dZ^2} + \frac{1/2}{Z}\frac{d}{dZ} - \frac{1}{4} + \frac{i\Lambda}{Z}\Big)\Phi(Z) = 0\,. \tag{20}$$

describing their solutions is a simple task (see in [18]).

8. Conclusion

We have we studied a generalized Duffin–Kemmer equation for spin 1 particle with two characteristics, anomalous magnetic moment and polarizability in presence of external uniform magnetic and electric fields. After separating the variables, we get the system of ten first order partial differential equations for 10 functions $f_i(r,z)$.

To describe the r-dependence of 10 functions $f_A(r,z), A = 1, ..., 10$, we applied the method by Fedorov – Gronskiy; so the complete 10-component wave function is decomposed into the sum of three projective constituents, dependence of each component on the polar coordinate r is determined by only one corresponding function, $F_i(r), i = 1,2,3$; these three basic functions are constructed in terms of the confluent hypergeometric functions, at this there arises the quantization rule due to the presence of magnetic field.

After that we derived a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved by using the elimination method. As the result, we find three separated second order differential equations, their solutions are constructed in the terms of the confluent hypergeometric functions.

Thus, the three types of solutions for a vector particle with two additional electromagnetic characteristics in presence of external uniform magnetic and electric fields are found.

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