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# Spin 2 Particle in Coulomb Field, the Non-relativistic Approximation

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The main goal of the present paper is to derive the non-relativistic system of equations for a spin 2 particle in presence of the external Coulomb field, then to solve it and to find the relevant energy spectra. We started with the known radial system of 39 equations derived for the case of a free particle and modified to take into account the presence of the Coulomb field. After eliminating the 28 components related to vector and 3-rank tensor, we get the system of second order for 11 components referring to scalar and symmetric tensor. In accordance with the parity restriction, it is divided in two subsystems, of 3 and 8 equations. For performing the non-relativistic approximation, we apply the method of projective operators constructed on the base of the matrix $\Gamma^0$ of the initial matrix equation. Depending on parity, we derive two non-relativistic sub-systems, of 2 and 3 linked differential equations. With the use of the linear similarity transformation they are reduced to five separate equations with Schrödiger type non-relativistic structure, and evident energy spectra. The case of minimal quantum number of the total angular momentum, $j = 0$, should be considered separately.

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## Introduction

The theory of massive and massless fields of spin-2, following the foundational work of Pauli and Fierz [1–3], has long attracted significant attention [4]–[36]. Several key aspects and challenges of this theory have been explored over the years. Most studies have been conducted within the framework of second-order differential equations, with a particular focus on the additional constraints required to preserve the five independent degrees of freedom for a spin-2 particle. This problem becomes even more intricate when extending the theory to curved Riemannian space-times. An additional complication arises when studying the massless spin-2 field. The well-known Pauli–Fierz solution for flat Minkowski space does not carry over to curved space-time. Extending the Pauli–Fierz prescription to a generally covariant form leads to unexpected constraints on space-time geometry: the Ricci tensor $R_{\alpha\beta}$ and the Riemann tensor $R_{\alpha\beta\rho\sigma}$ must vanish identically. To resolve this, a non-minimal interaction term involving the Riemann tensor can be introduced into the basic equations [30], allowing the constraints to be reduced to $R_{\alpha\beta} = 0$.

Another area of interest has been the problem of anomalous solutions in spin-2 theory. A technical alternative for studying spin-2 fields, both massive and massless, involves formulating first-order systems. This approach, based on the Gel'fand–Yaglom formalism [6], was first explored by Fedorov [7] and Regge [8]. Their work demonstrated that a spin-2 particle requires a 39-component set of tensors for its description. This formalism allows for the exploration of new physical questions related to degrees of freedom. For instance, for massless case, the 39-component matrix equation was solved in Minkowski space-time in [39] using cylindrical coordinates $t, r, \phi, z$ and a tetrad. Six linearly independent solutions were found. By applying the Pauli–Fierz approach, adjusted to the tetrad formalism, the gauge solutions were constructed using exact solutions for the massless spin-1 field. This yielded four independent gauge solutions and two gauge-free solutions for the spin-2 field, as expected from physical reasoning.

Additionally, F.I. Fedorov introduced a more general theory for the spin-2 particle based on a 50-component set of tensors. This theory, in the presence of external electromagnetic fields, describes a spin-2 particle with an anomalous magnetic moment [36, 40]. In Riemannian space-time, the reduced theory

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automatically incorporates non-minimal interaction terms involving the Ricci and Riemann tensors. One notable aspect of this theory is its allowance for a new massless limit for the spin-2 field [38]. This is particularly significant because the minimal Pauli–Fierz theory does not possess gauge symmetry in curved space-times with $R_{\alpha\beta} = 0$. However, the generalized theory exhibits gauge symmetry under these conditions, as demonstrated in [38].

In the present study, we focus on a specific problem within the 39-component theory: the non-relativistic approximation for a massive spin-2 particle in external Coulomb field. A similar problem was previously addressed for this particle in Cartesian coordinates [41], where a Pauli-like equation was derived in presence of electromagnetic fields.

In Section 1, the basic definitions and notations, including the structure of the 39-component wave function with spherical symmetry, are given. On solutions, the square and third projection of the total angular momentum are diagoanlized. For separating the angular dependence of the wave function we apply the Wigner $D$-functions.

In Section 2, we specify the known system of the radial equations, deriving previously for the case of a free particle, and modify it in order to take into account the presence of external Coulomb field. Besides, restrictions due to diagonalization of the spatial reflection operator are presented.

In section 3, we eliminate 28 variables related to 4-vector and 2-rank tensor, $\Phi_1, \Phi_{a[bc]}$, so producing eleven equations of the second order for the variables referring to scalar $\Phi$ and symmetric tensor $\Phi_{(ab)}$. Allowing for the parity restrictions, we derive two subsystem of three and eight second order radial equations.

In Section 4, we apply the general method for performing the non-relativistic approximation; it is based on distinguishing between large and small components of the wave function; the last are found by using three projective operators derived from the seventh-order minimal polynomial for the $39 \times 39$ matrix $\Gamma^0$ of the initial matrix equation.

In Section 5, from the system of three 2-nd order equations for states with the parity $P = (-1)^{j+1}$, we derive two liked radial equations with the non-relativistic structure; after performing a special linear transformation it reduces to two independent equations of the Schödinger type in presence of Coulomb field.

In Section 6, from the system of eight 2-nd order equations for states with the parity $P = (-1)^j$ we derive three linked radial equations with the non-relativistic structure; after performing a needed linear transformation, it leads to three independent equations of the Schödinger type in presence of Coulomb field.

So, we obtain five energy spectra, referring to a non-relativistic spin 2 particle in presence of the Coulomb field. The case of minimal value of the quantum number $j = 0$ is a special one, and it leads to only one Shrödinger type equation of the the known structure.

## 1. Initial equations

In [43], were performed calculations concerning the separation of the variables in the matrix equation describing a spin 2 particle in spherical coordinates. Below we recall only the most basic points. The wave function consists of scalar, 3-vector, symmetric tensor, and 3-rank tensor antisymmetric in two indices; in total of 39 components. On solutions with spherical symmetry, the following operators are diagonalized

$$\begin{gathered} \vec{J}^2 H = j(j+1)H, \quad J_3 H = mH; \quad \vec{J}^2_{(1)} H_1 = j(j+1)H_1, \quad J_3^{(1)} H_1 = mH_1; \\ \vec{J}^2_{(2)} H_2 = j(j+1)H_2, \quad J_3^{(2)} H_2 = mH_2; \quad \vec{J}^2_{(3)} H_3 = j(j+1)H_3, \quad J_3^{(3)} H_2 = mH_3. \end{gathered} \tag{1}$$

When separating the variables, we applied the Wigner $D$-functions technique $D^j_{-m,-s_3}(\phi, \theta, 0) = D_{-s_3}$; see in [44]; the known recurrent relations were used:

$$\begin{gathered} \partial_\theta D_0 = +\frac{1}{2} a\ D_{-1} - \frac{1}{2} a\ D_{+1}, \quad \frac{-m}{\sin\theta} D_0 = -\frac{1}{2} a\ D_{-1} - \frac{1}{2} a\ D_{+1}, \\ \partial_\theta D_{+1} = +\frac{1}{2} a\ D_0 - \frac{1}{2} b\ D_{+2}, \quad \frac{-m - \cos\theta}{\sin\theta} D_{+1} = -\frac{1}{2} a\ D_0 - \frac{1}{2} b\ D_{+2}, \\ \partial_\theta D_{-1} = +\frac{1}{2} b\ D_{-2} - \frac{1}{2} a\ D_0, \quad \frac{-m + \cos\theta}{\sin\theta} D_{-1} = -\frac{1}{2} b\ D_{-2} - \frac{1}{2} a\ D_0, \\ \partial_\theta D_{+2} = +\frac{1}{2} b\ D_{+1} - \frac{1}{2} c\ D_{+3}, \quad \frac{-m - 2\cos\theta}{\sin\theta} D_{+2} = -\frac{1}{2} b\ D_{+1} - \frac{1}{2} c\ D_{+3}, \\ \partial_\theta D_{-2} = +\frac{1}{2} c\ D_{-3} - \frac{1}{2} b\ D_{-1}, \quad \frac{-m + 2\cos\theta}{\sin\theta} D_{-2} = -\frac{1}{2} c\ D_{-3} - \frac{1}{2} b; D_{-1}, \end{gathered} \tag{2}$$

where $\sqrt{j(j+1)} = a$, $\sqrt{(j-1)(j+2)} = b$, $\sqrt{(j-2)(j+3)} = c$. We use the substitutions (the common multiplier $e^{-i\epsilon t}$ is omitted):

$$H = h(r)D_0, \quad H_1 = \begin{vmatrix} h_0(r)\ D_0 \\ h_1(r)\ D_{-1} \\ h_2(r)\ D_0 \\ h_3(r)\ D_{+1} \end{vmatrix}, \quad H_2 = \begin{vmatrix} f_1(r)\ D_{-2} \\ f_2(r)\ D_0 \\ f_3(r)\ D_{+2} \\ c_1(r)\ D_{+1} \\ c_2(r)\ D_0 \\ c_3(r)\ D_{-1} \\ d_1(r)\ D_{-1} \\ d_2(r)\ D_0 \\ d_3(r)\ D_{+1} \\ f_0(r)\ D_0 \end{vmatrix},$$

$$\varphi_0 = \begin{vmatrix} E_{10}(r)D_{-1} \\ E_{20}(r)D_0 \\ E_{30}(r)D_{+1} \\ B_{10}(r)D_{+1} \\ B_{20}(r)D_0 \\ B_{30}(r)D_{-1} \end{vmatrix}, \varphi_1 = \begin{vmatrix} E_{11}(r)D_{-2} \\ E_{21}(r)D_{-1} \\ E_{31}(r)D_0 \\ B_{11}(r)D_0 \\ B_{21}(r)D_{-1} \\ B_{31}(r)D_{-2} \end{vmatrix}, \varphi_2 = \begin{vmatrix} E_{12}(r)D_{-1} \\ E_{22}(r)D_0 \\ E_{32}(r)D_{+1} \\ B_{12}(r)D_{+1} \\ B_{22}(r)D_0 \\ B_{32}(r)D_{-1} \end{vmatrix}, \varphi_3 = \begin{vmatrix} E_{13}(r)D_0 \\ E_{23}(r)D_{+1} \\ E_{33}(r)D_{+2} \\ B_{13}(r)D_{+2} \\ B_{23}(r)D_{+1} \\ B_{33}(r)D_0 \end{vmatrix}. \tag{3}$$

We use the so called cyclic basis, in which the third projection of spin is diagonal matrix; The paper is devoted to the study of a spin 2 particle in presence of Coulomb field; we restrict ourselves to the non-relativistic approximation; similar approach was used for a spin 3/2 particle in [46].

## 2. Separation of the Variables

We start with the radial system in the cyclic basis (the presence of the external Coulomb field is reached by the change $\epsilon \Longrightarrow \epsilon + \frac{\alpha}{r}$) :

$$H, \qquad -\frac{\sqrt{2}a}{2r}(h_1 + h_3) - i(\epsilon + \frac{\alpha}{r})h_0 - h_2' - \frac{2}{2}h_2 - Mh = 0;$$

$$H_1, \qquad \sqrt{2}a\,(d_1 + d_3) + 2r\left(d_2' + i(\epsilon + \frac{\alpha}{r})f_0 - 3Mh_0\right) + 4d_2 - 3ir(\epsilon + \frac{\alpha}{r})h = 0,$$

$$\frac{2ac_2 - 3ah + 2bf_1}{6\sqrt{2}r} + \frac{1}{3}(c_3' + \frac{3c_3}{r} + i(\epsilon + \frac{\alpha}{r})d_1) - Mh_1 = 0,$$

$$\sqrt{2}a\,(c_1 + c_3) + 4c_2 + r(2i(\epsilon + \frac{\alpha}{r})d_2 + 2f_2' + 3h' - 6Mh_2) + 4f_2 = 0,$$

$$\frac{2ac_2 - 3ah + 2bf_3}{6\sqrt{2}r} + \frac{1}{3}(c_1' + \frac{3c_1}{r} + i(\epsilon + \frac{\alpha}{r})d_3) - Mh_3 = 0;$$

$H_2$

$$\frac{B_{31}}{r} - i(\epsilon + \frac{\alpha}{r})E_{11} - \frac{b\,(B_{21} + 2h_1)}{\sqrt{2}r} + B_{31}' - Mf_1 = 0,$$

$$4B_{11} - 4B_{33} + 4\,(E_{20} - 2h_2) + \sqrt{2}a\,(3B_{12} + B_{21} - B_{23} - 3B_{32} + E_{10} + E_{30} - 2\,(h_1 + h_3))$$
$$+2r\left(-i(\epsilon + \frac{\alpha}{r})\,(E_{13} + 3E_{22} + E_{31} + 2h_0) - B_{11}' + B_{33}' + E_{20}' + 6h_2'\right) - 8Mrf_2 = 0,$$

$$-2B_{13} + \sqrt{2}b\,(B_{23} - 2h_3) - 2r\left(i(\epsilon + \frac{\alpha}{r})E_{33} + Mf_3 + B_{13}'\right) = 0,$$

$$\sqrt{2}bB_{13} + 2B_{23} + \sqrt{2}a\,(B_{22} - B_{33} - 2h_2) - 4h_3 - 2r\left(2Mc_1 + i(\epsilon + \frac{\alpha}{r})\,(E_{23} + E_{32}) + B_{12}' - 2h_3'\right) = 0,$$

$$\sqrt{2}a\,(B_{12} + B_{21} - B_{23} - B_{32} - E_{10} - E_{30} - 2\,(h_1 + h_3)) - 2\,(2\,(E_{20} + 2h_2)$$
$$+r\left(4Mc_2 + i(\epsilon + \frac{\alpha}{r})\,(E_{13} + E_{22} + E_{31} - 2h_0) + B_{11}' - B_{33}' + E_{20}' - 2h_2'\right)\Big) = 0,$$

$$-4Mrc_3 + \sqrt{2}\,(aB_{11} - bB_{31} - a\,(B_{22} + 2h_2)) - 2\left(B_{21} + ir(\epsilon + \frac{\alpha}{r})\,(E_{12} + E_{21}) + 2h_1 - r\left(B_{32}' + 2h_1'\right)\right) = 0,$$

$$-4Mrd_1 + \sqrt{2}\,(bE_{11} + a\,(-B_{20} + E_{31} - 2h_0))$$
$$+2\left(B_{30} + E_{12} + 2E_{21} + r\left(-i(\epsilon + \frac{\alpha}{r})\,(E_{10} + 2h_1) + B_{30}' + E_{21}'\right)\right) = 0,$$

$$\sqrt{2}a\,(B_{10} - B_{30} + E_{12} + E_{32}) + 2\left(E_{13} + 2E_{22} + E_{31} + r\left(-2Md_2 - i(\epsilon + \frac{\alpha}{r})\,(E_{20} + 2h_2) + E_{22}' + 2h_0'\right)\right) = 0,$$

$$-2B_{10} + 4E_{23} + 2E_{32} + \sqrt{2}bE_{33} + \sqrt{2}a\,(B_{20} + E_{13} - 2h_0) - 2r\left(2Md_3 + i(\epsilon + \frac{\alpha}{r})\,(E_{30} + 2h_3) + B_{10}' - E_{23}'\right) = 0,$$

$$4B_{11}+\sqrt{2}a\left(B_{12}-B_{21}+B_{23}-B_{32}+3E_{10}+3E_{30}+2\left(h_1+h_3\right)\right)+2\left(-2B_{33}+6E_{20}+4h_2\right.$$
$$\left.+r\left(-4Mf_0+i(\epsilon+\frac{\alpha}{r})\left(E_{13}-E_{22}+E_{31}-6h_0\right)+B'_{11}-B'_{33}+3E'_{20}+2h'_2\right)\right)=0,$$

$H_3$

$$\frac{6c_3+\sqrt{2}\left(ac_2+3af_0+bf_1\right)}{r}+2\left(-2i(\epsilon+\frac{\alpha}{r})d_1-3ME_{10}+c'_3\right)=0,$$
$$4c_2+\sqrt{2}a\left(c_1+c_3\right)+4f_2+2r\left(-2i(\epsilon+\frac{\alpha}{r})d_2-3\left(ME_{20}+f'_0\right)+f'_2\right)=0,$$
$$\frac{6c_1+\sqrt{2}\left(ac_2+3af_0+bf_3\right)}{r}+2\left(-2i(\epsilon+\frac{\alpha}{r})d_3-3ME_{30}+c'_1\right)=0,$$
$$\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}+d'_3-MB_{10}=0,$$
$$\frac{a\left(d_3-d_1\right)}{\sqrt{2}r}-MB_{20}=0,\quad 2d_1+\sqrt{2}ad_2+2rd'_1+2rMB_{30}=0,$$
$$\frac{bd_1}{\sqrt{2}r}-i(\epsilon+\frac{\alpha}{r})f_1-ME_{11}=0,\quad -i(\epsilon+\frac{\alpha}{r})c_3-d'_1-ME_{21}=0,$$
$$2d_2+\sqrt{2}a\left(2d_1-d_3\right)-2r\left(3ME_{31}+i\left(\epsilon+\frac{\alpha}{r}\right)\left(3c_2+f_0\right)+d'_2\right)=0,$$
$$\sqrt{2}a\left(c_1-2c_3\right)-2\left(c_2+f_2\right)+2r\left(3MB_{11}+i(\epsilon+\frac{\alpha}{r})d_2-3c'_2+f'_2\right)=0,$$
$$+i(\epsilon+\frac{\alpha}{r})d_1+\frac{\sqrt{2}\left(2ac_2-bf_1\right)}{r}+c'_3-3MB_{21}=0,$$
$$\sqrt{2}bc_3+2f_1+2r\left(MB_{31}+f'_1\right)=0,\quad -i(\epsilon+\frac{\alpha}{r})c_3+\frac{d_1}{r}+\frac{ad_2}{\sqrt{2}r}-ME_{12}=0,$$
$$4d_2+\sqrt{2}a\left(d_1+d_3\right)-6MrE_{22}+2ir(\epsilon+\frac{\alpha}{r})\left(f_0-3f_2\right)-4rd'_2=0,$$
$$-i(\epsilon+\frac{\alpha}{r})c_1+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}-ME_{32}=0,\ 6MrB_{12}-6c_1+2ir(\epsilon+\frac{\alpha}{r})d_3+\sqrt{2}\left(ac_2-3af_2+bf_3\right)-4rc'_1=0,$$
$$\frac{a\left(c_1-c_3\right)}{\sqrt{2}r}-MB_{22}=0,\quad -6MrB_{32}-6c_3+2ir(\epsilon+\frac{\alpha}{r})d_1+\sqrt{2}\left(ac_2+bf_1-3af_2\right)-4rc'_3=0,$$
$$-2d_2+\sqrt{2}a\left(d_1-2d_3\right)+2r\left(3ME_{13}+i(\epsilon+\frac{\alpha}{r})\left(3c_2+f_0\right)+d'_2\right)=0,$$
$$-i(\epsilon+\frac{\alpha}{r})c_1-ME_{23}-d'_3=0,\quad \frac{bd_3}{\sqrt{2}r}-ME_{33}-i(\epsilon+\frac{\alpha}{r})f_3=0,$$
$$-MB_{13}+\frac{bc_1}{\sqrt{2}r}+\frac{f_3}{r}+f'_3=0,\quad -3MB_{23}-i(\epsilon+\frac{\alpha}{r})d_3+\frac{\sqrt{2}\left(bf_3-2ac_2\right)}{r}-c'_1=0,$$
$$\sqrt{2}a\left(c_3-2c_1\right)-2\left(c_2+f_2\right)+2r\left(-3MB_{33}+i(\epsilon+\frac{\alpha}{r})d_2-3c'_2+f'_2\right)=0.$$

The parity restrictions are known [43]:
$\underline{P=(-1)^{j+1},}$

$$h=0;\qquad h_0=0,\quad h_2=0,\quad h_3=-h_1;$$
$$f_0=0,\quad f_2=0,c_2=0,d_2=0,\quad f_3=-f_1,c_3=-c_1,d_3=-d_1;$$

$$\begin{aligned}\bar{E}_{30}=-\bar{E}_{10},\quad \bar{E}_{20}=0,\quad \bar{B}_{30}=+\bar{B}_{10};\\ \bar{E}_{32}=-\bar{E}_{12},\quad \bar{E}_{22}=0,\quad \bar{B}_{32}=+\bar{B}_{12};\\ \bar{E}_{13}=-\bar{E}_{31},\quad \bar{E}_{23}=-\bar{E}_{21},\quad \bar{E}_{33}=-\bar{E}_{11},\\ \bar{B}_{13}=+\bar{B}_{31},\quad \bar{B}_{23}=+\bar{B}_{21},\quad \bar{B}_{33}=+\bar{B}_{11};\end{aligned}\tag{4}$$

$$P=(-1)^j,\qquad \begin{aligned}\bar{B}_{30}=-\bar{B}_{10},\quad \bar{B}_{20}=0,\quad \bar{E}_{30}=+\bar{E}_{10};\\ \bar{B}_{32}=-\bar{B}_{12},\quad \bar{B}_{22}=0,\quad \bar{E}_{32}=+\bar{E}_{12};\\ \bar{B}_{13}=-\bar{B}_{31},\quad \bar{B}_{23}=-\bar{B}_{21},\quad \bar{B}_{33}=-\bar{B}_{11},\\ \bar{E}_{13}=+\bar{E}_{31},\quad \bar{E}_{23}=+\bar{E}_{21},\quad \bar{E}_{33}=+\bar{E}_{11};\end{aligned}\tag{5}$$

There exist a special case, when the initial substitutions should become more simple. At $j=0$ , the initial general substitution should be modified in accordance with the following constraints:

$$\bar{H}_1\qquad \bar{h}_1=0,\bar{h}_3=0,\qquad \bar{H}_2\qquad \bar{f}_1=0,\bar{f}_3=0,\quad \bar{c}_1=0,\bar{c}_3=0,\quad \bar{d}_1=0,\bar{d}_3=0,$$

$$\bar{H}_3\qquad \begin{aligned}\bar{E}_{10}=0\quad \bar{E}_{11}=0\quad \bar{E}_{12}=0\quad \bar{E}_{23}=0\\ \bar{E}_{30}=0\quad \bar{E}_{21}=0\quad \bar{E}_{32}=0\quad \bar{E}_{33}=0\\ \bar{B}_{10}=0\quad \bar{B}_{21}=0\quad \bar{B}_{12}=0\quad \bar{B}_{13}=0\\ \bar{B}_{30}=0\quad \bar{B}_{31}=0\quad \bar{B}_{32}=0\quad \bar{B}_{23}=0;\end{aligned}\tag{6}$$

## 3. Second Order Equations

Let us apply the notation $(\epsilon + \frac{\alpha}{r}) \Longrightarrow D_0$, then the above equations read

$$H, \qquad \sqrt{2}a\,(h_1+h_3)+2r\left(iD_0h_0+mh+h_2'\right)+4h_2=0;$$

$H_1$

$$\sqrt{2}a\,(d_1+d_3)+2r\left(iD_0f_0+d_2'-3mh_0\right)-3iD_0rh+4d_2=0,$$
$$\frac{2ac_2-3ah+2bf_1}{6\sqrt{2}r}+\frac{1}{3}\left(iD_0d_1+c_3'+\frac{3c_3}{r}\right)-mh_1=0,$$
$$\sqrt{2}a\,(c_1+c_3)+r\left(2iD_0d_2+2f_2'+3h'-6mh_2\right)+4c_2+4f_2=0,$$
$$\frac{2ac_2-3ah+2bf_3}{6\sqrt{2}r}+\frac{1}{3}\left(iD_0d_3+c_1'+\frac{3c_1}{r}\right)-mh_3=0;$$

$H_2$

$$\frac{B_{31}}{r}-iD_0E_{11}-mf_1-\frac{b\,(B_{21}+2h_1)}{\sqrt{2}r}+B_{31}'=0,$$
$$4B_{11}-4B_{33}-8mrf_2+4\,(E_{20}-2h_2)+\sqrt{2}a\,(3B_{12}+B_{21}-B_{23}-3B_{32}+E_{10}+E_{30}-2\,(h_1+h_3))$$
$$+2r\left(-iD_0\,(E_{13}+3E_{22}+E_{31}+2h_0)-B_{11}'+B_{33}'+E_{20}'+6h_2'\right)=0,$$
$$-2B_{13}+\sqrt{2}b\,(B_{23}-2h_3)-2r\left(iD_0E_{33}+mf_3+B_{13}'\right)=0,$$
$$\sqrt{2}bB_{13}+2B_{23}+\sqrt{2}a\,(B_{22}-B_{33}-2h_2)-4h_3-2r\left(2mc_1+iD_0\,(E_{23}+E_{32})+B_{12}'-2h_3'\right)=0,$$
$$\sqrt{2}a\,(B_{12}+B_{21}-B_{23}-B_{32}-E_{10}-E_{30}-2\,(h_1+h_3))$$
$$-2\left(2\,(E_{20}+2h_2)+r\left(4mc_2+iD_0\,(E_{13}+E_{22}+E_{31}-2h_0)+B_{11}'-B_{33}'+E_{20}'-2h_2'\right)\right)=0,$$
$$-4mrc_3+\sqrt{2}\,(aB_{11}-bB_{31}-a\,(B_{22}+2h_2))-2\left(B_{21}+irD_0\,(E_{12}+E_{21})+2h_1-r\left(B_{32}'+2h_1'\right)\right)=0,$$
$$-4mrd_1+\sqrt{2}\,(bE_{11}+a\,(-B_{20}+E_{31}-2h_0))+2\left(B_{30}+E_{12}+2E_{21}+r\left(-iD_0\,(E_{10}+2h_1)+B_{30}'+E_{21}'\right)\right)=0,$$
$$\sqrt{2}a\,(B_{10}-B_{30}+E_{12}+E_{32})+2\left(E_{13}+2E_{22}+E_{31}+r\left(-2md_2-iD_0\,(E_{20}+2h_2)+E_{22}'+2h_0'\right)\right)=0,$$
$$-2B_{10}+4E_{23}+2E_{32}+\sqrt{2}bE_{33}+\sqrt{2}a\,(B_{20}+E_{13}-2h_0)-2r\left(2md_3+iD_0\,(E_{30}+2h_3)+B_{10}'-E_{23}'\right)=0,$$
$$4B_{11}+\sqrt{2}a\,(B_{12}-B_{21}+B_{23}-B_{32}+3E_{10}+3E_{30}+2\,(h_1+h_3))$$
$$+2\left(-2B_{33}+6E_{20}+4h_2+r\left(-4mf_0+iD_0\,(E_{13}-E_{22}+E_{31}-6h_0)+B_{11}'-B_{33}'+3E_{20}'+2h_2'\right)\right)=0;$$

$H_3$

$$\frac{6c_3+\sqrt{2}\,(ac_2+3af_0+bf_1)}{r}+2\left(-2iD_0d_1-3mE_{10}+c_3'\right)=0,$$
$$4c_2+\sqrt{2}a\,(c_1+c_3)+4f_2+2r\left(-2iD_0d_2-3\left(mE_{20}+f_0'\right)+f_2'\right)=0,$$
$$\frac{6c_1+\sqrt{2}\,(ac_2+3af_0+bf_3)}{r}+2\left(-2iD_0d_3-3mE_{30}+c_1'\right)=0,$$
$$-mB_{10}+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}+d_3'=0, \qquad \frac{a\,(d_3-d_1)}{\sqrt{2}r}-mB_{20}=0,$$
$$2d_1+\sqrt{2}ad_2+2r\left(mB_{30}+d_1'\right)=0, \quad \frac{bd_1}{\sqrt{2}r}-mE_{11}-iD_0f_1=0, \quad -iD_0c_3-mE_{21}-d_1'=0,$$
$$2d_2+\sqrt{2}a\,(2d_1-d_3)-2r\left(3mE_{31}+iD_0\,(3c_2+f_0)+d_2'\right)=0,$$
$$\sqrt{2}a\,(c_1-2c_3)-2\,(c_2+f_2)+2r\left(3mB_{11}+iD_0d_2-3c_2'+f_2'\right)=0,$$
$$-3mB_{21}+iD_0d_1+\frac{\sqrt{2}\,(2ac_2-bf_1)}{r}+c_3'=0, \quad \sqrt{2}bc_3+2f_1+2r\left(mB_{31}+f_1'\right)=0,$$
$$-iD_0c_3+\frac{d_1}{r}+\frac{ad_2}{\sqrt{2}r}-mE_{12}=0, \quad 4d_2+\sqrt{2}a\,(d_1+d_3)-6mrE_{22}+2irD_0\,(f_0-3f_2)-4rd_2'=0,$$
$$-iD_0c_1+\frac{ad_2}{\sqrt{2}r}+\frac{d_3}{r}-mE_{32}=0, \quad 6mrB_{12}-6c_1+2irD_0d_3+\sqrt{2}\,(ac_2-3af_2+bf_3)-4rc_1'=0,$$
$$\frac{a\,(c_1-c_3)}{\sqrt{2}r}-mB_{22}=0, \quad -6mrB_{32}-6c_3+2irD_0d_1+\sqrt{2}\,(ac_2+bf_1-3af_2)-4rc_3'=0,$$
$$-2d_2+\sqrt{2}a\,(d_1-2d_3)+2r\left(3mE_{13}+iD_0\,(3c_2+f_0)+d_2'\right)=0,$$
$$-iD_0c_1-mE_{23}-d_3'=0, \quad \frac{bd_3}{\sqrt{2}r}-mE_{33}-iD_0f_3=0, \quad -mB_{13}+\frac{bc_1}{\sqrt{2}r}+\frac{f_3}{r}+f_3'=0,$$
$$-3mB_{23}-iD_0d_3+\frac{\sqrt{2}\,(bf_3-2ac_2)}{r}-c_1'=0,$$

$$\sqrt{2}a\,(c_3-2c_1)-2\,(c_2+f_2)+2r\left(-3mB_{33}+iD_0d_2-3c_2'+f_2'\right)=0.$$

From the system $H_1$we express the variables $h_0,h_1,h_2,h_3$ and substitute them into equations of the groups $H$ and $H_2$; also from equation of the group $H_3$ we express the variables $E_{i0},B_{i0};E_{i1},B_{i1};E_{i2},B_{i2};E_{i3},B_{i3}$ and substitute them into equation of the group $H_2$ (allowing for that the variables $h_0,h_1,h_2,h_3$ are eliminated as well).

Then we arrive at eleven equations of the second order for eleven variables $h,f_i,c_i,d_i,f_0$ (let the symbol $R_0$ designate the derivative $D_0$, in all 2nd order equations it should be locates in the right side and before the all involved variables):

0

$$\frac{2a^2}{3mr}c_2-\frac{a^2}{mr}h+\frac{i\sqrt{2}a}{3m}D_0d_1+\frac{i\sqrt{2}a}{3m}D_0d_3+\frac{i\sqrt{2}a}{3m}R_0d_1+\frac{i\sqrt{2}a}{3m}R_0d_3$$
$$+\frac{ab}{3mr}f_1+\frac{ab}{3mr}f_3+\frac{2\sqrt{2}a}{3m}D_rc_1+\frac{2\sqrt{2}a}{3m}D_rc_3+\frac{4\sqrt{2}a}{3mr}c_1$$
$$+\frac{4\sqrt{2}a}{3mr}c_3-r\frac{2}{3m}D_0R_0f_0+\frac{r}{m}D_0R_0h+\frac{2ri}{3m}D_0D_rd_2+\frac{4i}{3m}D_0d_2+\frac{2ir}{3m}D_rR_0d_2+\frac{4i}{3m}R_0d_2$$
$$+\frac{4}{3m}D_rc_2+\frac{4}{3mr}c_2+\frac{8}{3m}D_rf_2+\frac{2r}{3m}D_r^2f_2+\frac{4}{3mr}f_2+\frac{r}{m}D_r^2h+\frac{2}{m}D_rh+2mrh=0,$$

1

$$\frac{ab}{2mr^2}h-\frac{ab}{mr^2}c_2-\frac{\sqrt{2}b}{mr^2}c_3-\frac{ib}{\sqrt{2}mr}D_0d_1-\frac{ib}{\sqrt{2}mr}R_0d_1-mf_1-\frac{1}{m}D_0R_0f_1-\frac{\sqrt{2}b}{mr}D_rc_3-\frac{2}{mr}D_rf_1-\frac{1}{m}D_r^2f_1=0,$$

2

$$\frac{2ha^2}{mr}+\frac{2a^2}{mr}f_0+\frac{6a^2}{mr}f_2-\frac{2i\sqrt{2}a}{m}D_0d_1-\frac{2i\sqrt{2}a}{m}R_0d_1-\frac{2i\sqrt{2}a}{m}D_0d_3-\frac{2i\sqrt{2}a}{m}R_0d_3-\frac{2ba}{mr}f_1-\frac{2ba}{mr}f_3$$
$$+\frac{4\sqrt{2}a}{m}D_rc_1+\frac{4\sqrt{2}a}{m}D_rc_3-\frac{2r}{m}D_0R_0h-\frac{4r}{m}D_0R_0c_2-\frac{8}{mr}c_2-\frac{8i}{m}\;D_0d_2-\frac{8i}{m}R_0d_2$$
$$+\frac{2rD_0R_0f_0}{m}-8mrf_2-\frac{6rD_0R_0f_2}{m}-\frac{8f_2}{mr}-\frac{4D_rh}{m}+\frac{16D_rc_2}{m}+\frac{4irD_0D_rd_2}{m}$$
$$+\frac{4ir}{m}R_0D_rd_2-\frac{4}{m}D_rf_0+\frac{4}{m}D_rf_2+\frac{6r}{m}D_r^2h-\frac{4r}{m}D_r^2c_2-\frac{2r}{m}D_r^2f_0+\frac{6r}{m}D_r^2f_2=0,$$

3

$$\frac{ab}{mr}h-\frac{2\sqrt{2}b}{mr}c_1-\frac{2ab}{mr}c_2-\frac{i\sqrt{2}b}{m}D_0d_3-\frac{i\sqrt{2}b}{m}R_0d_3-2mrf_3-\frac{2r}{m}D_0R_0f_3-\frac{2\sqrt{2}b}{m}D_rc_1-\frac{4}{m}D_rf_3-\frac{2r}{m}D_r^2f_3=0,$$

4

$$\frac{c_1a^2}{mr}-\frac{2c_3a^2}{mr}+\frac{2\sqrt{2}ha}{mr}-\frac{4\sqrt{2}c_2a}{mr}-\frac{i\sqrt{2}D_0d_2a}{m}-\frac{i\sqrt{2}R_0d_2a}{m}-\frac{2\sqrt{2}D_rha}{m}+\frac{2\sqrt{2}D_rc_2a}{m}-\frac{2\sqrt{2}D_rf_2a}{m}$$
$$-4mrc_1-\frac{4r}{m}D_0R_0c_1+\frac{b^2}{mr}c_1-\frac{6}{mr}c_1-\frac{2i}{m}D_0d_3-\frac{2i}{m}R_0d_3+\frac{2ir}{m}D_0D_rd_3+\frac{2ir}{m}R_0D_rd_3+\frac{2\sqrt{2}b}{m}D_rf_3=0,$$

5

$$\frac{2a^2}{mr}h-\frac{2a^2}{mr}f_0+\frac{2a^2}{mr}f_2-\frac{4\sqrt{2}a}{mr}c_1-\frac{4\sqrt{2}a}{mr}c_3-\frac{2ba}{mr}f_1-\frac{2ba}{mr}f_3+\frac{2r}{m}D_0R_0h-8mrc_2-\frac{4r}{m}D_0R_0c_2-\frac{8}{mr}c_2-\frac{2r}{m}D_0R_0f_0$$
$$-\frac{2r}{m}D_0R_0f_2-\frac{8}{mr}f_2-\frac{4}{m}D_rh+\frac{4ir}{m}D_0D_rd_2+\frac{4ir}{m}D_rR_0d_2+\frac{4}{m}D_rf_0-\frac{4D_rf_2}{m}+\frac{2r}{m}D_r^2h-\frac{4r}{m}D_r^2c_2+\frac{2r}{m}D_r^2f_0+\frac{2r}{m}D_r^2f_2=0,$$

6

$$-\frac{2a^2}{mr}c_1+\frac{a^2}{mr}c_3+\frac{2\sqrt{2}a}{mr}h-\frac{4\sqrt{2}a}{mr}c_2-\frac{i\sqrt{2}a}{m}D_0d_2-\frac{i\sqrt{2}a}{m}R_0d_2-\frac{2\sqrt{2}a}{m}D_rh+\frac{2\sqrt{2}a}{m}D_rc_2-\frac{2\sqrt{2}a}{m}D_rf_2-4mrc_3$$
$$-\frac{4r}{m}D_0R_0c_3+\frac{b^2}{mr}c_3-\frac{6}{mr}c_3-\frac{2i}{m}D_0d_1-\frac{2i}{m}R_0d_1+\frac{2ir}{m}D_0D_rd_1+\frac{2ir}{m}D_rR_0d_1+\frac{2\sqrt{2}b}{m}D_rf_1=0,$$

7

$$\frac{a^2}{mr}d_1-\frac{2a^2}{mr}d_3+\frac{i\sqrt{2}a}{m}D_0h+\frac{i\sqrt{2}a}{m}R_0h-\frac{i\sqrt{2}a}{m}D_0c_2-\frac{i\sqrt{2}a}{m}R_0c_2-\frac{i\sqrt{2}a}{m}D_0f_0-\frac{i\sqrt{2}a}{m}R_0f_0-\frac{2\sqrt{2}a}{m}D_rd_2$$
$$-\frac{6i}{m}D_0c_3-\frac{6i}{m}R_0c_3-4mrd_1+\frac{b^2}{mr}d_1+\frac{2}{mr}d_1-\frac{i\sqrt{2}b}{m}D_0f_1-\frac{i\sqrt{2}b}{m}R_0f_1$$
$$-\frac{2ir}{m}D_0D_rc_3-\frac{2ir}{m}D_rR_0c_3-\frac{8}{m}D_rd_1-\frac{4r}{m}D_r^2d_1=0,$$

8

$$\frac{4a^2}{mr}d_2 - \frac{i\sqrt{2}a}{m}D_0c_1 - \frac{i\sqrt{2}a}{m}R_0c_1 - \frac{i\sqrt{2}a}{m}D_0c_3 - \frac{i\sqrt{2}a}{m}R_0c_3 + \frac{2\sqrt{2}a}{mr}d_1 + \frac{2\sqrt{2}a}{mr}d_3 + \frac{2\sqrt{2}a}{m}D_rd_1 + \frac{2\sqrt{2}a}{m}D_rd_3$$
$$-\frac{4i}{m}D_0c_2 - \frac{4i}{m}R_0c_2 - 4mrd_2 - \frac{4i}{m}D_0f_2 - \frac{4i}{m}R_0f_2 - \frac{2ir}{m}D_0D_rh$$
$$-\frac{2ir}{m}D_rR_0h + \frac{2ir}{m}D_0D_rf_0 + \frac{2ir}{m}D_rR_0f_0 - \frac{2ir}{m}D_0D_rf_2 - \frac{2ir}{m}D_rR_0f_2 = 0,$$

9

$$-\frac{2a^2}{mr}d_1 + \frac{a^2}{mr}d_3 + \frac{i\sqrt{2}a}{m}D_0h + \frac{i\sqrt{2}a}{m}R_0h - \frac{i\sqrt{2}a}{m}D_0c_2 - \frac{i\sqrt{2}a}{m}R_0c_2 - \frac{i\sqrt{2}a}{m}D_0f_0$$
$$-\frac{i\sqrt{2}a}{m}R_0f_0 - \frac{2\sqrt{2}a}{m}D_rd_2 - \frac{6i}{m}D_0c_1 - \frac{6i}{m}R_0c_1 - 4mrd_3 + \frac{b^2}{mr}d_3$$
$$+\frac{2}{mr}d_3 - \frac{i\sqrt{2}b}{m}D_0f_3 - \frac{i\sqrt{2}b}{m}R_0f_3 - \frac{2ir}{m}D_0D_rc_1 - \frac{2ir}{m}D_rR_0c_1 - \frac{8}{m}D_rd_3 - \frac{4r}{m}D_r^2d_3 = 0,$$

10

$$-\frac{2a^2}{mr}h + \frac{6a^2}{mr}f_0 + \frac{2a^2}{mr}f_2 + \frac{8\sqrt{2}a}{mr}c_1 + \frac{8\sqrt{2}a}{mr}c_3 - \frac{2i\sqrt{2}a}{m}D_0d_1 - \frac{2i\sqrt{2}a}{m}R_0d_1 - \frac{2i\sqrt{2}a}{m}D_0d_3 - \frac{2i\sqrt{2}a}{m}R_0d_3 + \frac{2ba}{mr}f_1$$
$$+\frac{2ba}{mr}f_3 + \frac{4\sqrt{2}a}{m}D_rc_1 + \frac{4\sqrt{2}D_rc_3a}{m} - \frac{6r}{m}D_0R_0h + \frac{4r}{m}D_0R_0c_2 + \frac{8}{mr}c_2 - \frac{8i}{m}D_0d_2 - \frac{8i}{m}R_0d_2 - 8mrf_0 + \frac{6r}{m}D_0R_0f_0$$
$$-\frac{2r}{m}D_0R_0f_2 + \frac{8}{mr}f_2 + \frac{4}{m}D_rh + \frac{16}{m}D_rc_2$$
$$-\frac{4ir}{m}D_0D_rd_2 - \frac{4ir}{m}D_rR_0d_2 - \frac{12}{m}D_rf_0 + \frac{12}{m}D_rf_2 + \frac{2r}{m}D_r^2h + \frac{4r}{m}D_r^2c_2 - \frac{6r}{m}D_r^2f_0 + \frac{2r}{m}D_r^2f_2 = 0.$$

Now we should take into account the parity restrictions.
Let $P = (-1)^{j+1}$,

$$h = 0,\ f_2 = 0,\ c_2 = 0,\ d_2 = 0,\ f_3 = -f_1,\ c_3 = -c_1,\ d_3 = -d_1; \tag{7}$$

then we get

0 (0=0)

$$+\frac{i\sqrt{2}a}{3m}D_0d_1 - \frac{i\sqrt{2}a}{3m}D_0d_1 + \frac{i\sqrt{2}a}{3m}R_0d_1 - \frac{i\sqrt{2}a}{3m}R_0d_1$$
$$+\frac{ab}{3mr}f_1 - \frac{ab}{3mr}f_1 + \frac{2\sqrt{2}a}{3m}D_rc_1 - \frac{2\sqrt{2}a}{3m}D_rc_1 + \frac{4\sqrt{2}a}{3mr}c_1 - \frac{4\sqrt{2}a}{3mr}c_1 = 0,$$

1 (3=1)

$$+\frac{\sqrt{2}b}{mr^2}c_1 - \frac{ib}{\sqrt{2}mr}D_0d_1 - \frac{ib}{\sqrt{2}mr}R_0d_1 - mf_1 - \frac{1}{m}D_0R_0f_1 + \frac{\sqrt{2}b}{mr}D_rc_1 - \frac{2}{mr}D_rf_1 - \frac{1}{m}D_r^2f_1 = 0,$$

3

$$-\frac{2\sqrt{2}b}{mr}c_1 + \frac{i\sqrt{2}b}{m}D_0d_1 + \frac{i\sqrt{2}b}{m}R_0d_1 + 2mrf_1 + \frac{2r}{m}D_0R_0f_1 - \frac{2\sqrt{2}b}{m}D_rc_1 + \frac{4}{m}D_rf_1 + \frac{2r}{m}D_r^2f_1 = 0, \quad (3=1),$$

2 (0=0)

$$-\frac{2i\sqrt{2}a}{m}D_0d_1 - \frac{2i\sqrt{2}a}{m}R_0d_1 + \frac{2i\sqrt{2}a}{m}D_0d_1 + \frac{2i\sqrt{2}a}{m}R_0d_1 - \frac{2ba}{mr}f_1 + \frac{2ba}{mr}f_1$$
$$+\frac{4\sqrt{2}a}{m}D_rc_1 - \frac{4\sqrt{2}a}{m}D_rc_1 = 0,$$

4 = 6

$$\frac{a^2}{mr}c_1 + \frac{2a^2}{mr}c_1 - 4mrc_1$$
$$-\frac{4r}{m}D_0R_0c_1 + \frac{b^2}{mr}c_1 - \frac{6}{mr}c_1 + \frac{2i}{m}D_0d_1 + \frac{2i}{m}R_0d_1 - \frac{2ir}{m}D_0D_rd_1 - \frac{2ir}{m}R_0D_rd_1 - \frac{2\sqrt{2}b}{m}D_rf_1 = 0,$$

6

$$-\frac{2a^2}{mr}c_1 - \frac{a^2}{mr}c_1 + 4mrc_1$$
$$+\frac{4r}{m}D_0R_0c_1 - \frac{b^2}{mr}c_1 + \frac{6}{mr}c_1 - \frac{2i}{m}D_0d_1 - \frac{2i}{m}R_0d_1 + \frac{2ir}{m}D_0D_rd_1 + \frac{2ir}{m}D_rR_0d_1 + \frac{2\sqrt{2}b}{m}D_rf_1 = 0,$$

5 (0=0)

$$-\frac{4\sqrt{2}a}{mr}c_1 + \frac{4\sqrt{2}a}{mr}c_1 - \frac{2ba}{mr}f_1 + \frac{2ba}{mr}f_1 = 0,$$

7 (9=7)

$$\frac{a^2}{mr}d_1+\frac{2a^2}{mr}d_1+\frac{6i}{m}D_0c_1+\frac{6i}{m}R_0c_1-4mrd_1+\frac{b^2}{mr}d_1+\frac{2}{mr}d_1-\frac{i\sqrt{2}b}{m}D_0f_1-\frac{i\sqrt{2}b}{m}R_0f_1$$
$$+\frac{2ir}{m}D_0D_rc_1+\frac{2ir}{m}D_rR_0c_1-\frac{8}{m}D_rd_1-\frac{4r}{m}D_r^2d_1=0,$$

9

$$-\frac{2a^2}{mr}d_1-\frac{a^2}{mr}d_1-\frac{6i}{m}D_0c_1-\frac{6i}{m}R_0c_1+4mrd_1-\frac{b^2}{mr}d_1$$
$$-\frac{2}{mr}d_1+\frac{i\sqrt{2}b}{m}D_0f_1+\frac{i\sqrt{2}b}{m}R_0f_1-\frac{2ir}{m}D_0D_rc_1-\frac{2ir}{m}D_rR_0c_1+\frac{8}{m}D_rd_1+\frac{4r}{m}D_r^2d_1=0,$$

8 (0 =0 )

$$-\frac{i\sqrt{2}a}{m}D_0c_1-\frac{i\sqrt{2}a}{m}R_0c_1+\frac{i\sqrt{2}a}{m}D_0c_1+\frac{i\sqrt{2}a}{m}R_0c_1+\frac{2\sqrt{2}a}{mr}d_1-\frac{2\sqrt{2}a}{mr}d_1+\frac{2\sqrt{2}a}{m}D_rd_1-\frac{2\sqrt{2}a}{m}D_rd_1=0,$$

10 (0=0)

$$+\frac{8\sqrt{2}a}{mr}c_1-\frac{8\sqrt{2}a}{mr}c_1-\frac{2i\sqrt{2}a}{m}D_0d_1-\frac{2i\sqrt{2}a}{m}R_0d_1+\frac{2i\sqrt{2}a}{m}D_0d_1+\frac{2i\sqrt{2}a}{m}R_0d_1+\frac{2ba}{mr}f_1$$
$$-\frac{2ba}{mr}f_1+\frac{4\sqrt{2}a}{m}D_rc_1-\frac{4\sqrt{2}D_ra}{m}c_1=0;$$

thus we have only three different equations 1,4,7:

$$\begin{gathered}\frac{\sqrt{2}b}{mr}\frac{1}{m}D_rc_1+\frac{\sqrt{2}b}{m^2r^2}c_1-\frac{i\sqrt{2}b}{mr}\frac{1}{m}D_0d_1-\frac{1}{m^2}D_0^2f_1-\frac{2}{mr}\frac{1}{m}D_rf_1-\frac{1}{m^2}D_r^2f_1-f_1=0,\\ \frac{3a^2+b^2-6}{m^2r^2}c_1-4c_1-\frac{4}{m^2}D_0^2c_1+\frac{4i}{mr}\frac{1}{m}D_0d_1-\frac{2i}{m^2}D_0D_rd_1-\frac{2i}{m^2}D_rD_0d_1-\frac{2\sqrt{2}b}{rm}\frac{1}{m}D_rf_1=0,\\ \frac{3a^2+b^2+2}{m^2r^2}d_1-4d_1-\frac{8}{mr}\frac{1}{m}D_rd_1-\frac{4}{m^2}D_r^2d_1\\ +\frac{12i}{mr}\frac{1}{m}D_0c_1+\frac{2i}{m^2}D_0D_rc_1+\frac{2i}{m^2}D_rD_0c_1-\frac{i2\sqrt{2}b}{rm}\frac{1}{m}D_0f_1=0,\end{gathered}\qquad(8)$$

Let $P=(-1)^j$,

$$f_3=+f_1,\quad c_3=+c_1,\quad d_3=+d_1;\qquad(9)$$

then we get eight 2-nd order equations:

0

$$\frac{2a^2}{3mr}c_2-\frac{a^2}{mr}h+\frac{i4\sqrt{2}a}{3m}D_0d_1+\frac{2ab}{3mr}f_1+\frac{4\sqrt{2}a}{3m}D_rc_1+\frac{4\sqrt{2}a}{3mr}c_1$$
$$+\frac{4\sqrt{2}a}{3mr}c_1-r\frac{2}{3m}D_0^2f_0+\frac{r}{m}D_0^2h+\frac{8i}{3m}D_0d_2+\frac{2ri}{3m}D_0D_rd_2+\frac{2ir}{3m}D_rD_0d_2$$
$$+\frac{4}{3m}D_rc_2+\frac{4}{3mr}c_2+\frac{8}{3m}D_rf_2+\frac{2r}{3m}D_r^2f_2+\frac{4}{3mr}f_2+\frac{r}{m}D_r^2h+\frac{2}{m}D_rh+2mrh=0,$$

1

$$\frac{ab}{2mr^2}h-\frac{\sqrt{2}b}{mr^2}c_1-\frac{ab}{mr^2}c_2-\frac{2ib}{\sqrt{2}mr}D_0d_1-mf_1-\frac{1}{m}D_0^2f_1-\frac{\sqrt{2}b}{mr}D_rc_1-\frac{2}{mr}D_rf_1-\frac{1}{m}D_r^2f_1=0,$$

3= 1

$$\frac{ab}{mr}h-\frac{2\sqrt{2}b}{mr}c_1-\frac{2ab}{mr}c_2-\frac{2i\sqrt{2}b}{m}D_0d_1-2mrf_3-\frac{2r}{m}D_0^2f_1-\frac{2\sqrt{2}b}{m}D_rc_1-\frac{4}{m}D_rf_1-\frac{2r}{m}D_r^2f_1=0,$$

2

$$\frac{2a^2}{mr}h+\frac{2a^2}{mr}f_0+\frac{6a^2}{mr}f_2-\frac{8i\sqrt{2}a}{m}D_0d_1-\frac{4ba}{mr}f_1$$
$$+\frac{8\sqrt{2}a}{m}D_rc_1-\frac{2r}{m}D_0^2h-\frac{4r}{m}D_0^2c_2-\frac{8}{mr}c_2-\frac{16i}{m}D_0d_2$$
$$+\frac{2rD_0D_0f_0}{m}-8mrf_2-\frac{6rD_0D_0f_2}{m}-\frac{8f_2}{mr}-\frac{4D_rh}{m}+\frac{16D_rc_2}{m}+\frac{8irD_0D_rd_2}{m}$$
$$-\frac{4}{m}D_rf_0+\frac{4}{m}D_rf_2+\frac{6r}{m}D_r^2h-\frac{4r}{m}D_r^2c_2-\frac{2r}{m}D_r^2f_0+\frac{6r}{m}D_r^2f_2=0,$$

4

$$\frac{b^2-6-a^2}{mr}c_1+\frac{2\sqrt{2}a}{mr}h-\frac{4\sqrt{2}a}{mr}c_2-\frac{2i\sqrt{2}a}{m}D_0d_2-\frac{2\sqrt{2}a}{m}D_rh+\frac{2\sqrt{2}a}{m}D_rc_2-\frac{2\sqrt{2}a}{m}D_rf_2$$
$$-4mrc_1-\frac{4r}{m}D_0^2c_1-\frac{4i}{m}D_0d_1+\frac{2ir}{m}D_0D_rd_1+\frac{2ir}{m}D_rD_0d_1+\frac{2\sqrt{2}b}{m}D_rf_1=0,$$

6=4

$$\frac{b^2-6-a^2}{mr}c_1+\frac{2\sqrt{2}a}{mr}h-\frac{4\sqrt{2}a}{mr}c_2-\frac{2i\sqrt{2}a}{m}D_0d_2-\frac{2\sqrt{2}a}{m}D_rh+\frac{2\sqrt{2}a}{m}D_rc_2-\frac{2\sqrt{2}a}{m}D_rf_2$$
$$-4mrc_3-\frac{4r}{m}D_0^2c_1-\frac{4i}{m}D_0d_1+\frac{2ir}{m}D_0D_rd_1+\frac{2ir}{m}D_rD_0d_1+\frac{2\sqrt{2}b}{m}D_rf_1=0,$$

5

$$\frac{2a^2}{mr}h-\frac{2a^2}{mr}f_0+\frac{2a^2}{mr}f_2-\frac{8\sqrt{2}a}{mr}c_1-\frac{4ba}{mr}f_1+\frac{2r}{m}D_0^2h-8mrc_2-\frac{4r}{m}D_0^2c_2-\frac{8}{mr}c_2-\frac{2r}{m}D_0^2f_0$$
$$-\frac{2r}{m}D_0^2f_2-\frac{8}{mr}f_2-\frac{4}{m}D_rh+\frac{4ir}{m}D_0D_rd_2+\frac{4ir}{m}D_rD_0d_2+\frac{4}{m}D_rf_0-\frac{4}{m}D_rf_2$$
$$+\frac{2r}{m}D_r^2h-\frac{4r}{m}D_r^2c_2+\frac{2r}{m}D_r^2f_0+\frac{2r}{m}D_r^2f_2=0,$$

10

$$-\frac{2a^2}{mr}h+\frac{6a^2}{mr}f_0+\frac{2a^2}{mr}f_2+\frac{16\sqrt{2}a}{mr}c_1-\frac{8i\sqrt{2}a}{m}D_0d_1+\frac{4ba}{mr}f_1$$
$$+\frac{8\sqrt{2}a}{m}D_rc_1-\frac{6r}{m}D_0^2h+\frac{4r}{m}D_0^2c_2+\frac{8}{mr}c_2-\frac{8i}{m}D_0d_2-\frac{8i}{m}R_0d_2-8mrf_0+\frac{6r}{m}D_0^2f_0$$
$$-\frac{2r}{m}D_0^2f_2+\frac{8}{mr}f_2+\frac{4}{m}D_rh+\frac{16}{m}D_rc_2$$
$$-\frac{4ir}{m}D_0D_rd_2-\frac{4ir}{m}D_rD_0d_2-\frac{12}{m}D_rf_0+\frac{12}{m}D_rf_2$$
$$+\frac{2r}{m}D_r^2h+\frac{4r}{m}D_r^2c_2-\frac{6r}{m}D_r^2f_0+\frac{2r}{m}D_r^2f_2=0,$$

7

$$\frac{b^2+2-a^2}{mr}d_1+\frac{2i\sqrt{2}a}{m}D_0h-\frac{2i\sqrt{2}a}{m}D_0c_2-\frac{2i\sqrt{2}a}{m}D_0f_0-\frac{2\sqrt{2}a}{m}D_rd_2-\frac{12i}{m}D_0c_1-4mrd_1-\frac{2i\sqrt{2}b}{m}D_0f_1$$
$$-\frac{2ir}{m}D_0D_rc_1-\frac{2ir}{m}D_rD_0c_1-\frac{8}{m}D_rd_1-\frac{4r}{m}D_r^2d_1=0,$$

9 =7

$$\frac{b^2+2-a^2}{mr}d_1+\frac{2i\sqrt{2}a}{m}D_0h-\frac{2i\sqrt{2}a}{m}D_0c_2-\frac{2i\sqrt{2}a}{m}D_0f_0-\frac{2\sqrt{2}a}{m}D_rd_2-\frac{12i}{m}D_0c_1-4mrd_1-\frac{2i\sqrt{2}b}{m}D_0f_1$$
$$-\frac{2ir}{m}D_0D_rc_1-\frac{2ir}{m}D_rD_0c_1-\frac{8}{m}D_rd_1-\frac{4r}{m}D_r^2d_1=0,$$

8

$$\frac{4a^2}{mr}d_2-\frac{4i\sqrt{2}a}{m}D_0c_1+\frac{4\sqrt{2}a}{mr}d_1+\frac{4\sqrt{2}a}{m}D_rd_1-\frac{8i}{m}D_0c_2$$
$$-4mrd_2-\frac{8i}{m}D_0f_2-\frac{2ir}{m}D_0D_rh-\frac{2ir}{m}D_rR_0h$$
$$+\frac{2ir}{m}D_0D_rf_0+\frac{2ir}{m}D_rD_0f_0-\frac{2ir}{m}D_0D_rf_2-\frac{2ir}{m}D_rD_0f_2=0;$$

Therefore, for states with the parity $P=(-1)^j$, we have eight different equations $0;\ 1-3,2;\ 4-6,5;\ 7-9,8;\ 10$:

$$\frac{2a^2}{3m^2r^2}c_2-\frac{a^2}{m^2r^2}h+\frac{i4\sqrt{2}a}{3rm}\frac{1}{m}D_0d_1\frac{2ab}{3m^2r^2}f_1+\frac{4\sqrt{2}a}{3mr}\frac{1}{m}D_rc_1+\frac{4\sqrt{2}a}{3m^2r^2}c_1$$
$$+\frac{4\sqrt{2}a}{3m^2r^2}c_1-\frac{2}{3mr}\frac{1}{m}D_0^2f_0+\frac{1}{m^2}D_0^2h+\frac{8i}{3mr}\frac{1}{m}D_0d_2+\frac{2i}{3m^2}D_0D_rd_2+\frac{2i}{3m^2}D_rD_0d_2$$
$$+\frac{4}{3mr}\frac{1}{m}D_rc_2+\frac{4}{3m^2r^2}c_2+\frac{8}{3mr}\frac{1}{m}D_rf_2+\frac{2}{3m^2}D_r^2f_2+\frac{4}{3m^2r^2}f_2+\frac{}{m^2}D_r^2h+\frac{2}{mr}\frac{1}{m}D_rh+2h=0,$$

1-3

$$\frac{ab}{2m^2r^2}h-\frac{\sqrt{2}b}{m^2r^2}c_1-\frac{ab}{m^2r^2}c_2-\frac{2ib}{\sqrt{2}mr}\frac{1}{m}D_0d_1$$
$$-f_1-\frac{1}{m^2}D_0^2f_1-\frac{\sqrt{2}b}{mr}\frac{1}{m}D_rc_1-\frac{2}{mr}\frac{1}{m}D_rf_1-\frac{1}{m^2}D_r^2f_1=0,$$

2

$$\frac{2a^2}{m^2r^2}h+\frac{2a^2}{m^2r^2}f_0+\frac{6a^2}{m^2r^2}f_2-\frac{8i\sqrt{2}a}{mr}\frac{1}{m}D_0d_1-\frac{4ba}{m^2r^2}f_1$$

$$+\frac{8\sqrt{2}a}{mr}\frac{1}{m}D_r c_1 - \frac{2}{m^2}D_0^2 h - \frac{4}{m^2}D_0^2 c_2 - \frac{8}{m^2 r^2}c_2 - \frac{16i}{mr}\frac{1}{m}D_0 d_2$$
$$+\frac{2}{m^2}D_0^2 f_0 - 8f_2 - \frac{6}{m^2}D_0^2 f_2 - \frac{8}{m^2r^2}f_2 - \frac{4}{mr}\frac{1}{m}D_r h + \frac{16}{mr}\frac{1}{m}D_r c_2 + \frac{8i}{m^2}D_0 D_r d_2$$
$$-\frac{4}{mr}\frac{1}{m}D_r f_0 + \frac{4}{mr}\frac{1}{m}D_r f_2 + \frac{6}{m^2}D_r^2 h - \frac{4}{m^2}D_r^2 c_2 - \frac{2}{m^2}D_r^2 f_0 + \frac{6}{m^2}D_r^2 f_2 = 0,$$

4-6

$$\frac{b^2-6-a^2}{m^2r^2}c_1 + \frac{2\sqrt{2}a}{m^2r^2}h - \frac{4\sqrt{2}a}{m^2r^2}c_2 - \frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0 d_2 - \frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r h + \frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r c_2 - \frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r f_2$$
$$-4c_1 - \frac{4}{m^2}D_0^2 c_1 - \frac{4i}{mr}\frac{1}{m}D_0 d_1 + \frac{2i}{m^2}D_0 D_r d_1 + \frac{2i}{m^2}D_r D_0 d_1 + \frac{2\sqrt{2}b}{mr}\frac{1}{m}D_r f_1 = 0,$$

5

$$\frac{2a^2}{m^2r^2}h - \frac{2a^2}{m^2r^2}f_0 + \frac{2a^2}{m^2r^2}f_2 - \frac{8\sqrt{2}a}{m^2r^2}c_1 - \frac{4ba}{m^2r^2}f_1$$
$$+\frac{2}{m^2}D_0^2 h - 8c_2 - \frac{4}{m^2}D_0^2 c_2 - \frac{8}{m^2r^2}c_2 - \frac{2}{m^2}D_0^2 f_0$$
$$-\frac{2}{m^2}D_0^2 f_2 - \frac{8}{m^2r^2}f_2 - \frac{4}{mr}\frac{1}{m}D_r h + \frac{4i}{m^2}D_0 D_r d_2 + \frac{4i}{m^2}D_r D_0 d_2 + \frac{4}{mr}\frac{1}{m}D_r f_0 - \frac{4}{mr}\frac{1}{m}D_r f_2$$
$$+\frac{2}{m^2}D_r^2 h - \frac{4}{m^2}D_r^2 c_2 + \frac{2}{m^2}D_r^2 f_0 + \frac{2}{m^2}D_r^2 f_2 = 0,$$

7-9

$$\frac{b^2+2-a^2}{m^2r^2}d_1 + \frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0 h - \frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0 c_2 - \frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0 f_0 - \frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r d_2$$
$$-\frac{12i}{mr}\frac{1}{m}D_0 c_1 - 4d_1 - \frac{2i\sqrt{2}b}{mr}\frac{1}{m}D_0 f_1 \frac{2i}{m^2}D_0 D_r c_1 - \frac{2i}{m^2}D_r D_0 c_1 - \frac{8}{mr}\frac{1}{m}D_r d_1 - \frac{4}{m^2}D_r^2 d_1 = 0,$$

8

$$\frac{4a^2}{m^2r^2}d_2 - \frac{4i\sqrt{2}a}{mr}\frac{1}{m}D_0 c_1 + \frac{4\sqrt{2}a}{m^2r^2}d_1 + \frac{4\sqrt{2}a}{mr}\frac{1}{m}D_r d_1$$
$$-\frac{8i}{mr}\frac{1}{m}D_0 c_2 - 4d_2 - \frac{8i}{mr}\frac{1}{m}D_0 f_2 - \frac{2i}{m^2}D_0 D_r h - \frac{2i}{m^2}D_r R_0 h +$$
$$+\frac{2i}{m^2}D_0 D_r f_0 + \frac{2i}{m^2}D_r D_0 f_0 - \frac{2i}{m^2}D_0 D_r f_2 - \frac{2i}{m^2}D_r D_0 f_2 = 0,$$

10

$$-\frac{2a^2}{m^2r^2}h + \frac{6a^2}{m^2r^2}f_0 + \frac{2a^2}{m^2r^2}f_2 + \frac{16\sqrt{2}a}{m^2r^2}c_1 - \frac{8i\sqrt{2}a}{mr}\frac{1}{m}D_0 d_1 + \frac{4ba}{m^2r^2}f_1$$
$$+\frac{8\sqrt{2}a}{mr}\frac{1}{m}D_r c_1 - \frac{6}{m^2}D_0^2 h + \frac{4}{m^2}D_0^2 c_2 + \frac{8}{m^2r^2}c_2 - \frac{8i}{mr}\frac{1}{m}D_0 d_2 - \frac{8i}{mr}\frac{1}{m}D_0 d_2 - 8f_0 + \frac{6}{m^2}D_0^2 f_0$$
$$-\frac{2}{m^2}D_0^2 f_2 + \frac{8}{m^2r^2}f_2 + \frac{4}{mr}\frac{1}{m}D_r h + \frac{16}{mr}\frac{1}{m}D_r c_2$$
$$-\frac{4i}{m^2}D_0 D_r d_2 - \frac{4i}{m^2}D_r D_0 d_2 - \frac{12}{mr}\frac{1}{m}D_r f_0$$
$$+\frac{12}{mr}\frac{1}{m}D_r f_2 \frac{2}{m^2}D_r^2 h + \frac{4}{m^2}D_r^2 c_2 - \frac{6}{m^2}D_r^2 f_0 + \frac{2}{m^2}D_r^2 f_2 = 0.$$

## 4. Projective Operators, Large and Small Components

In order to perform the non-relativistic approximation, we should find large and small components in the wave function. To this end, we are to use the matrix $\Gamma^0 = \Gamma$ [42].

Its explicit form in cyclic basis $\bar{\Gamma}^0$ was found in [43]:

$$\bar{\Gamma} = \begin{vmatrix} 0 & \bar{G}^0 & 0 & 0 \\ \frac{1}{2}\bar{\Delta}^0 & 0 & -\frac{1}{3}\bar{K}^0 & 0 \\ 0 & \bar{\Lambda}^0 & 0 & \frac{1}{2}\bar{B}^0 \\ 0 & 0 & \bar{F}^0 & 0 \end{vmatrix},$$

$$\bar{G}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \end{vmatrix}, \quad \bar{\Delta}^0 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \bar{K}^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix},$$

$$
\bar{B}_0 = \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0
\end{vmatrix}
$$

$$
\bar{\Lambda}^0 = \frac{1}{2} \begin{vmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \\
3 & 0 & 0 & 0
\end{vmatrix}, \quad
\bar{F}_0 = \begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0
\end{vmatrix},
$$

The matrix $\bar{\Gamma}$ satisfies the 7th order minimal equation

$$
\bar{\Gamma}^5(\bar{\Gamma}^2 - I) = 0; \tag{10}
$$

which permits us to introduce three projective operators

$$
\bar{P}_+ = \frac{1}{2}\bar{\Gamma}^5(\bar{\Gamma} + I) = \bar{P}_1, \quad \bar{P}_- = \frac{1}{2}\bar{\Gamma}^5(\bar{\Gamma} - I) = \bar{P}_2, \quad \bar{P}_0 = I - \bar{\Gamma}^6 = \bar{P}_3 \tag{11}
$$

with the needed properties $P_i^2 = P_i,\ P_1 + P_2 + P_3 = I,\ i = 1, 2, 3.$

We readily find explicit form of the projective constituents
so that

$$
\begin{aligned}
\Psi = \text{column}\{&h,\ h_0, h_1, h_2, h_3, \quad f_1, f_2, f_3, c_1, c_2, c_3, d_1, d_2, d_3, f_0, \\
&E_{10}, E_{20}, E_{30}, B_{10}, B_{20}, B_{30}, \quad E_{11}, E_{21}, E_{31}, B_{11}, B_{21}, B_{31}, \\
&E_{12}, E_{22}, E_{32}, B_{12}, B_{22}, B_{32}, \quad E_{13}, E_{23}, E_{33}, B_{13}, B_{23}, B_{33}\}
\end{aligned} \tag{12}
$$

$$
P_1\Psi = \left| \begin{array}{c}
0 \\
\\
0 \\ 0 \\ 0 \\ 0 \\
\\
\frac{1}{2}\left(E_{11}+f_1\right) \\
\frac{1}{6}\left(2c_2+E_{13}+2E_{22}+E_{31}+2f_2\right) \\
\frac{1}{2}\left(E_{33}+f_3\right) \\
\frac{1}{4}\left(2c_1+E_{23}+E_{32}\right) \\
\frac{1}{12}\left(2c_2+E_{13}+2E_{22}+E_{31}+2f_2\right) \\
\frac{1}{4}\left(2c_3+E_{12}+E_{21}\right) \\
0 \\ 0 \\ 0 \\ 0 \\
\\
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\
\frac{1}{2}\left(E_{11}+f_1\right) \\
\frac{1}{4}\left(2c_3+E_{12}+E_{21}\right) \\
\frac{1}{12}\left(2c_2+E_{13}+2E_{22}+E_{31}+2f_2\right) \\
0 \\ 0 \\ 0 \\
\frac{1}{4}\left(2c_3+E_{12}+E_{21}\right) \\
\frac{1}{6}\left(2c_2+E_{13}+2E_{22}+E_{31}+2f_2\right) \\
\frac{1}{4}\left(2c_1+E_{23}+E_{32}\right) \\
0 \\ 0 \\ 0 \\
\frac{1}{12}\left(2c_2+E_{13}+2E_{22}+E_{31}+2f_2\right) \\
\frac{1}{4}\left(2c_1+E_{23}+E_{32}\right) \\
\frac{1}{2}\left(E_{33}+f_3\right) \\
0 \\ 0 \\ 0
\end{array} \right| = \left| \begin{array}{c}
0 \\
\\
0 \\ 0 \\ 0 \\ 0 \\
\\
L_1 \\
L_2 \\
L_3 \\
L_4 \\
L_5 = \frac{1}{2}L_2 \\
L_6 \\
0 \\ 0 \\ 0 \\ 0 \\
\\
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\
L_7 = L_1 \\
L_8 = L_6 \\
L_9 = L_2 \\
0 \\ 0 \\ 0 \\
L_{10} = L_6 \\
L_{11} = L_2 \\
L_{12} = L_4 \\
0 \\ 0 \\ 0 \\
L_{13} = \frac{1}{2}L_2 \\
L_{14} = L_4 \\
L_{15} = L_3 \\
0 \\ 0 \\ 0
\end{array} \right| ,
$$

there exist only six large variables: $L_1, ..., L_6$ (for what follows it will be enough to follow only eleven functions referring to scalar and symmetric tensor)

$$
, \quad P_2\Psi = \left| \begin{array}{c} 0 \\ \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ \frac{1}{2}(f_1 - E_{11}) \\ \frac{1}{6}(2c_2 - E_{13} - 2E_{22} - E_{31} + 2f_2) \\ \frac{1}{2}(f_3 - E_{33}) \\ \frac{1}{4}(2c_1 - E_{23} - E_{32}) \\ \frac{1}{12}(2c_2 - E_{13} - 2E_{22} - E_{31} + 2f_2) \\ \frac{1}{4}(2c_3 - E_{12} - E_{21}) \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}(E_{11} - f_1) \\ \frac{1}{4}(-2c_3 + E_{12} + E_{21}) \\ \frac{1}{12}(-2c_2 + E_{13} + 2E_{22} + E_{31} - 2f_2) \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}(-2c_3 + E_{12} + E_{21}) \\ \frac{1}{6}(-2c_2 + E_{13} + 2E_{22} + E_{31} - 2f_2) \\ \frac{1}{4}(-2c_1 + E_{23} + E_{32}) \\ 0 \\ 0 \\ 0 \\ \frac{1}{12}(-2c_2 + E_{13} + 2E_{22} + E_{31} - 2f_2) \\ \frac{1}{4}(-2c_1 + E_{23} + E_{32}) \\ \frac{1}{2}(E_{33} - f_3) \\ 0 \\ 0 \\ 0 \end{array} \right| = \left| \begin{array}{c} 0 \\ \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 = \frac{1}{2}S_2 \\ S_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ S_7 = -S_1 \\ S_8 = -S_6 \\ S_9 = -\frac{1}{2}S_2 \\ 0 \\ 0 \\ 0 \\ S_{10} = -S_6 \\ S_{11} = -S_2 \\ S_{12} = -S_4 \\ 0 \\ 0 \\ 0 \\ S_{13} = -\frac{1}{2}S_2 \\ S_{14} = -S_4 \\ S_{15} = S_3 \\ 0 \\ 0 \\ 0 \end{array} \right| ,
$$

here there exist only five small variables $S_1, S_2, S_3, S_4, S_6$;

$$P_3\Psi = \left| \begin{array}{c} h \\ \\ h_0 \\ h_1 \\ h_2 \\ h_3 \\ \\ 0 \\ \frac{1}{3}(f_2 - 2c_2) \\ 0 \\ 0 \\ \frac{1}{3}(2c_2 - f_2) \\ 0 \\ d_1 \\ d_2 \\ d_3 \\ f_0 \\ \\ E_{10} \\ E_{20} \\ E_{30} \\ B_{10} \\ B_{20} \\ B_{30} \\ 0 \\ \frac{1}{2}(E_{21} - E_{12}) \\ \frac{1}{6}(-E_{13} - 2E_{22} + 5E_{31}) \\ B_{11} \\ B_{21} \\ B_{31} \\ \frac{1}{2}(E_{12} - E_{21}) \\ \frac{1}{3}(-E_{13} + E_{22} - E_{31}) \\ \frac{1}{2}(E_{32} - E_{23}) \\ B_{12} \\ B_{22} \\ B_{32} \\ \frac{1}{6}(5E_{13} - 2E_{22} - E_{31}) \\ \frac{1}{2}(E_{23} - E_{32}) \\ 0 \\ B_{13} \\ B_{23} \\ B_{33} \end{array} \right| = \left| \begin{array}{c} s_1 \\ \\ s_2 \\ s_3 \\ s_4 \\ s_5 \\ \\ s_6 = 0 \\ s_7 = 0 \\ s_8 = 0 \\ 0 \\ 0 \\ 0 \\ s_9 \\ s_{10} \\ s_{11} \\ s_{12} \\ \\ s_{13} \\ s_{14} \\ s_{15} \\ s_{16} \\ s_{17} \\ s_{18} \\ s_{19} = 0 \\ s_{20} \\ s_{21} \\ s_{22} \\ s_{23} \\ s_{24} \\ s_{25} = -s_{20} \\ s_{26} \\ s_{27} \\ s_{28} \\ s_{29} \\ s_{30} \\ s_{31} \\ s_{32} = -s_{27} \\ s_{33} = 0 \\ s_{34} \\ s_{35} \\ s_{36} \end{array} \right| .$$

From two groups referring to $P_1$ and $P_2$, we derive the following relations

$$\begin{aligned} f_1 &= 2L_1 - E_{11}, & f_1 &= 2S_1 + E_{11}, \\ 2c_2 + 2f_2 &= 6L_2 - (E_{13} + E_{31} + E_{22}), & 2c_2 + 2f_2 &= 6S_2 + (E_{13} + E_{31} + E_{22}), \\ f_3 &= 2L_3 - E_{33}, & f_3 &= 2S_3 + E_{33}, \\ 2c_1 &= 4L_4 - (E_{23} + E_{32}), & 2c_1 &= 4S_4 + (E_{23} + E_{32}), \\ 2c_3 &= 4L_6 - (E_{12} + E_{21}), & 2c_3 &= 4S_6 + (E_{12} + E_{21}). \end{aligned} \tag{13}$$

From the group $P_3$, if follows

$$h = s_1, \qquad h_0 = s_2, \quad h_1 = s_3, \quad h_2 = s_4, \quad h_3 = s_5,$$

$$f_2 = 2c_2, \quad d_1 = s_9, \quad d_2 = s_{10}, \quad d_3 = s_{11}, \quad f_0 = s_{12}.$$

Whence after simple combining, we arrive at expressions for eleven variables:

$$\begin{gathered} h = s_1, \quad f_1 = L_1 + S_1, \quad f_3 = L_3 + S_3, \quad f_2 = (L_2 + S_2), \\ c_1 = L_4 + S_4, \quad c_3 = L_6 + S_6, \quad c_2 = \frac{1}{2}(L_2 + S_2), \\ d_1 = s_9, \quad d_2 = s_{10}, \quad d_3 = s_{11}, \quad f_0 = s_{12}. \end{gathered} \tag{14}$$

Let us take into account the parity restrictions:

$\underline{P = (-1)^{j+1},}$

$$h = 0, \quad f_3 = -f_1, c_3 = -c_1, d_3 = -d_1, \quad f_2 = 0, c_2 = 0, d_2 = 0, \quad f_0 = 0,$$

so that
$\underline{P = (-1)^{j+1},}$

$$\begin{aligned} f_1 = -f_3 = L_1 + S_1, \quad c_1 = -c_3 = L_4 + S_4, \quad d_1 = -d_3 = s_9, \\ h = 0, \quad f_2 = 0, \quad c_2 = 0, \quad d_2 = 0, \quad f_0 = 0. \end{aligned} \tag{15}$$

## 5. The Non-relativistic Approximation, the Parity $P = (-1)^{j+1}$

Consider the states with the parity $P = (-1)^{j+1}$:

$$f_1 = (L_1 + S_1), \quad c_1 = (L_4 + S_4), \quad d_1 = s_9; \tag{16}$$

the above three equations take on the form

$$\frac{\sqrt{2}b}{mr}\frac{1}{m}D_r(L_4+S_4) + \frac{\sqrt{2}b}{m^2r^2}(L_4+S_4) - \frac{\sqrt{2}b}{mr}\frac{1}{m}iD_0 s_9 - \frac{1}{m^2}D_0^2(L_1+S_1)$$
$$-\frac{2}{mr}\frac{1}{m}D_r(L_1+S_1) - \frac{1}{m^2}D_r^2(L_1+S_1) - (L_1+S_1) = 0,$$
$$\frac{3a^2+b^2-6}{m^2r^2}(L_4+S_4) - 4(L_4+S_4) - \frac{4}{m^2}D_0^2(L_4+S_4) + \frac{4i}{mr}\frac{1}{m}D_0 s_9$$
$$-\frac{2i}{m^2}D_0D_r s_9 - \frac{2i}{m^2}D_rD_0 s_9 - \frac{2\sqrt{2}b}{rm}\frac{1}{m}D_r(L_1+S_1) = 0,$$
$$\frac{3a^2+b^2+2}{m^2r^2}s_9 - 4s_9 - \frac{8}{mr}\frac{1}{m}D_r s_9 - \frac{4}{m^2}D_r^2 s_9 + \frac{12i}{mr}\frac{1}{m}D_0(L_4+S_4)$$
$$+\frac{2i}{m^2}D_0D_r(L_4+S_4) + \frac{2i}{m^2}D_rD_0(L_4+S_4) - \frac{i2\sqrt{2}b}{rm}\frac{1}{m}D_0(L_1+S_1) = 0.$$

When performing the non-relativistic approximation, we should separate the rest energy by formal change $D_0 \Longrightarrow M + iD_0, M > 0$; besides, we should assume the following smallness orders for the wave function components and derivative operators

$$\begin{aligned} L \sim 1, \quad S_k \sim x, \quad s_k \sim x, \quad \frac{1}{m}D_r \sim x, \quad \frac{1}{mr} \sim x, \\ \frac{1}{m}iD_0 \Longrightarrow \frac{1}{m}(m+iD_0) = \frac{1}{m}\Big[m + (E+\frac{\alpha}{r})\Big] \sim (1+x^2), \\ \frac{1}{m^2}D_0^2 = \Big(-1 - \frac{2}{m}(E+\frac{\alpha}{r}) - \frac{1}{m^2}(E+\frac{\alpha}{r})^2\Big) \sim \big(-1-x^2-x^4\big). \end{aligned} \tag{17}$$

We will follow only the terms of orders $x^0, x, x^2$ and neglect the high order terms (note that in the order $x^0$ we should obtain identities). As the result, we derive (preserving only the orders $x$ and $x^2$) $x$

$$-\frac{2\sqrt{2}bL_1}{mr} + \frac{4D_rL_4}{m} + \frac{12L_4}{mr} - 4s_9 = 0,$$

$x^2$

$$\frac{\sqrt{2}bD_rL_4}{m^2r} + \frac{\sqrt{2}bL_4}{m^2r^2} - \frac{\sqrt{2}bs_9}{mr} + \frac{D_0^2L_1}{m^2} - \frac{D_r^2L_1}{m^2} - \frac{2D_rL_1}{m^2r} = 0,$$
$$\frac{3a^2L_4}{m^2r^2} + \frac{b^2L_4}{m^2r^2} - \frac{2\sqrt{2}bD_rL_1}{m^2r} + \frac{4\mathrm{D0}^2L_4}{m^2} - \frac{4D_rs_9}{m} - \frac{6L_4}{m^2r^2} + \frac{4s_9}{mr} = 0$$
$$-\frac{2\sqrt{2}bS_1}{mr} + \frac{4D_rS_4}{m} + \frac{12S_4}{mr} = 0$$

or $x$

$$-\frac{2\sqrt{2}bL_1}{mr} + \frac{4D_rL_4}{m} + \frac{12L_4}{mr} - 4s_9 = 0,$$

$x^2$

$$\frac{\sqrt{2}bD_rL_4}{m^2r} + \frac{\sqrt{2}bL_4}{m^2r^2} - \frac{\sqrt{2}bs_9}{mr} + \frac{2}{m}(E+\alpha/r)L_1 - \frac{D_r^2L_1}{m^2} - \frac{2D_rL_1}{m^2r} = 0,$$

$$\frac{3a^2L_4}{m^2r^2}+\frac{b^2L_4}{m^2r^2}-\frac{2\sqrt{2}bD_rL_1}{m^2r}+4\frac{2}{m}(E+\alpha/r)L_4-\frac{4D_rs_9}{m}-\frac{6L_4}{m^2r^2}+\frac{4s_9}{mr}=0,$$

$$-\frac{2\sqrt{2}bS_1}{mr}+\frac{4D_rS_4}{m}+\frac{12S_4}{mr}=0\quad\text{(we do not need this constraint on small variables).}$$

Expressing $s_9$ from the the first equations, and substituting result into the two remaining, we get

$$\frac{b^2L_1}{m^2r^2}-\frac{2\sqrt{2}bL_4}{m^2r^2}+\frac{2}{m}(E+\alpha/r)L_1-\frac{D_r^2L_1}{m^2}-\frac{2D_rL_1}{m^2r}=0,$$

$$\frac{3a^2L_4}{m^2r^2}+\frac{b^2L_4}{m^2r^2}-\frac{4\sqrt{2}bL_1}{m^2r^2}+4\frac{2}{m}(E+\alpha/r)L_4-\frac{4D_r^2L_4}{m^2}-\frac{8D_rL_4}{m^2r}+\frac{18L_4}{m^2r^2}=0.$$

Let us multiply these equation by $\frac{m}{2}$:

$$(E+\alpha/r)L_1+\frac{1}{2m}\Big[-D_r^2L_1-\frac{2D_rL_1}{r}+\frac{b^2L_1}{r^2}-\frac{2\sqrt{2}bL_4}{r^2}\Big]=0,$$

$$4(E+\alpha/r)L_4+\frac{1}{2m}\Big[-4D_r^2L_4-\frac{8D_rL_4}{r}+\frac{3a^2L_4}{r^2}+\frac{b^2L_4}{r^2}-\frac{4\sqrt{2}bL_1}{r^2}+\frac{18L_4}{r^2}\Big]=0.$$

With the use of a new variable, $4L_4=\bar{L}_4$, we get

$$(E+\alpha/r)L_1+\frac{1}{2m}\Big[-D_r^2L_1-\frac{2D_rL_1}{r}+\frac{b^2L_1}{r^2}-\frac{b\bar{L}_4}{\sqrt{2}r^2}\Big]=0,$$

$$(E+\alpha/r)\bar{L}_4+\frac{1}{2m}\Big[-D_r^2\bar{L}_4-\frac{2D_r\bar{L}_4}{r}+\frac{3a^2\bar{L}_4}{4r^2}+\frac{b^2\bar{L}_4}{4r^2}-\frac{4\sqrt{2}bL_1}{r^2}+\frac{18\bar{L}_4}{4r^2}\Big]=0.$$

This system may be rewritten differently

$$(E+\alpha/r)L_1+\frac{1}{2m}\Big[-(D_r^2L_1+\frac{2D_rL_1}{r})+\frac{1}{r^2}\Big(b^2L_1-\frac{b}{\sqrt{2}}\bar{L}_4\Big)\Big]=0,$$

$$(E+\alpha/r)\bar{L}_4+\frac{1}{2m}\Big[-(D_r^2\bar{L}_4+\frac{2D_r\bar{L}_4}{r})+\frac{1}{r^2}\Big(-4\sqrt{2}bL_1+(\frac{3a^2}{4}+\frac{b^2}{4}+\frac{18}{4})\bar{L}_4\Big]=0.$$

With the shortening notation

$$\Delta=r^2\Big[\frac{d^2}{dr^2}+2\frac{d}{dr}-2M(E+\alpha/r)\Big],$$

it reads

$$\Delta L_1=b^2L_1-\frac{b}{\sqrt{2}}\bar{L}_4,\quad \Delta L_2=-4\sqrt{2}bL_1+\frac{3a^2+b^2+18}{4}\bar{L}_4=0. \tag{18}$$

Taking in mind the identities

$$a=\sqrt{j(j+1)},\quad b=\sqrt{(j-1)(j+2)},\quad \frac{3a^2+b^2+18}{4}=j^2+j+4=c,$$

we get the matrix presentation of the system

$$\Delta\begin{vmatrix}L_1\\ \bar{L}_4\end{vmatrix}=\begin{vmatrix}b^2 & -\frac{b}{\sqrt{2}}\\ -4\sqrt{2}b & c\end{vmatrix}\begin{vmatrix}L_1\\ \bar{L}_4\end{vmatrix},\quad \Delta F=AF$$

Let us find the linear transformation reducing the system to a diagonal form

$$F'=SF,\quad S\Delta S^{-1}F'=SAS^{-1}F',\quad \Delta F'=(SAS^{-1})F',\quad (SAS^{-1})=A'=\begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{vmatrix}$$

this leads to

$$\begin{vmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{vmatrix}\begin{vmatrix}b^2 & -\frac{b}{\sqrt{2}}\\ -4\sqrt{2}b & c\end{vmatrix}=\begin{vmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{vmatrix}\begin{vmatrix}s_{11} & s_{12}\\ s_{21} & s_{22}\end{vmatrix}$$

Whence it follows

$$b^2s_{11}-4\sqrt{2}b\,s_{12}=\lambda_1\,s_{11},\quad -\frac{b}{\sqrt{2}}\,s_{11}+c\,s_{12}=\lambda_1\,s_{12},\quad \begin{vmatrix}(b^2-\lambda)) & -4\sqrt{2}b\\ -b/\sqrt{2} & (c-\lambda)\end{vmatrix}\begin{vmatrix}s_{11}\\ s_{12}\end{vmatrix}=0,$$

$$b^2\, s_{21} - 4\sqrt{2}b\, s_{22} = \lambda_2\, s_{21}, \quad -\frac{b}{\sqrt{2}}\, s_{21} + c\, s_{22} = \lambda_2\, s_{22}, \quad \left| \begin{array}{cc} (b^2-\lambda)) & -4\sqrt{2}b \\ -b/\sqrt{2} & (c-\lambda) \end{array} \right| \left| \begin{array}{c} s_{21} \\ s_{22} \end{array} \right| = 0.$$

From vanishing the determinant

$$\lambda^2 - \lambda(c+b^2) + b^2(c-4) = 0,$$

or (let $j(j+1) = x$)

$$\lambda^2 - 2\lambda(x+1) + x^2 - 2x = 0 \quad \Longrightarrow \quad [\lambda - (x+1)]^2 = 4x+1 = 4(j+1/2)^2;$$

so we arrive at

$$\lambda_1 = x + 1 + \sqrt{4(j+1/2)^2} = (j+1)(j+2), \quad (b^2 - \lambda_1)\, s_{11} - 4\sqrt{2}\, b\, s_{12} = 0, \quad s_{11} = 1,$$

$$\lambda_2 = x + 1 - \sqrt{4(j+1/2)^2} = (j-1)j, \lambda \quad (b^2 - \lambda_2)\, s_{21} - 4\sqrt{2}\, b\, s_{22} = 0, \quad s_{22} = 1,$$

In this way, we obtain two independent equations

$$\begin{gathered} (\Delta - \lambda_1)F_1 = 0, \quad \Big[\frac{d^2}{dr^2} + 2\frac{d}{dr} - 2M(E + \frac{\alpha}{r}) - \frac{(j+1)(j+2)}{r^2}\Big]F_1 = 0 \\ \Delta F_2 = \lambda_2 F_2, \quad \Big[\frac{d^2}{dr^2} + 2\frac{d}{dr} - 2M(E + \frac{\alpha}{r})\Big) - \frac{(j-1)j}{r^2}\Big]F_2 = 0; \end{gathered} \tag{19}$$

their solutions and energy spectra are well-know [47].

## 6. Nonrelartivistic Approximatoion, Parity $P = (-1)^j$

Let $\underline{P = (-1)^j}$:

$$\begin{gathered} h = s_1, \quad f_1 = (L_1 + S_1), \quad c_1 = (L_4 + S_4), \qquad d_1 = s_9, \\ f_2 = (L_2 + S_2), \quad c_2 = \frac{1}{2}(L_2 + S_2), \quad d_2 = s_{10}, \quad f_0 = s_{12}; \end{gathered} \tag{20}$$

the above 8 equation take the form

0

$$\begin{gathered} \frac{2a^2}{3m^2r^2}\frac{1}{2}(L_2+S_2) - \frac{a^2}{m^2r^2}s_1 + \frac{i4\sqrt{2}a}{3rm}\frac{1}{m}D_0 s_9 + \frac{2ab}{3m^2r^2}(L_1+S_1) \\ + \frac{4\sqrt{2}a}{3mr}\frac{1}{m}D_r(L_4+S_4) + \frac{4\sqrt{2}a}{3m^2r^2}(L_4+S_4) + \frac{4\sqrt{2}a}{3m^2r^2}(L_4+S_4) - \frac{2}{3mr}\frac{1}{m}D_0^2 f_0 + \frac{1}{m^2}D_0^2 s_1 \\ + \frac{8i}{3mr}\frac{1}{m}D_0 s_{10} + \frac{2i}{3m^2}D_0 D_r s_{10} + \frac{2i}{3m^2}D_r D_0 s_{10} \\ + \frac{4}{3mr}\frac{1}{m}D_r c_2 + \frac{4}{3m^2r^2}\frac{1}{2}(L_2+S_2) + \frac{8}{3mr}\frac{1}{m}D_r(L_2+S_2) + \frac{2}{3m^2}D_r^2(L_2+S_2) \\ + \frac{4}{3m^2r^2}(L_2+S_2) + \frac{1}{m^2}D_r^2 s_1 + \frac{2}{mr}\frac{1}{m}D_r s_1 + 2s_1 = 0, \end{gathered}$$

1-3

$$\begin{gathered} \frac{ab}{2m^2r^2}s_1 - \frac{\sqrt{2}b}{m^2r^2}(L_4+S_4) - \frac{ab}{m^2r^2}\frac{1}{2}(L_2+S_2) - \frac{2ib}{\sqrt{2}mr}\frac{1}{m}D_0 s_9 - (L_1+S_1) \\ - \frac{1}{m^2}D_0^2(L_1+S_1) - \frac{\sqrt{2}b}{mr}\frac{1}{m}D_r(L_4+S_4) - \frac{2}{mr}\frac{1}{m}D_r(L_1+S_1) - \frac{1}{m^2}D_r^2(L_1+S_1) = 0, \end{gathered}$$

2

$$\begin{gathered} \frac{2a^2}{m^2r^2}s_1 + \frac{2a^2}{m^2r^2}f_0 + \frac{6a^2}{m^2r^2}(L_2+S_2) - \frac{8i\sqrt{2}a}{mr}\frac{1}{m}D_0 s_9 - \frac{4ba}{m^2r^2}(L_1+S_1) \\ + \frac{8\sqrt{2}a}{mr}\frac{1}{m}D_r(L_4+S_4) - \frac{2}{m^2}D_0^2 s_1 - \frac{4}{m^2}D_0^2\frac{1}{2}(L_2+S_2) - \frac{8}{m^2r^2}\frac{1}{2}(L_2+S_2) - \frac{16i}{mr}\frac{1}{m}D_0 s_{10} \\ + \frac{2}{m^2}D_0^2 f_0 - 8(L_2+S_2) - \frac{6}{m^2}D_0^2(L_2+S_2) - \frac{8}{m^2r^2}(L_2+S_2) \\ - \frac{4}{mr}\frac{1}{m}D_r s_1 + \frac{16}{mr}\frac{1}{m}D_r\frac{1}{2}(L_2+S_2) + \frac{8i}{m^2}D_0 D_r s_{10} \\ - \frac{4}{mr}\frac{1}{m}D_r f_0 + \frac{4}{mr}\frac{1}{m}D_r(L_2+S_2) + \frac{6}{m^2}D_r^2 s_1 - \frac{4}{m^2}D_r^2\frac{1}{2}(L_2+S_2) - \frac{2}{m^2}D_r^2 f_0 + \frac{6}{m^2}D_r^2(L_2+S_2) = 0, \end{gathered}$$

4-6

$$\frac{b^2-6-a^2}{m^2r^2}(L_4+S_4)+\frac{2\sqrt{2}a}{m^2r^2}s_1-\frac{4\sqrt{2}a}{m^2r^2}\frac{1}{2}(L_2+S_2)-\frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0s_{10}$$
$$-\frac{2\sqrt{2}a}{mr}\frac{1}{m}D_rs_1+\frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r\frac{1}{2}(L_2+S_2)-\frac{2\sqrt{2}a}{mr}\frac{1}{m}D_r(L_2+S_2)$$
$$-4(L_4+S_4)-\frac{4}{m^2}D_0^2(L_4+S_4)-\frac{4i}{mr}\frac{1}{m}D_0s_9+\frac{2i}{m^2}D_0D_rs_9+\frac{2i}{m^2}D_rD_0s_9+\frac{2\sqrt{2}b}{mr}\frac{1}{m}D_r(L_1+S_1)=0,$$

5

$$\frac{2a^2}{m^2r^2}s_1-\frac{2a^2}{m^2r^2}f_0+\frac{2a^2}{m^2r^2}(L_2+S_2)-\frac{8\sqrt{2}a}{m^2r^2}(L_4+S_4)-\frac{4ba}{m^2r^2}(L_1+S_1)$$
$$+\frac{2}{m^2}D_0^2s_1-8\frac{1}{2}(L_2+S_2)-\frac{4}{m^2}D_0^2\frac{1}{2}(L_2+S_2)-\frac{8}{m^2r^2}\frac{1}{2}(L_2+S_2)-\frac{2}{m^2}D_0^2f_0$$
$$-\frac{2}{m^2}D_0^2(L_2+S_2)-\frac{8}{m^2r^2}(L_2+S_2)-\frac{4}{mr}\frac{1}{m}D_rs_1+\frac{4i}{m^2}D_0D_rs_{10}$$
$$+\frac{4i}{m^2}D_rD_0s_{10}+\frac{4}{mr}\frac{1}{m}D_rf_0-\frac{4}{mr}\frac{1}{m}D_r(L_2+S_2)$$
$$+\frac{2}{m^2}D_r^2s_1-\frac{4}{m^2}D_r^2\frac{1}{2}(L_2+S_2)+\frac{2}{m^2}D_r^2f_0+\frac{2}{m^2}D_r^2(L_2=S_2)=0,$$

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$$\frac{b^2+2-a^2}{m^2r^2}s_9+\frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0s_1-\frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0\frac{1}{2}(L_2+S_2)-\frac{2i\sqrt{2}a}{mr}\frac{1}{m}D_0f_0-\frac{2\sqrt{2}a}{mr}\frac{1}{m}D_rs_{10}$$
$$-\frac{12i}{mr}\frac{1}{m}D_0(L_4+S_4)-4s_9-\frac{2i\sqrt{2}b}{mr}\frac{1}{m}D_0(L_1+S_1)$$
$$-\frac{2i}{m^2}D_0D_r(L_4+S_4)-\frac{2i}{m^2}D_rD_0(L_4+S_4)-\frac{8}{mr}\frac{1}{m}D_rd_1-\frac{4}{m^2}D_r^2s_9=0,$$

8

$$\frac{4a^2}{m^2r^2}s_{10}-\frac{4i\sqrt{2}a}{mr}\frac{1}{m}D_0(L_4+S_4)+\frac{4\sqrt{2}a}{m^2r^2}s_9+\frac{4\sqrt{2}a}{mr}\frac{1}{m}D_rs_9$$
$$-\frac{8i}{mr}\frac{1}{m}D_0\frac{1}{2}(L_2+S_2)-4s_{10}-\frac{8i}{mr}\frac{1}{m}D_0(L_2+S_2)-\frac{2i}{m^2}D_0D_rs_1-\frac{2i}{m^2}D_rR_0s_1$$
$$+\frac{2i}{m^2}D_0D_rf_0+\frac{2i}{m^2}D_rD_0f_0-\frac{2i}{m^2}D_0D_r(L_2+S_2)-\frac{2i}{m^2}D_rD_0(L_2+S_2)=0,$$

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$$-\frac{2a^2}{m^2r^2}s_1+\frac{6a^2}{m^2r^2}f_0+\frac{2a^2}{m^2r^2}(L_2+S_2)+\frac{16\sqrt{2}a}{m^2r^2}(L_4+S_4)-\frac{8i\sqrt{2}a}{mr}\frac{1}{m}D_0s_9+\frac{4ba}{m^2r^2}(L_1+S_1)$$
$$+\frac{8\sqrt{2}a}{mr}\frac{1}{m}D_r(L_4+S_4)-\frac{6}{m^2}D_0^2s_1+\frac{4}{m^2}D_0^2\frac{1}{2}(L_2+S_2)+\frac{8}{m^2r^2}\frac{1}{2}(L_2+S_2)$$
$$-\frac{8i}{mr}\frac{1}{m}D_0s_{10}-\frac{8i}{mr}\frac{1}{m}D_0s_{10}-8f_0+\frac{6}{m^2}D_0^2f_0$$
$$-\frac{2}{m^2}D_0^2(L_2+S_2)+\frac{8}{m^2r^2}(L_2+S_2)+\frac{4}{mr}\frac{1}{m}D_rs_1+\frac{16}{mr}\frac{1}{m}D_r\frac{1}{2}(L_2+S_2)$$
$$-\frac{4i}{m^2}D_0D_rs_{10}-\frac{4i}{m^2}D_rD_0s_{10}-\frac{12}{mr}\frac{1}{m}D_rf_0+\frac{12}{mr}\frac{1}{m}D_r(L_2+S_2)$$
$$+\frac{2}{m^2}D_r^2s_1+\frac{4}{m^2}D_r^2\frac{1}{2}(L_2+S_2)-\frac{6}{m^2}D_r^2f_0+\frac{2}{m^2}D_r^2(L_2+S_2)=0.$$

No we are to take into consideration the smallness orders in accordance with the known rules:

$$L\sim 1,\quad S_k\sim x,\quad s_k\sim x,\quad \frac{1}{m}D_r\sim x,\quad \frac{1}{mr}\sim x,$$
$$\frac{1}{m}iD_0\Longrightarrow\frac{1}{m}(m+iD_0)=\frac{1}{m}\Big[m+(E+\frac{\alpha}{r})\Big]\sim(1+x^2),$$
$$\frac{1}{m^2}D_0^2=-1-\frac{2}{m}(E+\frac{\alpha}{r})-\frac{1}{m^2}(E+\frac{\alpha}{r})^2\sim -1-x^2-x^4\,;$$

as the result we get 1

$$x^3\Big(-\frac{a^2s_1}{m^2r^2}+\frac{a^2S_2}{3m^2r^2}+\frac{2abS_1}{3m^2r^2}+\frac{4\sqrt{2}aD_rS_4}{3m^2r}+\frac{8\sqrt{2}aS_4}{3m^2r^2}-\frac{D_0^2s_1}{m^2}$$
$$+\frac{D_r^2s_1}{m^2}+\frac{2D_r^2S_2}{3m^2}+\frac{2D_rs_1}{m^2r}+\frac{10D_rS_2}{3m^2r}+\frac{2S_2}{m^2r^2}\Big)$$
$$+x^2\Big(\frac{a^2L_2}{3m^2r^2}+\frac{2abL_1}{3m^2r^2}+\frac{4\sqrt{2}aD_rL_4}{3m^2r}+\frac{8\sqrt{2}aL_4}{3m^2r^2}+\frac{4\sqrt{2}as_9}{3mr}$$

$$+\frac{2D_r^2L_2}{3m^2}+\frac{10D_rL_2}{3m^2r}+\frac{4D_rs_{10}}{3m}+\frac{2L_2}{m^2r^2}+\frac{8s_{10}}{3mr}+\frac{2s_{12}}{3mr}\Big)$$
$$+x^4\left(\frac{4i\sqrt{2}aD_0s_9}{3m^2r}+\frac{2D_0^2s_{12}}{3m^2r}+\frac{8iD_0s_{10}}{3m^2r}+\frac{2iD_0D_rs_{10}}{3m^2}+\frac{2iD_rD_0s_{10}}{3m^2}\right)$$
$$+\frac{2D_0^2s_{12}x^6}{3m^2r}-\frac{D_0^2s_1x^5}{m^2}+s_1x=0,$$

2

$$x^2\left(-\frac{abL_2}{2m^2r^2}-\frac{\sqrt{2}bD_rL_4}{m^2r}-\frac{\sqrt{2}bL_4}{m^2r^2}-\frac{\sqrt{2}bs_9}{mr}+\frac{D_0^2L_1}{m^2}-\frac{D_r^2L_1}{m^2}-\frac{2D_rL_1}{m^2r}\right)$$
$$+x^3\left(\frac{abs_1}{2m^2r^2}-\frac{abS_2}{2m^2r^2}-\frac{\sqrt{2}bD_rS_4}{m^2r}-\frac{\sqrt{2}bS_4}{m^2r^2}+\frac{D_0^2S_1}{m^2}-\frac{D_r^2S_1}{m^2}-\frac{2D_rS_1}{m^2r}\right)$$
$$+x^4\left(\frac{D_0^2L_1}{m^2}-\frac{i\sqrt{2}bD_0s_9}{m^2r}\right)+\frac{D_0^2S_1x^5}{m^2}=0,$$

3

$$x^2\Big(\frac{6a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}-\frac{8\sqrt{2}as_9}{mr}+\frac{8D_0^2L_2}{m^2}+\frac{4D_r^2L_2}{m^2}+\frac{12\mathrm{Dr}L_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}\Big)$$
$$+x^3\Big(\frac{2a^2s_1}{m^2r^2}+\frac{2a^2s_{12}}{m^2r^2}+\frac{6a^2S_2}{m^2r^2}-\frac{4abS_1}{m^2r^2}+\frac{8\sqrt{2}aD_rS_4}{m^2r}+\frac{2D_0^2s_1}{m^2}-\frac{2\mathrm{D0}^2s_{12}}{m^2}$$
$$+\frac{8D_0^2S_2}{m^2}+\frac{6D_r^2s_1}{m^2}-\frac{2D_r^2s_{12}}{m^2}+\frac{4D_r^2S_2}{m^2}-\frac{4D_rs_1}{m^2r}-\frac{4D_rs_{12}}{m^2r}+\frac{12D_rS_2}{m^2r}-\frac{12S_2}{m^2r^2}\Big)+$$
$$+x^4\left(-\frac{8i\sqrt{2}aD_0s_9}{m^2r}+\frac{8D_0^2L_2}{m^2}-\frac{16i\mathrm{D0}s_{10}}{m^2r}+\frac{8iD_0D_rs_{10}}{m^2}\right)$$
$$+x^5\left(\frac{2D_0^2s_1}{m^2}-\frac{2D_0^2s_{12}}{m^2}+\frac{8D_0^2S_2}{m^2}\right)+(2s_1-2s_{12})\,x=0,$$

4

$$x^2\left(-\frac{a^2L_4}{m^2r^2}-\frac{\sqrt{2}aD_rL_2}{m^2r}-\frac{2\sqrt{2}aL_2}{m^2r^2}-\frac{2\sqrt{2}as_{10}}{mr}+\frac{b^2L_4}{m^2r^2}+\frac{2\sqrt{2}bD_rL_1}{m^2r}+\frac{4D_0^2L_4}{m^2}+\frac{4D_rs_9}{m}-\frac{6L_4}{m^2r^2}-\frac{4s_9}{mr}\right)$$
$$+x^3\left(-\frac{a^2S_4}{m^2r^2}-\frac{2\sqrt{2}aD_rs_1}{m^2r}-\frac{\sqrt{2}a\mathrm{Dr}S_2}{m^2r}+\frac{2\sqrt{2}as_1}{m^2r^2}-\frac{2\sqrt{2}aS_2}{m^2r^2}+\frac{b^2S_4}{m^2r^2}+\frac{2\sqrt{2}bD_rS_1}{m^2r}+\frac{4D_0^2S_4}{m^2}-\frac{6S_4}{m^2r^2}\right)$$
$$+x^4\left(-\frac{2i\sqrt{2}aD_0s_{10}}{m^2r}+\frac{4D_0^2L_4}{m^2}-\frac{4i\mathrm{D0}s_9}{m^2r}+\frac{2iD_0D_rs_9}{m^2}+\frac{2iD_rD_0s_9}{m^2}\right)+\frac{4D_0^2S_4x^5}{m^2}=0,$$

5

$$x^3\Big(\frac{2a^2s_1}{m^2r^2}-\frac{2a^2s_{12}}{m^2r^2}+\frac{2a^2S_2}{m^2r^2}-\frac{4abS_1}{m^2r^2}-\frac{8\sqrt{2}aS_4}{m^2r^2}-\frac{2D_0^2s_1}{m^2}+\frac{2D_0^2s_{12}}{m^2}+\frac{4D_0^2S_2}{m^2}$$
$$+\frac{2D_r^2s_1}{m^2}+\frac{2D_r^2s_{12}}{m^2}-\frac{4D_rs_1}{m^2r}+\frac{4D_rs_{12}}{m^2r}-\frac{4D_rS_2}{m^2r}-\frac{12S_2}{m^2r^2}\Big)$$
$$+x^2\left(\frac{2a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}-\frac{8\sqrt{2}aL_4}{m^2r^2}+\frac{4D_0^2L_2}{m^2}-\frac{4D_rL_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}\right)$$
$$+x^4\left(\frac{4D_0^2L_2}{m^2}+\frac{4iD_0D_rs_{10}}{m^2}+\frac{4iD_rD_0s_{10}}{m^2}\right)+x^5\left(-\frac{2D_0^2s_1}{m^2}+\frac{2D_0^2s_{12}}{m^2}+\frac{4D_0^2S_2}{m^2}\right)(2s_{12}-2s_1)\,x=0,$$

6

$$x^3\Big(-\frac{a^2s_9}{m^2r^2}-\frac{i\sqrt{2}aD_0L_2}{m^2r}-\frac{2\sqrt{2}aD_rs_{10}}{m^2r}+\frac{b^2s_9}{m^2r^2}-\frac{2i\sqrt{2}bD_0L_1}{m^2r}-\frac{2i\mathrm{D0Dr}L_4}{m^2}$$
$$-\frac{12iD_0L_4}{m^2r}-\frac{2iD_0D_rL_4}{m^2}-\frac{4D_r^2s_9}{m^2}-\frac{8D_rs_9}{m^2r}+\frac{2s_9}{m^2r^2}\Big)$$
$$+x^4\left(\frac{2i\sqrt{2}aD_0s_1}{m^2r}-\frac{2i\sqrt{2}aD_0s_{12}}{m^2r}-\frac{i\sqrt{2}aD_0S_2}{m^2r}-\frac{2i\sqrt{2}bD_0S_1}{m^2r}-\frac{2i\mathrm{D0}D_rS_4}{m^2}-\frac{12iD_0S_4}{m^2r}-\frac{2iD_0D_rS_4}{m^2}\right)$$
$$+x\left(-\frac{\sqrt{2}aL_2}{mr}-\frac{2\sqrt{2}bL_1}{mr}-\frac{4D_rL_4}{m}-\frac{12L_4}{mr}-4s_9\right)$$
$$+x^2\left(\frac{2\sqrt{2}as_1}{mr}-\frac{2\sqrt{2}as_{12}}{mr}-\frac{\sqrt{2}aS_2}{mr}-\frac{2\sqrt{2}bS_1}{mr}-\frac{4D_rS_4}{m}-\frac{12S_4}{mr}\right)=0,$$

7

$$x^3\left(\frac{4a^2s_{10}}{m^2r^2}-\frac{4i\sqrt{2}aD_0L_4}{m^2r}+\frac{4\sqrt{2}a\mathrm{Dr}s_9}{m^2r}+\frac{4\sqrt{2}as_9}{m^2r^2}-\frac{12iD_0L_2}{m^2r}-\frac{2i\mathrm{D0Dr}L_2}{m^2}-\frac{2iD_rD_0L_2}{m^2}\right)$$
$$+x^4\Big(-\frac{4i\sqrt{2}aD_0S_4}{m^2r}-\frac{12iD_0S_2}{m^2r}-\frac{2iD_0D_rs_1}{m^2}+\frac{2iD_0D_rs_{12}}{m^2}$$

$$-\frac{2iD_0D_rS_2}{m^2}-\frac{2iD_rD_0s_1}{m^2}+\frac{2iD_rD_0s_{12}}{m^2}-\frac{2iD_rD_0S_2}{m^2}\Big)$$
$$+x\left(-\frac{4\sqrt{2}aL_4}{mr}-\frac{4D_rL_2}{m}-\frac{12L_2}{mr}-4s_{10}\right)$$
$$+x^2\left(-\frac{4\sqrt{2}aS_4}{mr}-\frac{4D_rs_1}{m}+\frac{4D_rs_{12}}{m}-\frac{4D_rS_2}{m}-\frac{12S_2}{mr}\right)=0,$$

8

$$x^3\Big(-\frac{2a^2s_1}{m^2r^2}+\frac{6a^2s_{12}}{m^2r^2}+\frac{2a^2S_2}{m^2r^2}+\frac{4abS_1}{m^2r^2}+\frac{8\sqrt{2}aD_rS_4}{m^2r}+\frac{16\sqrt{2}aS_4}{m^2r^2}+\frac{6\mathrm{D0}^2s_1}{m^2}-\frac{6D_0^2s_{12}}{m^2}$$
$$+\frac{2D_r^2s_1}{m^2}-\frac{6D_r^2s_{12}}{m^2}+\frac{4D_r^2S_2}{m^2}+\frac{4D_rs_1}{m^2r}-\frac{12D_rs_{12}}{m^2r}+\frac{20D_rS_2}{m^2r}+\frac{12S_2}{m^2r^2}\Big)$$
$$+x^2\left(\frac{2a^2L_2}{m^2r^2}+\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}+\frac{16\sqrt{2}aL_4}{m^2r^2}-\frac{8\sqrt{2}as_9}{mr}+\frac{4D_r^2L_2}{m^2}+\frac{20D_rL_2}{m^2r}-\frac{8D_rs_{10}}{m}+\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}\right)$$
$$+x^4\left(-\frac{8i\sqrt{2}aD_0s_9}{m^2r}-\frac{16iD_0s_{10}}{m^2r}-\frac{4iD_0D_rs_{10}}{m^2}-\frac{4iD_rD_0s_{10}}{m^2}\right)$$
$$+x^5\left(\frac{6D_0^2s_1}{m^2}-\frac{6D_0^2s_{12}}{m^2}\right)+(6s_1-14s_{12})\,x=0.$$

Further we preserve only terms of orders $x$ and $x^2$:

$x$

$$s_1=0,\quad 2s_1-2s_{12}=0,\quad 2s_{12}-2s_1=0,$$
$$-\frac{\sqrt{2}aL_2}{mr}-\frac{2\sqrt{2}bL_1}{mr}-\frac{4D_rL_4}{m}-\frac{12L_4}{mr}-4s_9=0,$$
$$-\frac{4\sqrt{2}aL_4}{mr}-\frac{4D_rL_2}{m}-\frac{12L_2}{mr}-4s_{10}=0,\quad 6s_1-14s_{12}=0.$$

$x^2$

$$\frac{a^2L_2}{3m^2r^2}+\frac{2abL_1}{3m^2r^2}+\frac{4\sqrt{2}aD_rL_4}{3m^2r}+\frac{8\sqrt{2}aL_4}{3m^2r^2}+\frac{4\sqrt{2}as_9}{3mr}+\frac{2D_r^2L_2}{3m^2}+\frac{10D_rL_2}{3m^2r}+\frac{4D_rs_{10}}{3m}+\frac{2L_2}{m^2r^2}+\frac{8s_{10}}{3mr}+\frac{2s_{12}}{3mr}=0,$$
$$-\frac{abL_2}{2m^2r^2}-\frac{\sqrt{2}bD_rL_4}{m^2r}-\frac{\sqrt{2}bL_4}{m^2r^2}-\frac{\sqrt{2}bs_9}{mr}+\frac{D_0^2L_1}{m^2}-\frac{D_r^2L_1}{m^2}-\frac{2D_rL_1}{m^2r}=0,$$
$$\frac{6a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}-\frac{8\sqrt{2}as_9}{mr}+\frac{8D_0^2L_2}{m^2}+\frac{4D_r^2L_2}{m^2}+\frac{12D_rL_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}=0,$$
$$-\frac{a^2L_4}{m^2r^2}-\frac{\sqrt{2}aD_rL_2}{m^2r}-\frac{2\sqrt{2}aL_2}{m^2r^2}-\frac{2\sqrt{2}as_{10}}{mr}+\frac{b^2L_4}{m^2r^2}+\frac{2\sqrt{2}bD_rL_1}{m^2r}+\frac{4D_0^2L_4}{m^2}+\frac{4D_rs_9}{m}-\frac{6L_4}{m^2r^2}-\frac{4s_9}{mr}=0,$$
$$\frac{2a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}-\frac{8\sqrt{2}aL_4}{m^2r^2}+\frac{4D_0^2L_2}{m^2}-\frac{4D_rL_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}=0,$$
$$\frac{2\sqrt{2}as_1}{mr}-\frac{2\sqrt{2}as_{12}}{mr}-\frac{\sqrt{2}aS_2}{mr}-\frac{2\sqrt{2}bS_1}{mr}-\frac{4D_rS_4}{m}-\frac{12S_4}{mr}=0,$$
$$-\frac{4\sqrt{2}aS_4}{mr}-\frac{4D_rs_1}{m}+\frac{4D_rs_{12}}{m}-\frac{4D_rS_2}{m}-\frac{12S_2}{mr}=0,$$
$$\frac{2a^2L_2}{m^2r^2}+\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}+\frac{16\sqrt{2}aL_4}{m^2r^2}-\frac{8\sqrt{2}as_9}{mr}+\frac{4D_r^2L_2}{m^2}+\frac{20D_rL_2}{m^2r}-\frac{8D_rs_{10}}{m}+\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}=0,$$

or differently

$x$

$$s_1=0,$$
$$2s_1-2s_{12}=0,$$
$$2s_{12}-2s_1=0,$$
$$-\frac{\sqrt{2}aL_2}{mr}-\frac{2\sqrt{2}bL_1}{mr}-\frac{4D_rL_4}{m}-\frac{12L_4}{mr}-4s_9=0,$$
$$-\frac{4\sqrt{2}aL_4}{mr}-\frac{4D_rL_2}{m}-\frac{12L_2}{mr}-4s_{10}=0,$$
$$6s_1-14s_{12}=0;$$

$x^2$ (make the change $D_0\Longrightarrow\frac{2}{m}(E+\alpha/r)$)

$$\frac{a^2L_2}{3m^2r^2}+\frac{2abL_1}{3m^2r^2}+\frac{4\sqrt{2}aD_rL_4}{3m^2r}+\frac{8\sqrt{2}aL_4}{3m^2r^2}+\frac{4\sqrt{2}as_9}{3mr}+\frac{2D_r^2L_2}{3m^2}+\frac{10D_rL_2}{3m^2r}+\frac{4D_rs_{10}}{3m}+\frac{2L_2}{m^2r^2}+\frac{8s_{10}}{3mr}+\frac{2s_{12}}{3mr}=0,$$

$$-\frac{abL_2}{2m^2r^2}-\frac{\sqrt{2}bD_rL_4}{m^2r}-\frac{\sqrt{2}bL_4}{m^2r^2}-\frac{\sqrt{2}bs_9}{mr}+\frac{2}{m}(E+\alpha/r)L_1-\frac{D_r^2L_1}{m^2}-\frac{2D_rL_1}{m^2r}=0,$$

$$\frac{6a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}-\frac{8\sqrt{2}as_9}{mr}+8\frac{2}{m}(E+\alpha/r)L_2+\frac{4D_r^2L_2}{m^2}+\frac{12D_rL_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}=0,$$

$$-\frac{a^2L_4}{m^2r^2}-\frac{\sqrt{2}aD_rL_2}{m^2r}-\frac{2\sqrt{2}aL_2}{m^2r^2}-\frac{2\sqrt{2}as_{10}}{mr}+\frac{b^2L_4}{m^2r^2}+\frac{2\sqrt{2}bD_rL_1}{m^2r}+4\frac{2}{m}(E+\alpha/r)L_4+\frac{4D_rs_9}{m}-\frac{6L_4}{m^2r^2}-\frac{4s_9}{mr}=0$$

$$\frac{2a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}-\frac{8\sqrt{2}aL_4}{m^2r^2}+4\frac{2}{m}(E+\alpha/r)L_2-\frac{4D_rL_2}{m^2r}+\frac{8D_rs_{10}}{m}-\frac{12L_2}{m^2r^2}=0,$$

$$\frac{2\sqrt{2}as_1}{mr}-\frac{2\sqrt{2}as_{12}}{mr}-\frac{\sqrt{2}aS_2}{mr}-\frac{2\sqrt{2}bS_1}{mr}-\frac{4D_rS_4}{m}-\frac{12S_4}{mr}=0,$$

$$-\frac{4\sqrt{2}aS_4}{mr}-\frac{4D_rs_1}{m}+\frac{4D_rs_{12}}{m}-\frac{4D_rS_2}{m}-\frac{12S_2}{mr}=0,$$

$$\frac{2a^2L_2}{m^2r^2}+\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}+\frac{16\sqrt{2}aL_4}{m^2r^2}-\frac{8\sqrt{2}as_9}{mr}+\frac{4D_r^2L_2}{m^2}+\frac{20D_rL_2}{m^2r}-\frac{8D_rs_{10}}{m}+\frac{12L_2}{m^2r^2}-\frac{16s_{10}}{mr}=0$$

Expressing the small components from equations of order $x$ through large ones, and substituting result in equations of order $x^2$, we obtain

1

$$-\frac{a^2L_2}{3m^2r^2}-\frac{2abL_1}{3m^2r^2}-\frac{4\sqrt{2}aD_rL_4}{3m^2r}-\frac{4\sqrt{2}aL_4}{m^2r^2}-\frac{2D_r^2L_2}{3m^2}-\frac{10D_rL_2}{3m^2r}-\frac{6L_2}{m^2r^2}=0,$$

2

$$\frac{b^2L_1}{m^2r^2}+\frac{2\sqrt{2}bL_4}{m^2r^2}-\frac{D_r^2L_1}{m^2}-\frac{2D_rL_1}{m^2r}+\frac{2\alpha L_1}{mr}+\frac{2L_1E}{m}=0,$$

3

$$\frac{10a^2L_2}{m^2r^2}+\frac{4abL_1}{m^2r^2}+\frac{8\sqrt{2}aD_rL_4}{m^2r}+\frac{40\sqrt{2}aL_4}{m^2r^2}-\frac{4D_r^2L_2}{m^2}+\frac{4D_rL_2}{m^2r}+\frac{36L_2}{m^2r^2}+\frac{16\alpha L_2}{mr}+\frac{16L_2E}{m}=0,$$

4

$$\frac{3a^2L_4}{m^2r^2}+\frac{5\sqrt{2}aL_2}{m^2r^2}+\frac{b^2L_4}{m^2r^2}+\frac{2\sqrt{2}bL_1}{m^2r^2}-\frac{4D_r^2L_4}{m^2}-\frac{8D_rL_4}{m^2r}+\frac{6L_4}{m^2r^2}+\frac{8\alpha L_4}{mr}+\frac{8L_4E}{m}=0,$$

5

$$\frac{2a^2L_2}{m^2r^2}-\frac{4abL_1}{m^2r^2}-\frac{8\sqrt{2}aD_rL_4}{m^2r}-\frac{8\sqrt{2}aL_4}{m^2r^2}-\frac{8D_r^2L_2}{m^2}-\frac{28D_rL_2}{m^2r}-\frac{12L_2}{m^2r^2}+\frac{8\alpha L_2}{mr}+\frac{8L_2E}{m}=0,$$

$$-\frac{\sqrt{2}aS_2}{mr}-\frac{2\sqrt{2}bS_1}{mr}-\frac{4D_rS_4}{m}-\frac{12S_4}{mr}=0,\quad \text{not needed}$$

$$-\frac{4\sqrt{2}aS_4}{mr}-\frac{4D_rS_2}{m}-\frac{12S_2}{mr}=0,\quad \text{not needed}$$

6

$$\frac{6a^2L_2}{m^2r^2}+\frac{12abL_1}{m^2r^2}+\frac{24\sqrt{2}aD_rL_4}{m^2r}+\frac{56\sqrt{2}aL_4}{m^2r^2}+\frac{12D_r^2L_2}{m^2}+\frac{60D_rL_2}{m^2r}+\frac{60L_2}{m^2r^2}=0.$$

So we arrive at equations for $L_1, L_2, L_4$:

$$1\qquad \frac{2}{3}D_r^2L_2+\frac{10}{3r}D_rL_2+\frac{4\sqrt{2}a}{3r}D_rL_4+\frac{4\sqrt{2}a}{r^2}L_4+\frac{2ab}{3r^2}L_1+\frac{a^2+18}{3r^2}L_2=0,$$

$$6\qquad 12D_r^2L_2+\frac{60}{r}D_rL_2+\frac{6a^2+60}{r^2}L_2+\frac{24\sqrt{2}a}{r}D_rL_4+\frac{56\sqrt{2}a}{r^2}L_4\quad+\frac{12ab}{r^2}L_1=0.$$

$$3\qquad -4D_r^2L_2+\frac{4D_r}{r}L_2+16m(E+\frac{\alpha}{r})L_2+\frac{10a^2+36}{r^2}L_2+\frac{8\sqrt{2}a}{r}D_rL_4+\frac{40\sqrt{2}a}{r^2}L_4+\frac{4abL_1}{r^2}=0,$$

$$5\qquad 8D_r^2L_2+\frac{28}{r}D_rL_2-8m(E+\frac{\alpha}{r})L_2-\frac{2a^2-12}{r^2}L_2+\frac{8\sqrt{2}a}{r}D_rL_4+\frac{8\sqrt{2}a}{r^2}L_4+\frac{4abL_1}{r^2}-=0,$$

$$2\qquad \Big[D_r^2L_1+\frac{2}{r}D_rL_1-2m(E+\frac{\alpha}{r})L_1\Big]-\frac{b^2}{r^2}L_1-\frac{2\sqrt{2}b}{r^2}L_4=0,$$

$$4\qquad D_r^2L_4+\frac{8}{r}D_rL_4-8m(E+\frac{\alpha}{r})L_4-\frac{3a^2+b^2+6}{r^2}L_4-\frac{5\sqrt{2}a}{r^2}L_2-\frac{2\sqrt{2}b}{r^2}L_1=0,$$

let $4L_4=\bar{L}_4$, then we get

$$\bar{4}\qquad \Big[D_r^2\bar{L}_4+\frac{2}{r}D_r\bar{L}_4-2m(E+\frac{\alpha}{r})\bar{L}_4\Big]-\frac{3a^2+b^2+6}{4r^2}\bar{L}_4-\frac{5\sqrt{2}a}{r^2}L_2-\frac{2\sqrt{2}b}{r^2}L_1=0.$$

Equations 2 and $\bar{4}$ have the needed nonrelativistic structure

$$\Big[D_r^2+\frac{2}{r}D_r-2m(E+\frac{\alpha}{r})\Big]L_1=\Delta L_1,\quad \Big[D_r^2+\frac{2}{r}D_r-2m(E+\frac{\alpha}{r})\Big]\bar{L}_4=\Delta\bar{L}_4.$$

From equation 3, let us express, and substitute the result into equation 5 – this leads to

$$\Big[D_r^2 + \frac{2}{r}D_r - 2m(E+\frac{\alpha}{r})\Big]L_2 - \frac{a^2 L_2}{r^2} - \frac{6L_2}{r^2} - \frac{4\sqrt{2}aL_4}{r^2} = 0$$

Let us turn to equation 1 and express from this the variable $\frac{1}{r}D_r L_4$, and then substitute the result into equation 6; in this way we arrive ay the identity $0 \equiv 0$. Therefore, we have derived three equations with the needed structture (remembering that $4L_4 = \bar{L}_4$)

$$\begin{gathered}\Big[D_r^2 + \frac{2}{r}D_r - 2m(E+\frac{\alpha}{r})\Big]L_1 - \frac{b^2}{r^2}L_1 - \frac{\sqrt{2}b}{2r^2}\bar{L}_4 = 0,\\ \Big[D_r^2 + \frac{2}{r}D_r - 2m(E+\frac{\alpha}{r})\Big]L_2 - \frac{a^2+6}{r^2}L_2 - \frac{\sqrt{2}a}{r^2}\bar{L}_4 = 0,\\ \Big[D_r^2 + \frac{2}{r}D_r - 2m(E+\frac{\alpha}{r})\Big]\bar{L}_4 - \frac{3a^2+b^2+6}{4r^2}\bar{L}_4 - \frac{5\sqrt{2}a}{r^2}L_2 - \frac{2\sqrt{2}b}{r^2}L_1 = 0.\end{gathered} \tag{21}$$

In the matrix form it reads

$$\Delta = r^2\Big[D_r^2 + \frac{2}{r}D_r - 2m(E+\frac{\alpha}{r})\Big], \; L = \left|\begin{array}{c} L_1 \\ L_2 \\ \bar{L}_4 \end{array}\right|,$$

$$\Delta L = AL, \; A = \left|\begin{array}{ccc} b^2 & 0 & \frac{\sqrt{2}b}{2} \\ 0 & a^2+6 & \sqrt{2}a \\ 2\sqrt{2}b & 5\sqrt{2}a & \frac{3a^2+b^2+6}{4} \end{array}\right|, \; a = \sqrt{j(j+1)}, \quad b = \sqrt{(j-1)(j+2)},$$

$$F = SL = \left|\begin{array}{c} F_1 \\ F_2 \\ F_3 \end{array}\right|, \; \Delta F = SAS^{-1}F, \; S = \left|\begin{array}{ccc} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{array}\right|, \; SAS^{-1} = A' = \left|\begin{array}{ccc} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{array}\right|.$$

So we get $SA = A'S$, that is

$$\left|\begin{array}{ccc} s_{11}\left(b^2-\lambda_1\right) + 2\sqrt{2}bs_{13} & s_{12}\left(a^2-\lambda_1+6\right) + 5\sqrt{2}as_{13} & \frac{1}{4}s_{13}\left(3a^2+b^2-4\lambda_1+6\right) + \sqrt{2}as_{12} + \frac{bs_{11}}{\sqrt{2}} \\ s_{21}\left(b^2-\lambda_2\right) + 2\sqrt{2}bs_{23} & s_{22}\left(a^2-\lambda_2+6\right) + 5\sqrt{2}as_{23} & \frac{1}{4}s_{23}\left(3a^2+b^2-4\lambda_2+6\right) + \sqrt{2}as_{22} + \frac{bs_{21}}{\sqrt{2}} \\ s_{31}\left(b^2-\lambda_3\right) + 2\sqrt{2}bs_{33} & s_{32}\left(a^2-\lambda_3+6\right) + 5\sqrt{2}as_{33} & \frac{1}{4}s_{33}\left(3a^2+b^2-4\lambda_3+6\right) + \sqrt{2}as_{32} + \frac{bs_{31}}{\sqrt{2}} \end{array}\right| = 0.$$

Here we have the linear system of the same structure ($i = 1, 2, 3$):

$$s_{i1}\left(b^2-\lambda_i\right) + 2\sqrt{2}bs_{i3} = 0, \quad s_{i2}\left(a^2-\lambda_i+6\right) + 5\sqrt{2}as_{i3} = 0,$$

$$\frac{bs_{i1}}{\sqrt{2}} + \sqrt{2}as_{i2} + \frac{1}{4}s_{i3}\left(3a^2+b^2+6-4\lambda_i\right) = 0.$$

From vanishing the determinant of the last system, we derive

$$\lambda^3 - \left(3j^2+3j+5\right)\lambda^2 + \left(3j^4+6j^3+j^2-2j-4\right)\lambda -$$

$$-(j-2)(j-1)(j+2)(j+3)\left(j^2+j+1\right) = 0, \quad \lambda_1\lambda_2\lambda_3 > 0.$$

Let us study the roots numerically, for fixed values of $j = 1, 2, ..., 10$:

| | | | |
|---|---|---|---|
| $j = 1$ | 0 | 0.376525 | 10.6235 |
| $j = 2$ | 0 | 5.15571 | 17.8443 |
| $j = 3$ | 2.54917 | 11.2483 | 27.2025 |
| $j = 4$ | 7.10818 | 19.283 | 38.6088 |
| $j = 5$ | 13.6632 | 29.3 | 52.0368 |
| $j = 6$ | 22.2141 | 41.3096 | 67.4763 |
| $j = 7$ | 32.7618 | 55.3156 | 84.9226 |
| $j = 8$ | 45.3072 | 71.3196 | 104.373 |
| $j = 9$ | 59.8508 | 89.3223 | 125.827 |
| $j = 10$ | 76.3931 | 109.324 | 149.283 |

The first line of the table is meaningless. However, we should recall that the case $j = 0$ is special, because the initial general substitution should be modified. Evidently, the case of $j = 0$ should be examined additionally and as a separate one.

Instead of the parameter $\lambda$, let us introduce more helpful parameter, $\lambda = L(L+1)$. Then the main algebraic equation takes on the form

$$(j-2)(j-1)(j+2)(j+3)\left(j^2+j+1\right)+$$
$$+\left(4-j\left(3(j+2)j^2+j-2\right)\right)L+(9-j(j+1)(3j(j+1)-5))L^2+$$
$$+(6j(j+1)+9)L^3+(3j(j+1)+2)L^4-3L^5-L^6=0,$$

then the numerical study gives

| | | | | | | |
|---|---|---|---|---|---|---|
| $j=0$ | $-3.$ | $-1.24037-0.89i$ | $-1.24037+0.89i$ | $0.240369-0.89i$ | $0.240369+0.89i$ | $2.$ |
| $j=1$ | $-3.7975$ | $-1.29153$ | $-1.$ | $0.$ | $0.291533$ | $2.7975$ |
| $j=2$ | $-4.75374$ | $-2.82502$ | $-1.$ | $0.$ | $1.82502$ | $3.75374$ |
| $j=3$ | $-5.73952$ | $-3.89092$ | $-2.17307$ | $1.17307$ | $2.89092$ | $4.73952$ |
| $j=4$ | $-6.73368$ | $-4.91962$ | $-3.2126$ | $2.2126$ | $3.91962$ | $5.73368$ |
| $j=5$ | $-7.73096$ | $-5.93599$ | $-4.23004$ | $3.23004$ | $4.93599$ | $6.73096$ |
| $j=6$ | $-8.7296$ | $-6.94668$ | $-5.23963$ | $4.23963$ | $5.94668$ | $7.7296$ |
| $j=7$ | $-9.7289$ | $-7.95423$ | $-6.24559$ | $5.24559$ | $6.95423$ | $8.7289$ |
| $j=8$ | $-10.7286$ | $-8.95988$ | $-7.2496$ | $6.2496$ | $7.95988$ | $9.72855$ |
| $j=9$ | $-11.7284$ | $-9.96427$ | $-8.25247$ | $7.25247$ | $8.96427$ | $10.7284$ |
| $j=10$ | $-12.7284$ | $-10.9678$ | $-9.2546$ | $8.2546$ | $9.96778$ | $11.7284$ |

Tree last columns give positive values for $L_1, L_2, L_3$; therefore, after performing the linear transformation, for states with the parity $P = (-1)^j$ we obtain three separate equations

$$\begin{aligned}
&\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-2m(E+\frac{\alpha}{r})-L_1(L_1+1)\Big]F_1=0,\\
&\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-2m(E+\frac{\alpha}{r})-L_2(L_2+1)\Big]F_2=0,\\
&\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-2m(E+\frac{\alpha}{r})-L_3(L_3+1)\Big]F_3=0;
\end{aligned} \tag{22}$$

solutions for these equations are well-known [47].

## 7. States with $j = 0$

For the case $j = 0$, we present only the final result. Solutions with states with the parity $P = (-1)^{j+}$ do not exist. For states with the parity $P = (-1)^j$, the 5 independent components are divided into large ($L$) and small ($S$) parts as follows

$$h=S_1,\quad f_2=L_2+S_2,\quad c_2=\frac{L_2+S_2}{2},\quad d_2=S_{10},\quad f_0=S_{12}. \tag{23}$$

After performing the needed calculation (similar to described in the above), we arrive at a second order equation for the function $L_2(r) = f(r)$:

$$\Big[\frac{d^2}{dr^2}+\frac{4}{r}\frac{d}{dr}+\frac{2mE}{3}+\frac{2m\alpha}{3}\frac{1}{r}\Big]L_2=0,\ E_n=-\frac{m\alpha^2}{6}\frac{1}{(2+n)^2},\quad n=0,1,2,\ldots. \tag{24}$$

## 8. Conclusions

In this study, we have studied the non-relativistic approc=ximation for a spin-2 particle in external Coulomb field. The use of parity constraints led to a natural separation of the system into subsystems of 3 and 8 equations for states with parities $P = (-1)^{j+1}$ and $P = (-1)^j$, respectively. For the subsystem corresponding to $P = (-1)^{j+1}$, we have reduced the problem to two independent Schrodinger-type equations in the presence of a Coulomb field. For the subsystem corresponding to $P = (-1)^j$, have reduced the problem to three independent Schrodinger-type equations in the presence of a Coulomb field. Consequently, we have obtained five distinct energy spectra corresponding to the non-relativistic spin-2 particle in the presence of a Coulomb field. For the special case $j = 0$, we have reduce the problem to a single Schrodinger equation with a hydrogen-like spectrum.

## References


[1] M. Fierz. Über die relativistische theorie Kraftefreier Teilchen mit beliebigem Spin. Helv. Phys. Acta. 1939. **12**. 3-37.

[2] W. Pauli, M. Fierz. Über relativistische Feldleichungen von Teilchen mit beliebigem Spin im elektromagnetishen Feld. Helv. Phys. Acta. 1939. **12**. 297-300.

[3] M. Fierz, W. Pauli. On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. Proc. Roy. Soc. London. 1939. **A173**. 211-232.

[4] L. de Broglie. Sur l'interprétation de certaines équations dans la théorie des particules de spin 2. C. R. Acad. Sci. Paris. 1941. **212**. 657-659.

[5] E. Wild. On first order wave equations for elementary particles without subsidiary conditions. Proc. Roy. Soc. London. 1947. bf A191. 253-268.

[6] I.M. Gelfand, A.M. Yaglom. General relativistically invariant equations and infinitedimensional representations of the Lorentz group. J. Experimental and Theoretical Physics. 1948. **18**. 703-733 (in Russian).

[7] F.I. Fedorov. On the theory of the spin 2 particle. Proceedings of Belorussian State University. Ser. phys.-math. 1951. no. 12. 156-173.

[8] T. Regge. On properties of the particle with spin 2. Nuovo Cimento. 1957. **5**, no 2. 325-326.

[9] H.A. Buchdahl. On the compatibility of relativistic wave equations for particles of higher spin in the presence of a gravitational field. Nuovo Cim. 1958. **10**. 96-103.

[10] H.A. Buchdahl. On the compatibility of relativistic wave equations in Riemann spaces. Nuovo Cim. 1962. **25**. 486-496.

[11] L.A. Shelepin. Covariant theory of relativictic wave equations. Nucl. Phys. 1962. **33**, no 4. 580-593.

[12] G. Velo, D. Zwanziger. Noncausality and other defects of interaction Lagrangians for particles with spin one and higher. Phys. Rev. 1969. **188**, no 5. 2218-2222.

[13] C. Aragone, S. Deser. Constraints on gravitationally coupled tensor fields. Nuovo Cim. 1971. **A3**. 709-720.

[14] G. Velo. Anomalous behavior of a massive spin two charged particle in an external electromagnetic field. Nucl. Phys. 1972. **B43**. 389-401.

[15] C.R. Hagen. Minimal electromagnetic coupling of spin-two fields. Phys. Rev. 1972. **D6**, no 4. 984-987.

[16] J.F. Reilly. Minimal coupling in spin-2 field. Nucl. Phys. 1974. **B1**, no 70. 356-364.

[17] P.M. Mathews, M. Seetharaman, J. Prabhakaran. Inconsistencies in the symmetric tensor field description of spin-2 particles in an external homogeneous magnetic field. Phys. Rev. 1976. **D14**. 1021-1031.

[18] M. Kobayashi, A. Shamaly. Minimal electromagnetic coupling for massive spin-2 fields. Phys. Rev. 1978. **D 17**, no 8. 2179.

[19] C. Aragone, S. Deser. Consistency problems of spin-2 gravity coupling. Nuovo Cim. 1980. **B57**. 33-49.

[20] P.M. Mathews, T.R. Govindarajan, M. Seetharaman, J. Prabhakaran. Relativistic wave equations coupled to external fields: an algebraic study of the problem of constraints. J. Math.Phys. 1980. **21**. 1495-1505.

[21] W. Cox. First-order formulation of massive spin-2 field theories. J. Phys. A. 1982. **15**. 253-268.

[22] R.M. Wald. Spin-two fields and general covariance. Phys. Rev. 1986. **D33**, no 12. 3613-3625.

[23] R.K. Loide. On conformally covariant spin-3/2 and spin-2 equations. J. Phys. A. 1986. **19**, no 5. 827-829.

[24] A. Higuchi. Forbidden mass range for spin-2 field theory in de Sitter space-time. Nucl. Phys. 1987. **B282**. 397.

[25] A.A. Bogush, V.V. Kisel. On description of anomalous magnetic moment of a massive particle with spin 2 in the theory of relativistic wave equations. Izvestia Vuzov. SSSR. Fizika. 1988. **31**, no 3. 11-16 (in Russian).

[26] M.A. Vasiliev. More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions. Phys. Lett. 1992. **B285**. 225-234.

[27] I.L. Buchbinder, V.A. Krykhtin, V.D. Pershin. On consistent equations for massive spin-2 field coupled to gravity in string theory. Phys. Lett. 1999. **B466**.216-226.

[28] I.L. Buchbinder, D.M. Gitman, V.A. Krykhtin, V.D. Pershin. Equations of motion for massive spin 2 field coupled to gravity. Nucl. Phys. 2000. **B584**. 615-640.

[29] H. Casini, R. Montemayor, L.F. Urrutia. Duality for symmetric second rank tensors. The linearized gravitational field. Phys. Rev. 2003. **D68**. 065011.

[30] V.M. Red'kov, N.G. Tokarevskaya, V.V. Kisel. Graviton in a curved space-time background and gauge symmetry. Nonlinear Phenomena in Complex Systems. 2003. **6**, no 3. 772-778.

[31] Angnis Margreta Schmidt-May. Classically Consistent Theories of Interacting Spin-2 Fields. Doctoral Thesis in Theoretical Physics. Stockholm, Sweden. 2013.

[32] L. Bernard, C. Deffayet, A. Schmidt-May and M. von Strauss. Linear spin-2 fields in most general backgrounds. Phys. Rev. 2016. **D93**. 084020.

[33] Masafumi Fukuma, Hikaru Kawai, Katsuta Sakai, and JunjiYamamoto. Massive higher spin fields in curved spacetime and necessity of non-minimal couplings. Prog. Theor. Exp. Phys. 2016. **073B02**.

[34] Koenigstein, F. Giacosa, and D.H. Rischke. Classical and quantum theory of the massive spin-two field. Ann. Phys. (Amsterdam). 2016. **368**. 16.

[35] Charles Mazuet, Mikhail S. Volkov. Massive spin-2 field in arbitrary spacetimes, the detailed derivation. Phys. Rev. 2017. **D96**. 124023.

[36] V.V. Kisel, E.M. Ovsiyuk, O.V. Veko, Y.A. Voynova, V. Balan, V.M. Red'kov. Elementary Particles with Internal Structure in External Fields. I. General Theory; II. Physical Problems. New York: Nova Science Publishers Inc., 2018. New York: Nova Science Publishers Inc., 2018.

[37] A. Ivashkevich, A. Buryy, E. Ovsiyuk, V. Balan, V. Kisel, V. Red'kov. On the matrix equation for a spin 2 particle in pseudo-Riemannian space-time, tetrad method. Proceedings of Balkan Society of Geometers. 2021. **28**. 1-23.

[38] I.G. Dudko, O.A. Semenyuk, V.V Kisel, V.M. Red'kov. Spin 2 particle with anomalous magnetic moment in Riemann space-time, restriction to massless case, gauge symmetry. Nonlinear Phenomena in Complex Systems. 2022. **25**, no 2. 286-296.

[39] Ivashkevich A. V., Red'kov V. M., Ishkhanyan A. M. Massless spin 2 field in 50-component approach, exact solutions with cylindrical symmetry, eliminating the gauge degrees of freedom. Proceedings of the National Academy of Sciences of Belarus. Physics and Mathematics series. 2024. **60**, no. 2. 132-145.

[40] A.V. Ivashkevich, A.V. Bury, V.M. Red'kov, V.V. Kisel. On new form of the 50-component theory for spin 2 particle with anomalous magnetic moment in the basis of tensors of 2-nd and 3-rd ranks. Nonlinear Dynamics and Applications. 2023. **29**. 289–330.

[41] A.V. Ivashkevich, A.V. Bury, E.M. Ovsiyuk, V.V. Kisel, V.M. Red'kov. Nonrelativistic approximation in 39-component theory for a spin 2 particle. Proceedings of the Komi Science Centre of the Ural Branch of the Russian Academy of Sciences. Physical and Mathematical Sciences. 2024. **5(71)**. 46-57.

[42] Nonrelativistic Approximation in the Theory of a Spin-2 Particle with Anomalous Magnetic Moment Alina Ivashkevich, Viktor Red'kov, and Artur Ishkhanyan // Axioms. – 2025. Vol. 14(1), 35.

[43] On the matrix equation for a spin 2 particle in pseudo-Riemannian space-time. II. Separating the variables in spherical coordinates / A. Ivashkevich, A. Bury, E. Ovsiyuk, V. Balan, V. Kisel, V. Red'kov // Proceedings of Balkan Society of Geometers. – 2022. – P. 12–33.

[44] Varshalovich, D.A.; Moskalev, A.N.; Hersonskiy, V.K., Quantum theory of angular moment. Nauka, Leningrad, 1975 (in Russian).

[45] V.M. Red'kov. Tetrad formalism, spherical symmetry and Schrödinger basis. Minsk, Belarussian Science Publ., 2011. 339 p. (in Russian).

[46] Ivashkevich, A.V. Spin 3/2 particle in the Coulomb field, the non-relativistic approximation. A.V. Ivashkevich, V.M. Red'kov, A.M. Ishkhanyan // Physics of Particles and Nuclei. – 2025. - Vol. 26, no 6 [8 pages].

[47] L. D. Landau, E. M. Lifshitz. Quantum Mechanics. Nonrelativistic theory. 3rd Edition. Elsevier, 2013. 688 p.