JONES TYPE BISPINORS IN POLARIZATION OPTICS

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Abstract. We shall present some facts relevant to solving the problems of polarization optics in the frames of Stokes's vector and Jones's spinor formalisms. It is known that the completely polarized light can be described by Stokes 4-dimensional vector or alternatively by Jones complex 2-dimensional spinor. It is known that the Stokes formalism may be extended to a partially polarized light, but Jones approach does not. In the present paper, we introduce the concept of 4-dimensional Jones type spinor, first for a completely polarized light. This approach is extended to the partially polarized light. From 4-dimensional spinor follow both 4-vector and antisymmetric tensor of Stokes type. Stokes vector depends on four parameters, whereas the Stokes tensor depends on five parameters. By this reason, we can assume that the Stokes tensor contains more information about the partially polarized light than the Stokes vector. We have found relationships between the four components of Stokes vector and five components of Stokes tensor in analytical form, they are studied numerically as well. In addition, we shortly discuss relationships between polarization of the light and space models with spinor structure.

Key words: Completely and partially polarized light, Stokes vector, Stokes antisymmetric tensor, Jones bispinor, the Lorentz group theory, the spinor structure.

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1. INTRODUCTION

The main line of evolution in theoretical tools of the polarization optics [1]–[22] seems to be quite independent on relativistic symmetry methods, developed in the physics of elementary particles. However, it was noticed by many authors [23]–[29] that these two branches of physics employ the very similar mathematical techniques, with distinctions in notation and physical emphasis; also see recent publications [30]–[33]. In the present paper, we shall focus on the unity of the mathematical foundations of particle physics and polarized light optics. Keeping in mind the notable differences between the properties of isotropic and time-like vectors in Special Relativity, we should expect the similar major differences in describing the completely polarized and partially polarized light. To clarify this point, we shall consider these two cases separately.

2. POLARIZATION OF THE LIGHT, STOKES–MUELLER FORMALISM

To elucidate how mathematical facts on the Euclidean rotation and relativistic Lorentz groups may be applied to problems of polarization optics, let us shortly recall some definitions.

For a plane electromagnetic wave spreading along the axis z, at an arbitrary fixed point z we have the following behavior of the field variables (for simplicity follow only electric components)

$$E^1 = N\cos\omega t, E^2 = M\cos(\omega t+\Delta), E^3 = 0, N\geq 0, M\geq 0, \Delta\in[-\pi,\pi]. \quad (1)$$

If the amplitudes N, M and the phase shift Δ are not changed in the measuring process, the Stokes parameters are equal to (I is the intensity)

$$S_0 = I = N^2+M^2, S_3 = N^2-M^2, S_1 = 2NM\cos\Delta, S_2 = 2NM\sin\Delta; \quad (2)$$

the identity holds $S_a S^a = I^2 - \mathbf{S}^2 = 0$. This means that $\mathbf{S} = I\mathbf{n}, \mathbf{n}^2 = 1$, where $\mathbf{n}$ stands for any 3-vector. In other words, for the completely polarized light, the Stokes 4-vector is an isotropic one. For natural light, the Stokes parameters are trivial, $S^a_{nat} = (I_{nat}, 0, 0, 0)$.

When summing two non-coherent light waves, their Stokes parameters behave in accordance with the rule: $I_1 + I_2, \mathbf{S}_1 + \mathbf{S}_2$. The partially polarized light can be obtained as the superposition of natural and completely polarized light:

$$S^a_{nat} = (I_{nat}, 0, 0, 0),\ S^a_{pol} = (I_{pol}, I_{pol}\mathbf{n}),\ S^a = (I_{nat}+I_{pol})(1, \frac{I_{pol}}{I_{nat}+I_{pol}}\mathbf{n}).$$

With the notations

$$I = I_{nat}+I_{pol}\ ,\ \ p = \frac{I_{pol}}{I_{nat}+I_{pol}},$$

the Stokes 4-vector of the partially polarized light is specified as

$$S_a = (I,\ Ip\,\mathbf{n}),\ \ S_a S^a = I^2(1-p^2)\geq 0,\ \ p\in[0,\ 1]. \quad (3)$$

The properties of Stokes 4-vectors for the completely and partially polarized light may be considered as isomorphic to the behavior of the isotropic and time-like vectors with respect to the Lorentz group in Special Relativity.

3. JONES 2-DIMENSIONAL FORMALISM

Let us recall the Jones approach, and consider its connection with the concept of spinor for rotation and Lorentz groups. It is convenient to start with a relativistic 2-spinor Ψ, representation of the special linear group $SL(2,C)$, covering for the Lorentz group L [38]:

$$\Psi = \begin{vmatrix} \psi^1 \\ \psi^2 \end{vmatrix}, \;\; \Psi' = B(k)\Psi\,, \;\; B(k) = k_0 + k_j \sigma^j, \;\; \det B = k_0^2 - \mathbf{k}^2 = 1; \qquad (4)$$

the symbols σ^j denote the Pauli matrices, (k_0, k_j) are complex-valued parameters. A 2-rank spinor $\Psi \otimes \Psi^*$ can be presented as follows

$$\Psi \otimes \Psi^* = \frac{1}{2}\,(S_a\,\bar{\sigma}^a) = \frac{1}{2}\,(S_0 - S_j\,\sigma^j), \;\; \bar{\sigma}^a = (I, -\sigma_j)\,. \qquad (5)$$

The spinor nature of Ψ generates a definite transformation for coefficients S_a

$$S\,'_a\,\bar{\sigma}^a = S_a\,B(k)\bar{\sigma}^a B^{+}(k)\,, \qquad (6)$$

whence, using the well-known relations in the Lorentz group theory, we obtain

$$B(k)\bar{\sigma}^a B^{+}(k) = \bar{\sigma}^b L_b{}^a \quad \Longrightarrow \quad S'_b = L_b{}^a S_a,$$

$$L_b{}^a(k,\,k^*) = \bar{\delta}_b^c\,(-\delta_c^a\,k^n k_n^* + k_c k^{a*} + k_c^* k^a + i\,\epsilon_c{}^{anm}\,k_n k_m^*\,)\,,$$

the modified Kronecker symbol is

$$\bar{\delta}_b^c = \begin{cases} +1, & c = b = 0, \\ -1, & c = b = 1,2,3. \end{cases}$$

If we restrict ourselves to the $SU(2)$ group, we get [38]

$$k_0 = n_0,\; k_j = -in_j,\; n_0^2 + \vec{n}^2 = +1,\; B(n) = n_0 - in_j\sigma_j,$$

$$L(\pm n) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-2(n_2^2+n_3^2) & -2n_0n_3 + 2n_1n_2 & 2n_0n_2 + 2n_1n_3 \\ 0 & 2n_0n_3 + 2n_1n_2 & 1-2(n_1^2+n_3^2) & -2n_0n_1 + 2n_2n_3 \\ 0 & -2n_0n_2 + 2n_1n_3 & 2n_0n_1 + 2n_2n_3 & 1-2(n_1^2+n_2^2) \end{vmatrix}.$$

Now, let us introduce the Jones polarization 2-spinor Ψ:

$$\Psi = \begin{vmatrix} Ne^{i\alpha} \\ Me^{i\beta} \end{vmatrix}, \quad N \geq 0,\; M \geq 0,$$

whence it follows

$$\Psi \otimes \Psi^* = \begin{vmatrix} N^2 & NMe^{-i(\beta-\alpha)} \\ NMe^{+i(\beta-\alpha)} & M^2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} S_0 + S_3 & S_1 - iS_2 \\ S_1 + iS_2 & S_0 - S_3 \end{vmatrix}, \qquad (7)$$

that is

$$S_1 = 2NM\cos(\beta-\alpha)\ , \quad S_2 = 2NM\sin(\beta-\alpha)\ ,$$
$$S_3 = N^2 - M^2\ , \quad S_0 = N^2 + M^2 = \sqrt{S_1^2+S_2^2+S_3^2} = S\ . \tag{8}$$

The formulas (8) should be compared with equations (2), written as

$$S_0 = N^2+M^2 = \sqrt{S_1^2+S_2^2+S_3^2}, \quad S_3 = N^2-M^2,$$
$$S_1 = 2NM\ \cos\Delta, \quad S_2 = 2NM\sin\Delta, \quad \beta-\alpha=\Delta. \tag{9}$$

Instead of α and β, one may introduce the new variables $\Delta = \beta-\alpha$, $\gamma=\beta+\alpha$; accordingly, the spinor Ψ reads

$$\Psi = e^{i\gamma/2} \left| \begin{array}{c} N\,e^{-i\Delta/2} \\ M\,e^{+i\Delta/2} \end{array} \right| = e^{i\gamma/2} \left| \begin{array}{c} \sqrt{(S+S_3)/2}\ e^{-i\Delta/2} \\ \sqrt{(S-S_3)/2}\ e^{+i\Delta/2} \end{array} \right|; \tag{10}$$

the last formula (up to the phase multiplier $e^{i\gamma/2}$) coincides with the Jones spinor. The inverse relations to (8), (9) read

$$2N^2 = S+S_3, \quad 2M^2 = S-S_3, \quad \tan\Delta = S_2/S_1; \tag{11}$$

these relations correlate with the formulas defining the parabolic coordinates

$$\xi = r+z, \qquad \eta = r-z, \qquad \tan\phi = y/x. \tag{12}$$

Thus, the evident isomorphism exists between the parameters (N,M,Δ) and the parabolic coordinates (ξ,η,ϕ) in the effective space of Stokes 3-vector:

$$\xi = 2N^2,\ \eta = 2M^2,\ \phi=\Delta; \qquad x=S_1,\ y=S_2,\ z=S_3. \tag{13}$$

Let us find a space spinor Ψ_{space} [38] related to the Cartesian coordinates

$$\Psi_{space} = \left| \begin{array}{c} N\,e^{i\alpha} \\ M\,e^{i\beta} \end{array} \right|, \quad \Psi_{space}\otimes\Psi^*_{space} = \frac{1}{2} \left| \begin{array}{cc} r+z & x-iy \\ x+iy & r-z \end{array} \right|; \tag{14}$$

then we produce formulas similar to (10):

$$\Psi_{space} = e^{-\gamma/2} \left| \begin{array}{c} \sqrt{r+z}e^{-i\phi/2} \\ \sqrt{r-z}e^{+i\phi/2} \end{array} \right| = e^{-\gamma/2} \left| \begin{array}{c} \sqrt{\xi}e^{-i\phi/2} \\ \sqrt{\eta}e^{+i\phi/2} \end{array} \right|, e^{i\phi} = \frac{x+iy}{\sqrt{x^2+y^2}}. \tag{15}$$

The spinor Ψ (or Ψ_{space}) has obvious peculiarities: at the whole axis $S_1=S_2=0$ (or at $x=y=0$), its defining relations assume the ambiguity $(0+i0)/0$. It should be mentioned that the polarization singularities, attracting attention [10], should be associated with the appearance of this ambiguity.

Also, one can pay special attention to the multiplayers $e^{+i\phi/2}$ and $e^{-i\phi/2}$ in the expression for the Jones spinor, which leads to $(\pm)$-ambiguity at the values $\phi=0$ and $\phi=+2\pi$ or $\Delta=0$ and $\Delta=+2\pi$. This is an old problem with spinors applied to the description of 3-vectors, and it can be overcome within the framework of space spinor structure [34]–[35]. Does the spinor group topology is relevant to the Jones complex formalism or not – this issue remains open for both theory and experiments.

Besides, we might pay attention to the fact that, usually, the space vector (x, y, z) is not assumed to be a pseudovector, but the construction of (x, y, z) according to (14) leads to such a pseudovector model. In this connection we can recall the Cartan's classification [36]–[37] for the nonrelativistic 2-spinors with respect to spinor P-reflection: namely, the simplest irreducible representations of the unitary extended group $\tilde{SU}(2)$, are 2-component spinors of two types. According to Cartan, there exist two ways to construct a 3-vector in terms of 2-spinors

$$\begin{aligned} &1)\ \Psi_{space} \otimes \Psi^{*}_{space} = r + x_j\,\sigma^j,\ r = +\sqrt{x_j\,x_j}\ ,\ x_j \text{ is a pseudovector;} \\ &2)\ \Psi'_{space} \otimes \Psi'_{space} = (y_j + i x_j)\sigma^j \sigma^2,\ y_j, x_j \text{ are vectors.} \end{aligned} \tag{16}$$

The variant 1 provides us with the possibility to build a spinor model for the pseudovector 3-space, whereas variant 2 leads to a spinor model of the proper vector 3-space. These spinors, Ψ_{space} and Ψ'_{space} respectively, turn out to be different functions of Cartesian coordinates [38]. In particular, the second spinor model corresponding to a vector space (variant 2) is described by two spinors $\Psi'_{space}(\mathbf{x})$, each covering a vector half-space

$$\begin{aligned} &x_3 > 0\ ,\ \Psi'^{+}_{space} = \left| \begin{array}{c} \sqrt{r - (x^2+y^2)^{1/2}}\ e^{-i\phi/2} \\ \sqrt{r + (x^2+y^2)^{1/2}}\ e^{+i\phi/2} \end{array} \right|,\ e^{i\phi/2} = \sqrt{\frac{x+iy}{\sqrt{x^2+y^2}}}; \\ &x_3 < 0,\ \Psi'^{-}_{space} = i \left| \begin{array}{c} \sqrt{r - (x^2+y^2)^{1/2}}\ e^{-i\sigma/2} \\ \sqrt{r + (x^2+y^2)^{1/2}}\ e^{+i\sigma/2} \end{array} \right|,\ e^{i\sigma/2} = -i\sqrt{\frac{x+iy}{\sqrt{x^2+y^2}}}. \end{aligned} \tag{17}$$

In the context of polarization optics, instead of (17), we have

$$\begin{aligned} &S_3 > 0\ ,\ \Psi'_{+} = \left| \begin{array}{c} \sqrt{S - (S_1^2+S_2^2)^{1/2}} e^{-i\Delta/2} \\ \sqrt{S + (S_1^2+S_2^2)^{1/2}} e^{+i\Delta/2} \end{array} \right|,\ e^{i\Delta/2} = \sqrt{\frac{S^1+iS^2}{\sqrt{S_1^2+S_2^2}}}, \\ &S_3 < 0\ ,\ \Psi'_{-} = i \left| \begin{array}{c} \sqrt{S - (S_1^2+S_2^2)^{1/2}}\ e^{-i\sigma/2} \\ \sqrt{S + (S_1^2+S_2^2)^{1/2}}\ e^{+i\sigma/2} \end{array} \right|,\ e^{i\sigma/2} = -i\sqrt{\frac{S^1+iS^2}{\sqrt{S_1^2+S_2^2}}}. \end{aligned} \tag{18}$$

Finally, let us write down the two different formulas for Stokes 3-vectors:

traditional $\Psi(\mathbf{S})$

$$S_1 = \sqrt{\frac{NM}{2}}\ \cos\Delta,\ \ S_2 = \sqrt{\frac{NM}{2}}\ \sin\Delta,\ S_3 = N^2 - M^2;$$

alternative $\Psi'(\mathbf{S})$

$$S_1 = \sqrt{2 \mid M'^2 - N'^2 \mid} \cos\Delta, S_2 = \sqrt{2 \mid M'^2 - N'^2 \mid} \sin\Delta, S_3 = \pm\sqrt{N'M'}.$$

In general, the properties of any optical medium in the context of polarization optics are determined only by the medium itself, so we cannot anticipate the type of behavior of Stokes 3-vectors: of true-vector or of pseudovector nature.

4. 4-SPINORS AND COMPLETELY POLARIZED LIGHT

Let us start with the well-known relations [38], [39] between a 2-rank bispinor and simplest tensors. Such a bispinor $U = \Psi \otimes \Psi$ can be decomposed into the scalar S, vector S_b, pseudoscalar $\tilde{S}$, pseudovector $\tilde{S}_b$, and skew-symmetric tensor S_{ab}, according to the formula

$$U = \Psi \otimes \Psi = (-i\, S + \gamma^b\, S_b + i\, \sigma^{ab}\, S_{ab} + \gamma^5\, \tilde{S} + i\, \gamma^b \gamma^5\, \tilde{S}_b) E^{-1}; \qquad (19)$$

to handle with relativistic tensors, we assume the use of the metrical tensor with signature $(+,-,-,-)$; for instance, $S_a = g_{ab} S^b$, and so on. We shall refer all the subsequent considerations to the spinor basis

$$U = \left| \begin{array}{cc} \xi^{\alpha\beta} & \Delta^{\alpha}{}_{\dot{\beta}} \\ H_{\dot{\alpha}}{}^{\beta} & \eta_{\dot{\alpha}\dot{\beta}} \end{array} \right| , \; \gamma^a = \left| \begin{array}{cc} 0 & \bar{\sigma}^a \\ \sigma^a & 0 \end{array} \right| , \; \gamma^5 = \left| \begin{array}{cc} -I & 0 \\ 0 & +I \end{array} \right| ,$$
$$\sigma^{ab} = \frac{1}{4} \left| \begin{array}{cc} \bar{\sigma}^a \sigma^b - \bar{\sigma}^b \sigma^a & 0 \\ 0 & \sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a \end{array} \right| = \left| \begin{array}{cc} \Sigma^{ab} & 0 \\ 0 & \bar{\Sigma}^{ab} \end{array} \right| ; \qquad (20)$$

the symbol E in (19) stands for a bispinor metrical matrix,

$$E = \left| \begin{array}{cc} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{array} \right| = \left| \begin{array}{cc} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{array} \right| .$$

The inverses of the relations (19) have the form

$$S_a = \frac{1}{4}\, \text{Sp}\, [E\gamma_a U] \, , \; \tilde{S}_a = \frac{1}{4i}\, \text{Sp}\, [E\gamma^5 \gamma_a U] \, ,$$
$$S = \frac{i}{4}\, \text{Sp}\, [EU] \, , \quad \tilde{S} = \frac{1}{4}\, \text{Sp}\, [E\gamma^5 U] \, , \quad S_{mn} = -\frac{1}{2i}\, \text{Sp}\, [E\sigma_{mn} U] \, . \qquad (21)$$

First, let us detail the vector S_a, applying the 2×2 block form

$$S_a = \frac{1}{2} \text{Sp} \left| \begin{array}{cc} i\sigma^2 \bar{\sigma}_a H & i\sigma^2 \bar{\sigma}_a \eta \\ -i\sigma^2 \sigma_a \xi & -i\sigma^2 \sigma_a \Delta \end{array} \right| ,$$

$$S_0 = \frac{1}{2}\, [\, (H_{\dot{2}}{}^1 - H_{\dot{1}}{}^2) - (\Delta^2{}_{\dot{1}} - \Delta^1{}_{\dot{2}})\,] = \xi^1 \eta_{\dot{2}} - \xi^2 \eta_{\dot{1}},$$
$$S_1 = \frac{1}{2}\, [\, (H_{\dot{1}}{}^1 - H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} - \Delta^2{}_{\dot{2}})\,] = \xi^1 \eta_{\dot{1}} - \xi^2 \eta_{\dot{2}},$$

$$S_2 = \frac{i}{2}\,[\,(H_{\dot{1}}{}^1 + H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} + \Delta^2{}_{\dot{2}})\,] = i\,(\xi^1 \eta_{\dot{1}} + \xi^2 \eta_{\dot{2}}),$$

$$S_3 = -\frac{1}{2}\,[\,(H_{\dot{2}}{}^1 + H_{\dot{1}}{}^2) + (\Delta^2{}_{\dot{1}} + \Delta^1{}_{\dot{2}})\,] = -\,(\xi^1 \eta_{\dot{2}} + \xi^2 \eta_{\dot{1}}).$$

Similarly, for the pseudovector we have

$$\tilde{S}_a = \frac{1}{2}\mathrm{Sp}\left| \begin{array}{cc} -i\sigma^2 \bar{\sigma}_a H & -i\sigma^2 \bar{\sigma}_a \eta \\ -i\sigma^2 \sigma_a \xi & -i\sigma^2 \sigma_a \Delta \end{array} \right|,$$

$$\tilde{S}_0 = \frac{1}{2}\,[\,-(H_{\dot{2}}{}^1 - H_{\dot{1}}{}^2) - (\Delta^2{}_{\dot{1}} - \Delta^1{}_{\dot{2}})\,] \equiv 0,$$

$$\tilde{S}_1 = \frac{1}{2}\,[\,-(H_{\dot{1}}{}^1 - H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} - \Delta^2{}_{\dot{2}})\,] \equiv 0,$$

$$\tilde{S}_2 = \frac{i}{2}\,[\,-(H_{\dot{1}}{}^1 + H_{\dot{2}}{}^2) + (\Delta^1{}_{\dot{1}} + \Delta^2{}_{\dot{2}})\,] \equiv 0,$$

$$\tilde{S}_3 = -\frac{1}{2}\,[\,-(H_{\dot{2}}{}^1 + H_{\dot{1}}{}^2) + (\Delta^2{}_{\dot{1}} + \Delta^1{}_{\dot{2}})] \equiv 0.$$

In the same manner, we obtain expressions for scalar and pseudoscalar

$$S = \frac{i}{4}[(\xi^{21} - \xi^{12}) - (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})\,] \equiv 0, \tilde{S} = \frac{i}{4}[-(\xi^{21} - \xi^{12}) - (\eta_{\dot{2}\dot{1}} + \eta_{\dot{1}\dot{2}})] \equiv 0;$$

and for the skew-symmetric tensor

$$S^{01} = \frac{i}{4}[(\xi^1\xi^1 - \xi^2\xi^2) + (\eta_{\dot{1}}\eta_{\dot{1}} - \eta_{\dot{2}}\eta_{\dot{2}})], S^{23} = \frac{1}{4}[(\xi^1\xi^1 - \xi^2\xi^2) - (\eta_{\dot{1}}\eta_{\dot{1}} - \eta_{\dot{2}}\eta_{\dot{2}})],$$

$$S^{02} = -\frac{1}{4}[(\xi^1\xi^1 + \xi^2\xi^2) + (\eta_{\dot{1}}\eta_{\dot{1}} + \eta_{\dot{2}}\eta_{\dot{2}})],\ S^{12} = -\frac{1}{2}[\xi^1\xi^2 - \eta_{\dot{1}}\eta_{\dot{2}}],$$

$$S^{31} = -\frac{1}{4i}[(\xi^1\xi^1 + \xi^2\xi^2) - (\eta_{\dot{1}}\eta_{\dot{1}} + \eta_{\dot{2}}\eta_{\dot{2}})],\ S^{03} = -\frac{i}{2}[\xi^1\xi^2 + \eta_{\dot{1}}\eta_{\dot{2}}].$$

After collecting the results together, we get

$$\Psi = \left| \begin{array}{c} \xi^\alpha \\ \eta_{\dot{\alpha}} \end{array} \right|,\ \Psi \otimes \Psi \implies S = 0,\ \tilde{S} = 0,\ \tilde{S}_a = 0,\ S_a \neq 0,\ S_{mn} \neq 0\,. \qquad (22)$$

In order that the vector and the tensor be both real, one should impose additional restrictions (of Majorana type). The first possibility is

$$\eta = +i\,\sigma^2\,\xi^* \quad \Longrightarrow \quad \eta_{\dot{1}} = +\,\xi^{2*},\ \ \eta_{\dot{2}} = -\,\xi^{1*}; \qquad (23)$$

this implies*

$$
\begin{aligned}
S_0 = -(\xi^1\xi^{1*} + \xi^2\xi^{2*}) < 0,\ \ S_3 = (\xi^1\xi^{1*} - \xi^2\xi^{2*}),\\
S_1 = (\xi^1\xi^{2*} + \xi^2\xi^{1*}), \quad S_2 = i\,(\xi^1\xi^{2*} - \xi^2\xi^{1*}),\\
S^{01} = \frac{i}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) + (\xi^{2*}\xi^{2*} - \xi^{1*}\xi^{1*})\,],\\
S^{23} = \frac{1}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) - (\xi^{2*}\xi^{2*} - \xi^{1*}\xi^{1*})\,],\\
S^{02} = -\frac{1}{4}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) + (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,],\\
S^{31} = -\frac{1}{4i}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) - (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,],\\
S^{03} = -\frac{i}{2}\,(\,\xi^1\xi^2 - \xi^{2*}\xi^{1*}),\ \ S^{12} = -\frac{1}{2}\,[\,\xi^1\xi^2 + \xi^{2*}\xi^{1*}].
\end{aligned} \tag{24}
$$

Exists an alternative possibility

$$
\eta = -i\,\sigma^2\,\xi^* \quad \Longrightarrow \quad \eta_{\dot 1} = -\xi^{2*},\ \ \eta_{\dot 2} = +\xi^{1*}, \tag{25}
$$

which implies†

$$
\begin{aligned}
S_0 = (\xi^1\xi^{1*} + \xi^2\xi^{2*}) > 0, \quad S_3 = -(\xi^1\xi^{1*} - \xi^2\xi^{2*}),\\
S_1 = -(\xi^1\xi^{2*} + \xi^2\xi^{1*}), \quad S_2 = -i\,(\xi^1\xi^{2*} - \xi^2\xi^{1*}),
\end{aligned} \tag{26}
$$

and

$$
\begin{aligned}
S^{01} = \frac{i}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) + (\xi^{2*}\,\xi^{2*} - \xi^{1*}\xi^{1*})\,],\\
S^{23} = \frac{1}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) - (\xi^{2*}\xi^{2*} - \xi^{1*}\xi^{1*})],\\
S^{02} = -\frac{1}{4}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) + (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,],\\
S^{31} = -\frac{1}{4i}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) - (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,],\\
S^{03} = -\frac{i}{2}\,(\,\xi^1\xi^2 - \xi^{2*}\xi^{1*}),\ S^{12} = -\frac{1}{2}\,[\,\xi^1\xi^2 + \xi^{2*}\xi^{1*}].
\end{aligned} \tag{27}
$$

The case (26) seems to be appropriate to describe the Stokes 4-vector and additionally to determine the Stokes 2-rank tensor (27). So we obtain expressions for

*Note that $S_0 < 0$.
†Note that $S_0 > 0$.

Stokes 4-vectors and Stokes 4-tensors:

$$\Psi = \left| \begin{array}{c} \xi \\ \eta = -i\,\sigma^2\,\xi^* \end{array} \right|, \;\; \Psi \otimes \Psi \;\; \Longrightarrow \;\; S_a \neq 0, \;\; S_{mn} \neq 0,$$

$$\begin{gathered} S_0 = (\xi^1 \xi^{1*} + \xi^2 \xi^{2*}) > 0, \quad S_3 = -(\xi^1 \xi^{1*} - \xi^2 \xi^{2*}), \\ S_1 = -(\xi^1 \xi^{2*} + \xi^2 \xi^{1*}), \quad S_2 = -i\,(\xi^1 \xi^{2*} - \xi^2 \xi^{1*}), \\ a_1 = S^{01} = \frac{i}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) + (\xi^{2*}\xi^{2*} - \xi^{1*}\xi^{1*})\,], \\ b_1 = S^{23} = \frac{1}{4}\,[\,(\xi^1\xi^1 - \xi^2\xi^2) - (\xi^{2*}\xi^{2*} - \xi^{1*}\xi^{1*})\,], \\ a_2 = S^{02} = -\frac{1}{4}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) + (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,], \\ b_2 = S^{31} = -\frac{1}{4i}\,[\,(\xi^1\xi^1 + \xi^2\xi^2) - (\xi^{2*}\xi^{2*} + \xi^{1*}\xi^{1*})\,], \\ a_3 = S^{03} = -\frac{i}{2}\,(\,\xi^1\xi^2 - \xi^{2*}\,\xi^{1*}\,)\,, \quad b_3 = S^{12} = -\frac{1}{2}\,(\,\xi^1\xi^2 + \xi^{2*}\xi^{1*}). \end{gathered} \tag{28}$$

We can calculate the main invariant for the 4-vector, which turns out to be equal to zero

$$S_0 S_0 - S_j S_j = (\xi^1 \xi^{1*})\,(\xi^2 \xi^{2*}) - (\xi^1 \xi^{2*})\,(\xi^2 \xi^{1*}) \equiv 0; \tag{29}$$

so the quantity S_a indeed may be considered as a Stokes 4-vector for the completely polarized light. In turn, the 4-tensor S_{mn}, constructed from the Jones bispinor Ψ, will be called a Stokes tensor. Let us calculate the two invariants for S_{mn}:

$$\begin{gathered} I_1 = -\frac{1}{2}\,S^{mn} S_{mn} = \mathbf{a}^2 - \mathbf{b}^2 = \frac{1}{16}[-2(\xi^1\xi^1 - \xi^2\xi^2)^2 - 2(\xi^{1*}\xi^{1*} - \xi^{2*}\xi^{2*}) \\ +2(\xi^1\xi^1 + \xi^2\xi^2)^2 + 2(\xi^{1*}\xi^{1*} + \xi^{2*}\xi^{2*}) - 8(\xi^1)^2(\xi^2)^2 - 8(\xi^{1*})^2(\xi^{2*})^2] \equiv 0, \end{gathered}$$

and the second invariant is

$$\begin{gathered} I_2 = \frac{1}{4}\,\epsilon_{abmn} S^{ab} S^{mn} = \mathbf{a} \cdot \mathbf{b} = \frac{i}{16}[(\xi^1\xi^1 - \xi^2\xi^2)^2 - (\xi^{1*}\xi^{1*} - \xi^{2*}\xi^{2*})^2 \\ -(\xi^1\xi^1 + \xi^2\xi^2)^2 + (\xi^{1*}\xi^{1*} + \xi^{2*}\xi^{2*})^2 + 4(\,\xi^1\xi^2 - \xi^{2*}\xi^{1*})(\xi^1\xi^2 + \xi^{2*}\xi^{1*})] \\ = \frac{i}{16}[(-4\xi^1\xi^1\xi^2\xi^2 + 4\xi^{1*}\xi^{1*}\xi^{2*}\xi^{2*}) + (4\xi^1\xi^1\xi^2\xi^2 - 4\xi^{1*}\xi^{1*}\xi^{2*}\xi^{2*})] \equiv 0. \end{gathered}$$

We can describe the Jones bispinor with the use of parameters M, N, α, β:

$$\begin{gathered} \Psi = \left| \begin{array}{r} N\,e^{i\alpha} \\ M\,e^{i\beta} \\ -M\,e^{-i\beta} \\ N\,e^{-i\alpha} \end{array} \right|, \;\; \Psi \otimes \Psi \;\Longrightarrow\; S_a \neq 0, \;\; S_{mn} \neq 0, \\ S_0 = M^2 + N^2, \;\; S_3 = M^2 - N^2, \\ S_1 = -2MN\cos(\alpha - \beta), \;\; S_2 = 2MN\sin(\alpha - \beta), \end{gathered} \tag{30}$$

which coincide with (8); and

$$
\begin{aligned}
a_1 = S^{01} &= -\frac{1}{2}(N^2 \sin 2\alpha - M^2 \sin 2\beta),\\
b_1 = S^{23} &= +\frac{1}{2}(N^2 \cos 2\alpha - M^2 \cos 2\beta),\\
a_2 = S^{02} &= -\frac{1}{2}(N^2 \cos 2\alpha + M^2 \cos 2\beta),\\
b_2 = S^{31} &= -\frac{1}{2}(N^2 \sin 2\alpha + M^2 \sin 2\beta),\\
a_3 = S^{03} = +NM \sin(\alpha+\beta), &\; b_3 = S^{12} = -NM \cos(\alpha+\beta).
\end{aligned}
\tag{31}
$$

Instead of the Stokes tensor S_{ab}, one may introduce a complex Stokes 3-vector, $\mathbf{s} = \mathbf{a} + i\mathbf{b}$:

$$
\begin{aligned}
s_1 = a^1 + ib^1 = S^{01} + iS^{23} &= \frac{i}{2}\,(\xi^1\xi^1 - \xi^2\xi^2),\\
s_2 = a^2 + ib^2 = S^{02} + iS^{31} &= -\frac{1}{2}\,(\xi^1\xi^1 + \xi^2\xi^2),\\
s_3 = a^3 + ib^3 = S^{03} + iS^{12} &= -i\,\xi^1\xi^2;
\end{aligned}
\tag{32}
$$

whence it follows

$$
s_1 + is_2 = -i\,\xi^2\xi^2, \quad s_1 - is_2 = +i\,\xi^1\xi^1, \quad s_3 = -i\,\xi^1\xi^2,
\tag{33}
$$

so that

$$
\xi^1 = \sqrt{-i(s_1 - is_2)},\; \xi^2 = \sqrt{i(s_1 + is_2)},\; s_3 = -i\sqrt{s_1^2 + s_2^2}.
\tag{34}
$$

Therefore, there exists possibility to express the components $(S^{03} + iS^{12})$ through the quantities $(S^{01} + iS^{23})$ and $(S^{02} + iS^{31})$. In other words, among the six components of the Stokes tensor only four are independent, and two remaining are expressed through the four independent.

It may be noted that the quantity s transforms as a vector under the complex rotation group $SO(3,C)$; recall that this group is isomorphic to the Lorentz group L [38]. In other words, instead of the Stokes tensor formalism one can apply other technique, based on the use of complex 3-vector:

$$
\mathbf{s} = \mathbf{a} + i\mathbf{b} = \frac{1}{2}\left|\begin{array}{c} i\,(N^2 e^{2i\alpha} - M^2 e^{2i\beta})\\ -\,(N^2 e^{2i\alpha} + M^2 e^{2i\beta})\\ -2i\,NM\,e^{i(\alpha+\beta)} \end{array}\right|;
\tag{35}
$$

this complex vector is isotropic, the condition $\mathbf{s}^2 = 0$ provides us with two constraints (see (34)).

5. ON JONES 4-SPINORS FOR PARTIALLY POLARIZED LIGHT

Now, let us examine other possibility to construct tensors from 4-spinors [38], [39]:

$$\Psi \otimes \Psi^c = \begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} \otimes \begin{vmatrix} +D^* \\ -C^* \\ -B^* \\ +A^* \end{vmatrix} = \begin{vmatrix} AD^* & -AC^* & -AB^* & +AA^* \\ BD^* & -BC^* & -BB^* & +BA^* \\ CD^* & -CC^* & -CB^* & +CA^* \\ DD^* & -DC^* & -DB^* & +DA^* \end{vmatrix}, \tag{36}$$

where Ψ^c is a charge conjugated bispinor. We readily obtain expressions for the components of the tensors equivalent to $\Psi \otimes \Psi^c$:

the scalar and the pseudo-scalar (they are both imaginary)

$$\Phi = -\frac{1}{4i}(AC^* + BD^* + CA^* + DB^*), \tilde{S} = -\frac{1}{4}(AC^* + BD^* - CA^* - DB^*);$$

the 4-vector (it is real)

$$S^0 = \frac{1}{4}(AA^* + BB^* + DD^* + CC^*),\; S^3 = \frac{1}{4}(AA^* - BB^* + DD^* - CC^*),$$

$$S^1 = \frac{1}{4}(AB^* + BA^* - CD^* - DC^*), S^2 = -\frac{i}{4}(-AB^* + BA^* + CD^* - DC^*);$$

the pseudo-vector (it is imaginary)

$$\tilde{S}^0 = \frac{1}{4i}\,(AA^* + BB^* - DD^* - CC^*), \tilde{S}^3 = \frac{1}{4i}(AA^* - BB^* - DD^* + CC^*),$$

$$\tilde{S}^1 = \frac{1}{4i}(AB^* + BA^* + CD^* + DC^*),\; \tilde{S}^2 = -\frac{1}{4}(-AB^* + BA^* - CD^* + DC^*);$$

the skew-symmetric tensor (it is real)

$$S^{01} = \frac{i}{4}(AD^* + BC^* - CB^* - DA^*),\; S^{23} = \frac{1}{4}(AD^* + BC^* + CB^* + DA^*),$$

$$S^{02} = -\frac{1}{4}(AD^* - BC^* - CB^* + DA^*),\; S^{31} = \frac{i}{4}(AD^* - BC^* + CB^* - DA^*),$$

$$S^{03} = -\frac{i}{4}(-AC^* + BD^* + CA^* - DB^*), S^{12} = -\frac{1}{4}(-AC^* + BD^* - CA^* + DB^*),$$

$$s_1 = \frac{i}{2}(AD^* + BC^*),\; s_2 = -\frac{1}{2}\,(AD^* - BC^*),\; s_3 = \frac{i}{2}(AC^* - BD^*).$$

Allowing for the identities

$$S_0^2 - S_3^2 = \frac{1}{16}(AA^* + DD^*)(BB^* + CC^*),$$

$$S_1^2 + S_2^2 = \frac{1}{16}(AB^* - CD^*)(A^*B - C^*D),$$

we find the invariant of S^a:

$$S^a S_a = \frac{1}{16}(AC^* + BD^*)\,(A^*C + B^*D) = \frac{1}{16}\mid AC^* + BD^* \mid^2 \geq 0\,, \qquad (37)$$

which means that the vector is time-like. Bearing in mind that $S^0 > 0$, we conclude that the 4-vector S^a can be considered as a Stokes vector of partially polarized light. The corresponding complex 3-vector $\mathbf{s}$ is not isotropic:

$$\mathbf{s}^2 = -\frac{1}{4}\,(\xi^1\eta_1^* + \xi^2\eta_2^*)^2 = -\frac{1}{4}\,(AC^* + BD^*)^2 \neq 0\,. \qquad (38)$$

With the use of the explicit form of the (imaginary) pseudo-vector $\tilde{S}^a$,

$$\tilde{S}^0 = \frac{AA^* - DD^* + BB^* - CC^*}{4i}, \quad \tilde{S}^3 = \frac{AA^* - DD^* - BB^* - CC^*}{4i},$$
$$\tilde{S}^1 = \frac{AB^* + CD^* + BA^* + DC^*}{4i}, \quad \tilde{S}^2 = \frac{AB^* + CD^* - BA^* + DC^*}{4i},$$

for its invariant we find

$$\tilde{S}_0^2 - \tilde{S}_1^2 - \tilde{S}_2^2 - \tilde{S}_1^3 = \frac{1}{4}(AC^* + BD^*)(A^*C + B^*D) > 0\,. \qquad (39)$$

This means that the corresponding real 4-pseudo-vector $i\tilde{S}^a$ cannot be considered as being of Stokes type.

Let us establish the explicit form of the above Stokes tensors, applying the following parametrization of the initial 4-spinor:

$$\Psi = \begin{vmatrix} A \\ B \\ C \\ D \end{vmatrix} = \begin{vmatrix} a\,e^{i\alpha} \\ b\,e^{i\beta} \\ c\,e^{is} \\ d\,e^{it} \end{vmatrix}. \qquad (40)$$

We readily derive (follow only Stokes 4-vector and Stokes tensor)

$$S^0 = \frac{1}{4}\,(a^2 + b^2 + c^2 + d^2), \quad S^3 = \frac{1}{4}\,(a^2 - b^2 - c^2 + d^2),$$
$$S^1 = \frac{ab\,\cos(\alpha - \beta) - cd\,\cos(s - t)}{2}, \quad S^2 = \frac{ab\,\sin(\beta - \alpha) + cd\,\sin(s - t)}{2},$$

and

$$S^{01} = -\frac{ad\sin(\alpha - t) + bc\,\sin(\beta - s)}{2}, \quad S^{23} = \frac{ad\cos(\alpha - t) + bc\cos(\beta - s)}{2},$$
$$S^{02} = -\frac{ad\cos(\alpha - t) - bc\cos(\beta - s)}{2}, \quad S^{31} = -\frac{ad\sin(\alpha - t) - bc\sin(\beta - s)}{2},$$
$$S^{03} = \frac{-ac\sin(\alpha - s) + bd\sin(\beta - t)}{2}, \quad S^{12} = \frac{ac\cos(\alpha - s) - bd\cos(\beta - t)}{2}.$$

These tensors are specified by the following parameters

$$S^{a} \Longrightarrow a,b,c,d,\alpha-\beta,s-t;\; S^{ab} \Longrightarrow a,b,c,d,\alpha-t,\beta-s,\alpha-s,\beta-t. \qquad (41)$$

The matrix $\Psi\otimes\Psi^{c}$ explicitly reads

$$U=\Psi\otimes\Psi^{c}=\left|\begin{array}{cccc} ade^{i(\alpha-t)} & -ace^{i(\alpha-s)} & -abe^{i(\alpha-\beta)} & a^{2} \\ bde^{i(\beta-t)} & -bce^{i(\beta-s)} & -b^{2} & abe^{-i(\alpha-\beta)} \\ cde^{i(s-t)} & -c^{2} & -bce^{-i(\beta-s)} & ace^{-i(\alpha-s)} \\ d^{2} & -cde^{-i(s-t)} & -bde^{-i(\beta-t)} & ade^{-i(\alpha-t)} \end{array}\right|, \qquad (42)$$

the initial 4-spinor may be factorized as follow

$$\Psi=\left|\begin{array}{cccc} e^{i(\alpha+\beta)/2} & 0 & 0 & 0 \\ 0 & e^{i(\alpha+\beta)/2} & 0 & 0 \\ 0 & 0 & e^{i(s+t)/2} & 0 \\ 0 & 0 & 0 & e^{i(s+t)/2} \end{array}\right|\Psi^{(0)},\ \Psi^{(0)}=\left|\begin{array}{c} a\,e^{i(\alpha-\beta)/2} \\ b\,e^{-(\alpha-\beta)/2} \\ c\,e^{i(s-t)/2} \\ d\,e^{-i(s-t)/2} \end{array}\right|. \qquad (43)$$

6. STOKES VECTOR S^{a} AND TENSOR S^{ab}, PARTIALLY POLARIZED LIGHT

Having in mind the identities

$$\frac{\alpha+\beta}{2}=(\frac{\alpha+\beta}{4}+\frac{s+t}{4})+(\frac{\alpha+\beta}{4}-\frac{s+t}{4})=\gamma+\Gamma,$$

$$\frac{s+t}{2}=(\frac{\alpha+\beta}{4}+\frac{s+t}{4})-(\frac{\alpha+\beta}{4}-\frac{s+t}{4})=\gamma-\Gamma\,,$$

we can present the Jones bispinor (43) differently

$$\Psi=e^{i\gamma}\left|\begin{array}{cccc} e^{i\Gamma} & 0 & 0 & 0 \\ 0 & e^{i\Gamma} & 0 & \\ 0 & 0 & e^{-i\Gamma} & 0 \\ 0 & 0 & 0 & e^{-i\Gamma} \end{array}\right|\Psi^{(0)}=e^{i\gamma}\ \exp\,(i\Gamma\gamma^{5})\ \Psi^{(0)}\,. \qquad (44)$$

Correspondingly, the components of the Stokes tensor read

$$S^{01}=-\frac{ad\sin(\alpha-t)+bc\sin(\beta-s)}{2},S^{23}=\frac{ad\cos(\alpha-t)+bc\cos(\beta-s)}{2},$$
$$S^{02}=-\frac{ad\cos(\alpha-t)-bc\cos(\beta-s)}{2},S^{31}=-\frac{ad\sin(\alpha-t)-bc\sin(\beta-s)}{2},$$
$$S^{03}=\frac{-ac\sin(\alpha-s)+bd\sin(\beta-t)}{2},S^{12}=\frac{ac\cos(\alpha-s)-bd\cos(\beta-t)}{2}.$$

7. THE MINIMAL JONES BISPINOR

Let us detail some facts concerning the Stokes 4-vector. It is convenient to introduce new parametrization for parameters a, b, c, d:

$$\begin{aligned} 2(S_0+S_3) = a^2+d^2 \Longrightarrow a = \sqrt{2(S_0+S_3)}\cos X, d = \sqrt{2(S_0+S_3)}\sin X; \\ 2(S_0-S_3) = b^2+c^2 \Longrightarrow b = \sqrt{2(S_0-S_3)}\cos Y,\ c = \sqrt{2(S_0-S_3)}\sin Y, \end{aligned} \tag{45}$$

where $X, Y \in [0, \pi/2]$. Dependence of the bispinor $\Psi^{(0)}$ on the parameters X, Y can be described as follows

$$\Psi^{(0)} = \left| \begin{array}{cccc} \cos X & 0 & 0 & 0 \\ 0 & \cos Y & 0 & 0 \\ 0 & 0 & \sin Y & 0 \\ 0 & 0 & 0 & \sin X \end{array} \right| \left| \begin{array}{c} \sqrt{2(S_0+S_3)}\ e^{+i(\alpha-\beta)/2} \\ \sqrt{2(S_0-S_3)}\ e^{-i(\alpha-\beta)/2} \\ \sqrt{2(S_0-S_3)}\ e^{+i(s-t)/2} \\ \sqrt{2(S_0+S_3)}\ e^{-i(s-t)/2} \end{array} \right| . \tag{46}$$

Parameters X, Y are not measurable quantities. The simplest way to restrict the freedom in $\Psi^{(0)}$ is to set $X = Y = \pi/4$; in this way we obtain the minimal description of the Jones bispinor

$$a = d = \sqrt{S_0+S_3}\ , \quad b = c = \sqrt{S_0-S_3}\ , \quad \Psi^{min} = \left| \begin{array}{c} a\ e^{i\tau/2} \\ b\ e^{-i\tau/2} \\ b\ e^{i\sigma/2} \\ a\ e^{-i\sigma/2} \end{array} \right| . \tag{47}$$

The formulas defining the Stokes 4-vector take on the form

$$\begin{aligned} S_0 = \frac{1}{2}(a^2+b^2),\ S_3 = \frac{1}{2}(a^2-b^2) \Longrightarrow a = \sqrt{S_0+S_3},\ b = \sqrt{S_0-S_3}, \\ S_1 = \frac{1}{2}[ab\cos(\alpha-\beta) - cd\cos(s-t)] = \frac{1}{2}\sqrt{S_0^2-S_3^2}(\cos\tau - \cos\sigma), \\ S_2 = \frac{1}{2}\ [-ab\sin(\alpha-\beta) + cd\sin(s-t)] = \frac{1}{2}\sqrt{S_0^2-S_3^2}(-\sin\tau + \sin\sigma), \\ S_1 + iS_2 = \frac{1}{2}\sqrt{S_0^2-S_3^2}(e^{i\sigma} - e^{i\tau}),\ S_1 - iS_2 = \frac{1}{2}\sqrt{S_0^2-S_3^2}(e^{-i\sigma} - e^{-i\tau}). \end{aligned} \tag{48}$$

Equations from the fourth row may be solved with respect to $x = e^{i\sigma}, y = e^{i\tau}$:

$$\frac{2(S_1+iS_2)}{\sqrt{S_0^2-S_3^2}} = x - y, \quad \frac{2(S_1-iS_2)}{\sqrt{S_0^2-S_3^2}} = \frac{1}{x} - \frac{1}{y}. \tag{49}$$

The last equation has two solutions

$$x_1 = \frac{+(S_1^2+S_2^2) - i\sqrt{(S_1^2+S_2^2)(S_0^2-S_1^2-S_2^2-S_3^2)}}{(S_1 - iS_2)\sqrt{S_0^2-S_3^2}} = e^{i\sigma_1},$$
$$y_1 = \frac{-(S_1^2+S_2^2) - i\sqrt{(S_1^2+S_2^2)(S_0^2-S_1^2-S_2^2-S_3^2)}}{(S_1 - iS_2)\sqrt{S_0^2-S_3^2}} = e^{i\tau_1};$$
$$x_2 = \frac{+(S_1^2+S_2^2) + i\sqrt{(S_1^2+S_2^2)(S_0^2-S_1^2-S_2^2-S_3^2)}}{(S_1 - iS_2)\sqrt{S_0^2-S_3^2}} = e^{i\sigma_2},$$
$$y_2 = \frac{-(S_1^2+S_2^2) + i\sqrt{(S_1^2+S_2^2)(S_0^2-S_1^2-S_2^2-S_3^2)}}{(S_1 - iS_2)\sqrt{S_0^2-S_3^2}} = e^{i\tau_2}. \tag{50}$$

Therefore we obtain two different solutions

$$e^{i\sigma_2/2} = i\ ei\tau_1/2,\ e^{-i\sigma_2/2} = -i\ e{-}i\tau_1/2;$$
$$e^{i\tau_2/2} = i\ ei\sigma_1/2, e^{-i\tau_2/2} = -i\ e{-}i\sigma_1/2. \tag{51}$$

In other words, two different minimal bispinors Ψ_1^{min} and Ψ_2^{min} lead to one the same Stokes vector:

$$\Psi_1^{min} = \begin{vmatrix} a\,e^{+i\tau_1/2} \\ b\,e^{-i\tau_1/2} \\ b\,e^{+i\sigma_1/2} \\ a\,e^{-i\sigma_1/2} \end{vmatrix}, \quad \Psi_2^{min} = \begin{vmatrix} a\,e^{+i\tau_2/2} \\ b\,e^{-i\tau_2/2} \\ b\,e^{+i\sigma_2/2} \\ a\,e^{-i\sigma_2/2} \end{vmatrix}$$
$$= \begin{vmatrix} ia\,e^{+i\sigma_1/2} \\ -ib\,e^{-i\sigma_1/2} \\ ib\,e^{+i\tau_1/2} \\ -ia\,e^{-i\tau_1/2} \end{vmatrix} = \begin{vmatrix} a\,e^{i(\sigma_1+\pi)/2} \\ b\,e^{-i(\sigma_1+\pi)/2} \\ b\,e^{i(\tau_1+\pi)/2} \\ a\,e^{-i(\tau_1+\pi)/2} \end{vmatrix}, \ \tau_2 = \sigma_1 + \pi,\ \sigma_2 = \tau_1 + \pi. \tag{52}$$

The Stokes vector S_a determines only two differences $\alpha - \beta = \tau,\ s - t = \sigma$, but it does not fix separately four parameters α, β, s, t. Similar effect may be seen in 2-dimensional Jones spinor – see relation (10). Below we will see that the components of Stokes tensor S^{ab} describe the freedom in choosing parameters α, β and s, t.

The angular parameters are determined by the explicit formulas

$$e^{i\sigma_1} = \cos\sigma_1 + i\sin\sigma_1,$$

$$\cos\sigma_1 = \frac{+S_1(S_1^2+S_2^2) + S_2\sqrt{(S_1^2+S_2^2)(S_0^2-S_1^2-S_2^2-S_3^2)}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}},$$

$$\sin\sigma_1 = \frac{S_2(S_1^2+S_2^2) - S_1\sqrt{\left(S_1^2+S_2^2\right)\left(S_0^2-S_1^2-S_2^2-S_3^2\right)}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}};$$

$$e^{i\tau_1} = \cos\tau_1 + i\sin\tau_1,$$

$$\cos\tau_1 = \frac{-S_1(S_1^2+S_2^2) + S_2\sqrt{\left(S_1^2+S_2^2\right)\left(S_0^2-S_1^2-S_2^2-S_3^2\right)}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}},$$

$$\sin\tau_1 = \frac{-S_2(S_1^2+S_2^2) - S_1\sqrt{\left(S_1^2+S_2^2\right)\left(S_0^2-S_1^2-S_2^2-S_3^2\right)}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}};$$

the second solution is determined by the first one:

$$e^{i\sigma_2} = -e^{i\tau_1}, \;\; e^{i\tau_2} = -e^{i\sigma_1}.$$

From two relations

$$\cos\sigma + \cos\tau = \frac{2S_2\sqrt{\left(S_1^2+S_2^2\right)\left[(S_0^2-S_3^2)-(S_1^2+S_2^2)\right]}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}},$$
$$\cos\sigma - \cos\tau = \frac{2S_1\sqrt{\left(S_1^2+S_2^2\right)\left[(S_0^2-S_3^2)-(S_1^2+S_2^2)\right]}}{(S_1^2+S_2^2)\sqrt{S_0^2-S_3^2}}, \tag{53}$$

it follows the constraint

$$S_2 = \frac{\cos\sigma+\cos\tau}{\cos\sigma-\cos\tau} S_1. \tag{54}$$

Besides, we readily get the identity $S_0^2 - S_3^2 = a^2b^2$. Let us introduce the notation $S_1^2 + S_2^2 = A^2$, then using Eq. (54) we obtain

$$A^2 = S_1^2\Big[1+(\frac{\cos\sigma+\cos\tau}{\cos\sigma-\cos\tau})^2\Big] \implies A = S_1\sqrt{1+\frac{(\cos\sigma+\cos\tau)^2}{(\cos\sigma-\cos\tau)^2}}. \tag{55}$$

Allowing for this expression for A, we derive (see (53))

$$\cos\sigma - \cos\tau = S_1\frac{2\sqrt{a^2b^2-A^2}}{Aab} \implies (\cos\sigma-\cos\tau)Aab = 2S_1\sqrt{a^2b^2-A^2},$$

or

$$(\cos\sigma-\cos\tau)ab\sqrt{1+\frac{(\cos\sigma+\cos\tau)^2}{(\cos\sigma-\cos\tau)^2}} = 2\sqrt{a^2b^2-A^2}.$$

Squaring the last equation, we get

$$2a^2b^2(\cos^2\sigma+\sin^2\sigma) = 4a^2b^2 - 4A^2 \implies A = ab\sqrt{1-\frac{1}{2}(\cos^2\sigma+\sin^2\sigma)}.$$

We can rewrite expression for A in (55) differently

$$A = S_1 \frac{\sqrt{2(\cos^2\sigma + \cos^2\tau)}}{\cos\sigma - \cos\tau}. \tag{56}$$

Then we obtain the expression for S_1

$$S_1 = ab\sqrt{1 - \frac{1}{2}(\cos^2\sigma + \sin^2\sigma)}\ \frac{\cos\sigma - \cos\tau}{\sqrt{2(\cos^2\sigma + \cos^2\tau)}}; \tag{57}$$

further we obtain the similar expression for S_2

$$S_2 = ab\sqrt{1 - \frac{1}{2}(\cos^2\sigma + \sin^2\sigma)}\ \frac{\cos\sigma + \cos\tau}{\sqrt{2(\cos^2\sigma + \cos^2\tau)}}. \tag{58}$$

The formulas (57) and (58) determine the correspondence between S_1, S_2 and angular parameters σ, τ.

8. STOKES TENSOR

Taking into account definitions (45), we find new representation for four components of the Stokes tensor (setting $X = Y = \pi/4$):

$$\begin{aligned} S^{01} &= -\frac{1}{2}[\,(S_0 + S_3)\ \sin(\alpha - t) + (S_0 - S_3)\ \sin(\beta - s)\,], \\ S^{23} &= \frac{1}{2}[\,(S_0 + S_3)\ \cos(\alpha - t) + (S_0 - S_3)\ \cos(\beta - s)\,], \\ S^{02} &= -\frac{1}{2}[\,(S_0 + S_3)\ \cos(\alpha - t) - (S_0 - S_3)\ \cos(\beta - s)\,], \\ S^{31} &= -\frac{1}{2}[\,(S_0 + S_3)\ \sin(\alpha - t) - (S_0 - S_3)\ \sin(\beta - s)\,], \end{aligned} \tag{59}$$

here we can see only combinations $(\alpha - t)$ and $(\beta - s)$; two remaining components are

$$\begin{aligned} S^{03} &= \frac{1}{2}\sqrt{S_0^2 - S_3^2}[-\sin(\alpha - s) + \sin(\beta - t)], \\ S^{12} &= \frac{1}{2}\sqrt{S_0^2 - S_3^2}[\cos(\alpha - s) - \cos(\beta - t)], \end{aligned} \tag{60}$$

where we can see only combinations $(\alpha - s)$ and $(\beta - t)$. Therefore, the Stokes tensor depends on four angular parameters, whereas the Stokes vector depends on two angular parameters τ, σ.

Further we will apply the notations

$$\alpha - t = \rho,\ \beta - s = \delta,\ \alpha - s = \mu,\ \beta - t = \nu.$$

Then the above relations may be presented shorter

$$
\begin{aligned}
S^{01} &= -\frac{1}{2}[(S_0+S_3)\sin\rho + (S_0-S_3)\sin\delta],\\
S^{23} &= \frac{1}{2}[(S_0+S_3)\cos\rho + (S_0-S_3)\cos\delta],\\
S^{02} &= -\frac{1}{2}[(S_0+S_3)\cos\rho - (S_0-S_3)\cos\delta],\\
S^{31} &= -\frac{1}{2}[(S_0+S_3)\sin\rho - (S_0-S_3)\sin\delta];
\end{aligned}
\qquad (61)
$$

$$
S^{03} = \frac{1}{2}\sqrt{S_0^2-S_3^2}[-\sin\mu+\sin\nu],\ S^{12} = \frac{1}{2}\sqrt{S_0^2-S_3^2}[\cos\mu-\cos\nu]. \qquad (62)
$$

It is evident that from (61) we can find expressions for $\cos\rho, \sin\rho$ and $\cos\delta, \sin\delta$:

$$
\begin{aligned}
\cos\rho &= \frac{S^{23}-S^{02}}{S_0+S_3}, \sin\rho = -\frac{S^{01}+S^{31}}{S_0+S_3},\ e^{i\rho} = \frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{01}+S^{31}}{S_0+S_3},\\
\cos\delta &= \frac{S^{23}+S^{02}}{S_0-S_3},\ \sin\delta = \frac{S^{31}-S^{01}}{S_0-S_3},\ e^{i\delta} = \frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3},
\end{aligned}
\qquad (63)
$$

from the two last relations in (63) follow constraints for modulus

$$
\Big|\frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{01}+S^{31}}{S_0+S_3}\Big| = 1, \qquad \Big|\frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3}\Big| = 1.
$$

So, we get the explicit expressions for parameters ρ and δ:

$$
\begin{aligned}
\rho &= \arg\Big(\frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{31}+S^{01}}{S_0+S_3}\Big), \quad \tan\rho = -\frac{S^{31}+S^{01}}{S^{23}-S^{02}};\\
\delta &= \arg\Big(\frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3}\Big), \quad \tan\delta = +\frac{S^{31}-S^{01}}{S^{23}+S^{02}}\ .
\end{aligned}
\qquad (64)
$$

Let us additionally study the linear relationships between the angular parameters. In the first place, we can eliminate the variables α and s:

$$
\alpha - t = \rho \quad \Longrightarrow \quad \beta+\tau-t-\rho = 0,
$$

$$
\beta - s = \delta \quad \Longrightarrow \quad \beta - t - \sigma - \delta = 0,
$$

$$
\alpha - s = \mu \quad \Longrightarrow \quad \beta+\tau-t-\sigma-\mu = 0, \quad \beta - t = \nu;
$$

in this way we derive the linear constraints

$$
\rho = \nu+\tau, \quad \delta = \nu-\sigma, \quad \mu = \nu+\tau-\sigma, \qquad (65)
$$

where $\nu = \beta - t$. Therefore, there exists only one independent parameter ν; however we have two different expressions for this parameter:

$$
(1) \quad \nu_1 = \rho_1 - \tau, \qquad (2) \quad \nu_2 = \delta_2 + \sigma;
$$

in explicit form they read

$$(1)\ \nu_1 = \rho_1 - \tau = \arg\Big(\frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{31}+S^{01}}{S_0+S_3}\Big) - \tau, \tan\rho_1 = -\frac{S^{31}+S^{01}}{S^{23}-S^{02}};$$
$$(2)\ \nu_2 = \delta_2 + \sigma = \arg\Big(\frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3}\Big) + \sigma, \tan\delta_2 = +\frac{S^{31}-S^{01}}{S^{23}+S^{02}}. \qquad (66)$$

For parameters ρ and δ we have expressions

$$(1) \quad \begin{aligned} \rho_1 &= \arg\Big(\frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{31}+S^{01}}{S_0+S_3}\Big), \\ \delta_1 &= \arg\Big(\frac{S^{23}-S^{02}}{S_0+S_3} - i\frac{S^{31}+S^{01}}{S_0+S_3}\Big) - \tau - \sigma = \rho_1 - (\tau+\sigma); \end{aligned} \qquad (67)$$

$$(2) \quad \begin{aligned} \delta_2 &= \arg\Big(\frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3}\Big), \\ \rho_2 &= \arg\Big(\frac{S^{23}+S^{02}}{S_0-S_3} + i\frac{S^{31}-S^{01}}{S_0-S_3}\Big) + \sigma + \tau = \delta_2 + \sigma + \tau. \end{aligned} \qquad (68)$$

Let us turn two non-used relations

$$S^{03} = \frac{1}{2}\sqrt{S_0^2 - S_3^2}(-\sin\mu + \sin\nu), \ S^{12} = \frac{1}{2}\sqrt{S_0^2 - S_3^2}(\cos\mu - \cos\nu). \qquad (69)$$

Taking in mind $\nu = \rho - \tau$, $\mu = \rho - \sigma$, we rewrite system (69) in the form

$$\begin{aligned} S^{03} &= \frac{2S_2(S^{02}+S^{23}) - \sqrt{S_0^2-S_3^2}(S^{01}-S^{31})(\cos\sigma - \cos\tau)}{2(S_0-S_3)}, \\ S^{12} &= -\frac{\sqrt{S_0^2-S_3^2}(S^{02}+S^{23})(\cos\sigma-\cos\tau) + 2S_2(S^{01}-S^{31})}{2(S_0-S_3)}. \end{aligned} \qquad (70)$$

Substituting $(\cos\tau - \cos\sigma)$ and $(-\sin\tau + \sin\sigma)$ from (48):

$$S_1 = \frac{1}{2}\sqrt{S_0^2-S_3^2}(\cos\tau - \cos\sigma), \ \ S_2 = \frac{1}{2}\sqrt{S_0^2-S_3^2}(-\sin\tau + \sin\sigma),$$

we obtain

$$\begin{aligned} S^{03} &= \frac{S_1(S^{01}-S^{31}) + S_2(S^{02}+S^{23})}{S_0-S_3}, \\ S^{12} &= \frac{S_2(S^{31}-S^{01}) + S_1(S^{02}+S^{23})}{S_0-S_3}. \end{aligned} \qquad (71)$$

The last formulas allow us to calculate the components S^{03} and S^{12} through four components S^{01}, S^{31}, S^{02}, S^{23} and the known S_0, S_1, S_2, S_3.

9. NUMERICAL SIMULATION

Example 1. For given a, b, τ_1, σ_1 (see (47)):

$$(\Psi_1^{min}) \qquad a = 1, \;\; b = 2, \;\; \tau_1 = \frac{\pi}{3}, \;\; \sigma_1 = \frac{\pi}{6},$$

we find (see (48))

$$S_0 = \frac{5}{2}, \quad S_3 = -\frac{3}{2}, \quad S_1 = \frac{1}{2}(1-\sqrt{3}), \quad S_2 = \frac{1}{2}(1-\sqrt{3});$$

the corresponding degree of polarization equals

$$p = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} = \frac{1}{5}\sqrt{17 - 4\sqrt{3}} \approx 0.635;$$

so, the Stokes 4-vector reads

$$\Big(S_0, S_1, S_2, S_3\Big) \approx \Big(2.5, -0.366, -0.366, -1.5\Big). \tag{72}$$

In order to determine the Stokes tensor, we should fix two additional parameters δ, ρ (they do not influence the Stokes 4-vector); let they be

$$\delta = \frac{5\pi}{12}, \;\; \rho = -\frac{\pi}{12};$$

then we find four main components of Stokes tensor (see (61))

$$S^{01} = -\frac{5+3\sqrt{3}}{4\sqrt{2}}, \;\; S^{02} = \frac{3\sqrt{3}-5}{4\sqrt{2}}, \;\; S^{23} = \frac{5\sqrt{3}-3}{4\sqrt{2}}, \;\; S^{31} = \frac{3+5\sqrt{3}}{4\sqrt{2}}.$$

and calculate two remaining ones (see (71))

$$S^{03} = \frac{\sqrt{2-\sqrt{3}}}{2}, \;\; S^{12} = \frac{\sqrt{3}-3}{2\sqrt{2}}.$$

Thus, the Stokes tensor is given as

$$\begin{aligned} S^{01} \approx -1.802, \quad S^{23} \approx 1.001, \\ S^{02} \approx 0.0347, \quad S^{31} \approx 2.061, \\ S^{03} \approx 0.259, \quad S^{12} \approx -0.448. \end{aligned} \tag{73}$$

Let us consider the second alternative case (see (52)):

$$(\Psi_2^{min}) \qquad a = 1, \;\; b = 2, \;\; \tau_2 = \frac{\pi}{6} + \pi, \;\; \sigma_2 = \frac{\pi}{3} + \pi;$$

we readily verify that these parameters lead to the same Stokes vector and the same Stokes tensor.

Example 2. For given a, b, τ_1, σ_1 (see (47)):

$$(\Psi_1^{min}) \qquad a = 1, \;\; b = \frac{1}{2}, \;\; \tau_1 = \frac{\pi}{4}, \;\; \sigma_1 = \frac{\pi}{8},$$

we find (see (48))

$$S_0=\frac{5}{8},\ \ S_1=\frac{1}{8}(\sqrt{2}-2\cos\frac{\pi}{8}),\ \ S_2=\frac{1}{8}(2\sin\frac{\pi}{8}-\sqrt{2}),\ \ S_3=\frac{3}{8}.$$

Thus, the components of Stokes vector are

$$S_0=0.625,\ \ S_1\approx-0.054,\ \ S_2\approx-0.081,\ \ S_3=0.375, \tag{74}$$

and degree polarization is

$$p=\frac{1}{5}\sqrt{17-2\sqrt{2}\csc\frac{\pi}{8}}\approx 0.620.$$

In order to determine the Stokes tensor, we fix two additional parameters, let they be

$$\delta=\frac{7\pi}{12},\ \ \rho=-\frac{\pi}{12};$$

further we obtain (see (61), (71))

$$S^{01}=\frac{3\sqrt{3}-5}{16\sqrt{2}},\ \ S^{02}=-\frac{3+5\sqrt{3}}{16\sqrt{2}},\ \ S^{23}=\frac{5+3\sqrt{3}}{16\sqrt{2}},\ \ S^{31}=\frac{5\sqrt{3}-3}{16\sqrt{2}},$$

and

$$S^{03}=\frac{\sqrt{2+\sqrt{2-\sqrt{3}}}-1}{8},\ S^{12}=\frac{\left(\sqrt{6}+\sqrt{2}-2\right)\csc\left(\frac{\pi}{8}\right)-4\sqrt{3}}{32}.$$

So the Stokes tensor is given as

$$\begin{aligned} S^{01}&\approx 0.009, \quad S^{23}\approx 0.451,\\ S^{02}&\approx -0.515, \quad S^{31}\approx 0.250,\\ S^{03}&\approx 0.073, \quad S^{12}\approx -0.064. \end{aligned} \tag{75}$$

Let us consider the second alternative case (see (52)):

$$(\Psi_2^{min}) \qquad a=1,\ \ b=2,\ \ \tau_2=\frac{\pi}{8}+\pi,\ \ \sigma_2=\frac{\pi}{4}+\pi;$$

we readily verify that this parameters lead to the same Stokes vector and the same Stokes tensor.

10. PLOTS FOR STOKES VECTORS

We start with the general formulas

$$S_0=\frac{1}{2}(a^2+b^2),\ S_3=\frac{1}{2}(a^2-b^2),$$

$$S_1=\frac{1}{2}\sqrt{S_0^2-S_3^2}(\cos\sigma-\cos\tau),\ S_2=\frac{1}{2}\sqrt{S_0^2-S_3^2}(\cos\sigma+\cos\tau).$$

Case 1. Let

$$a = b = 1, \quad \tau \in (-\pi, +\pi), \quad \sigma \in (-\pi, +\pi),$$

then $S_0 = 1,\ S_3 = 0$, and the component $S_1(\tau, \sigma) = \frac{1}{2}(\cos\sigma - \cos\tau)$ (Fig. 1), the component $S_2(\tau, \sigma) = \frac{1}{2}(\sin\sigma - \sin\tau)$ (Fig. 2), the degree of polarization $p(\tau, \sigma) = \frac{1}{2}\sqrt{2 - 2\cos(\sigma - \tau)}$ (Fig. 3).

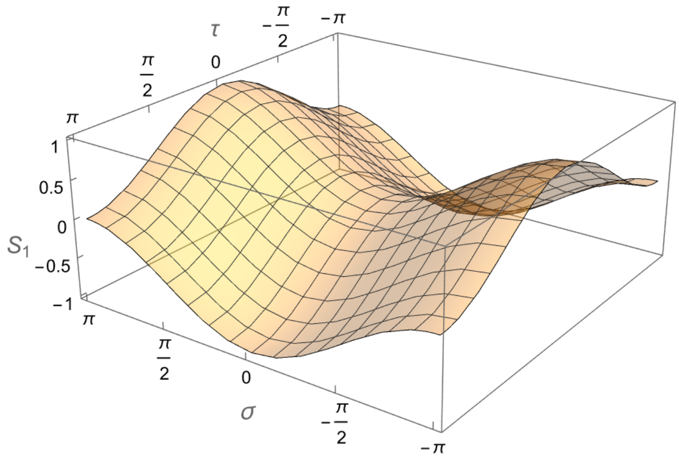

Fig. 1 – $S_1(\tau, \sigma),\ S_0 = 1,\ S_3 = 0.$

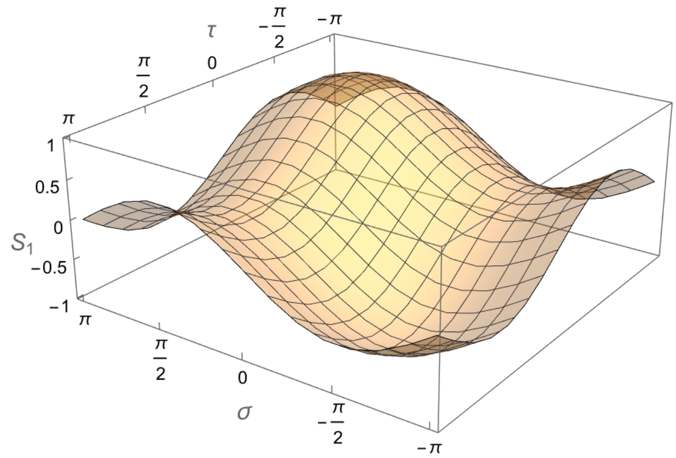

Fig. 2 – $S_2(\tau, \sigma),\ S_0 = 1,\ S_3 = 0.$

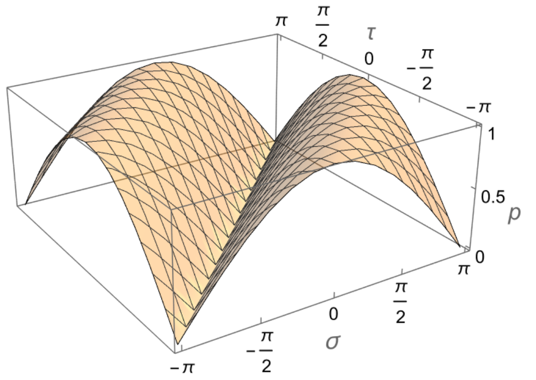

Fig. 3 – $p(\tau,\sigma),\ S_0=1,\ S_3=0.$

Case 2. Let

$$a\in[0,1],\ \ b\in[0,1],\ \ \tau=0,\ \ \sigma=\pi/6.$$

Then in the case 2 we have

$$S_0=\frac{1}{2}\left(a^2+b^2\right),\ \ S_3=\frac{1}{2}\left(a^2-b^2\right),\ \ S_1=-\frac{1}{4}\left(\sqrt{3}-2\right)ab,\ \ S_2=\frac{ab}{4}.$$

Then the degree of polarization is given as (Fig. 4)

$$p=\frac{\sqrt{a^4-\sqrt{3}a^2b^2+b^4}}{a^2+b^2}.$$

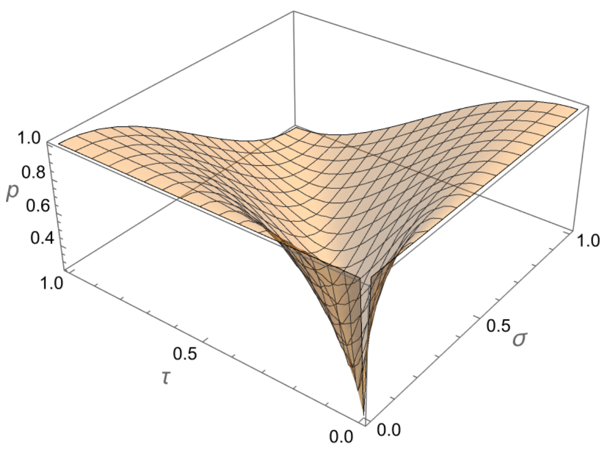

Fig. 4 – $p(a,b),\ \tau=0,\ \sigma=\pi/6.$

11. PARAMETRIZATION OF STOKES TENSOR

As shown in the above, the Stokes vector is parameterized by four independent parameters $S_a \iff a, b, \tau, \sigma$; whereas the Stokes tensor is parameterized by five independent parameters

$$S^{ab} \quad \iff \quad a, b, \tau, \sigma, \rho \;\; (\delta = \rho - \sigma - \tau). \tag{76}$$

Therefore, we can assume that the Stokes tensor S^{ab} contains more information about the partially polarized light than the Stokes vector S_a. In order to make numerical study, let us fix for parameters (see Example 1)

$$(\Psi_1^{min}) \qquad a = 1, \;\; b = 2, \;\; \tau_1 = \frac{\pi}{3}, \;\; \sigma_1 = \frac{\pi}{6}.$$

Using the formulas (61), (65), (71), we obtain (see Figs. 5-6)

$$S^{01} = -2\sin(\delta) - \frac{\cos\delta}{2}, \;\; S^{02} = \frac{1}{2}(\sin\delta + 4\cos\delta), \;\; S^{23} = 2\cos\delta - \frac{\sin\delta}{2},$$

$$S^{31} = 2\sin\delta - \frac{\cos\delta}{2}, \;\; S^{03} = \frac{1-\sqrt{3}}{2}(\cos\delta - \sin\delta), \;\; S^{12} = \frac{1-\sqrt{3}}{2}(\sin\delta + \cos\delta);$$

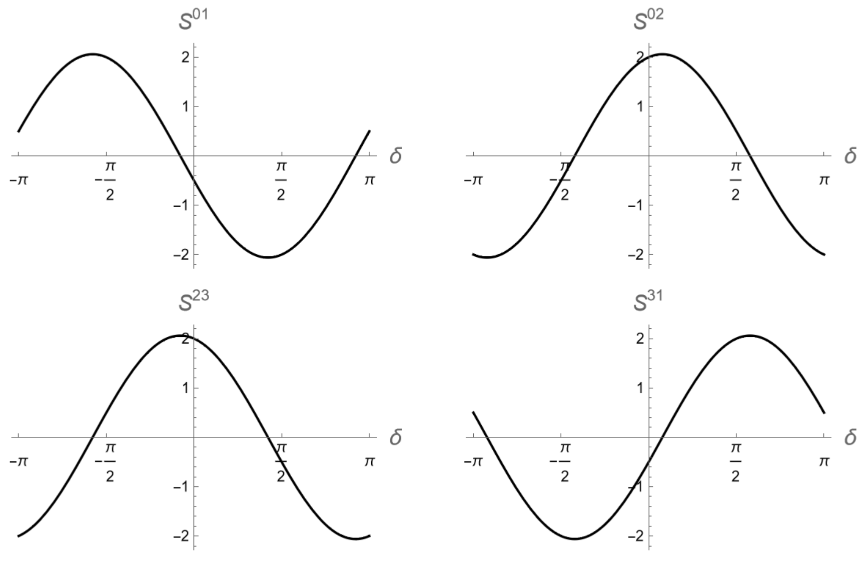

Fig. 5 – Plots $S^{01}(\delta), S^{02}(\delta), S^{23}(\delta), S^{31}(\delta)$, $a = 1, b = 2, \tau = \pi/3, \sigma = \pi/6$.

12. CONCLUSIONS

We have detailed some facts of the theory of the Lorentz group which can be relevant for solving several problems of light polarization in the frames of the vector

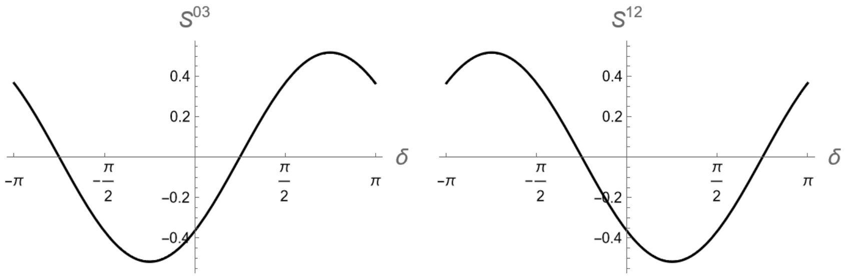

Fig. 6 – Plots $S^{03}(\delta), S^{12}(\delta)$, $a = 1, b = 2, \tau = \pi/3, \sigma = \pi/6$.

approach by Stokes, and by the spinor approach by Jones.

The concept of 4-dimensional Jones spinor has been introduced, first for a completely polarized light. They determine corresponding 4-vector and antisymmetric two-rank tensor of Stokes type. They both are isotropic. The antisymmetric tensor is equivalent to a complex three-dimensional vector, which in the frames of the Lorentz symmetry is the vector representation of the complex rotation group SO(3,C).

This approach is extended to the partially polarized light. We have introduced the concept of Jones-type 4-spinor, and have found expressions for Stokes vector and Stokes antisymmetric tensor. The analytical results are illustrated by several numerical examples.

Stokes vector depends on four parameters, whereas the Stokes tensor depends on five parameters. By this reason, we assume that the Stokes tensor contains more information about the partially polarized light than the Stokes vector.

It should be noted that in the paper we used some arguments related to the theory of the Lorentz group, however because the main results were obtained within the algebraic calculations, they are partly applicable for much more general situations in which the Lorentz symmetry does not play any role. For those cases, the combinations like $S^a S_a$, $S^{ab} S_{ab}$ are not invariant ones.

We shortly discuss relationships between the Jones type 4-spinor and the concept of space models with spinor structure.

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