

Article

The Role of Symmetry Aspects in Considering the Spin-1 Particle with Two Additional Electromagnetic Characteristics in the Presence of Both Magnetic and Electric Fields

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Abstract

In this paper, we study a generalized Duffin–Kemmer equation for a spin-1 particle with two characteristics, anomalous magnetic moment and polarizability in the presence of external uniform magnetic and electric fields. After separating the variables, we obtained a system of 10 first-order partial differential equations for 10 functions $f_A(r, z)$. To resolve this complicated problem, we first took into account existing symmetry in the structure of the derived system. The main step consisted of applying a special method for fixing the r -dependence of ten functions $f_A(r, z)$, $A = 1, \dots, 10$. We used the approach of Fedorov–Gronskiy, according to which the complete 10-component wave function is decomposed into the sum of three projective constituents. The dependence of each component on the polar coordinate r is determined by only one corresponding function, $F_i(r)$, $i = 1, 2, 3$. These three basic functions are constructed in terms of confluent hypergeometric functions, and in this process a quantization rule arises due to the presence of a magnetic field. In fact, this approach is a step-by-step algebraization of the systems of equations in partial derivatives. After that, we derived a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system was solved using the elimination method and with the help of special linear combined with the involved functions. As a result, we found three separated second-order differential equations, and their solutions were constructed in the terms of the confluent hypergeometric functions. Thus, in this paper, the three types of solutions for a vector particle with two additional electromagnetic characteristics in the presence of both external uniform magnetic and electric fields.

Keywords: spin-1 particle; anomalous magnetic moment; polarizability; magnetic field; electric field; cylindrical symmetry; projective operators; systems of equations in partial derivatives; exact solutions



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1. Introduction

The theory of spin-1 particles has an extensive history [1–30], and is closely related to Lorentz group symmetry. In addition to the classical and simplest model for a vector particle, there exist more complicated models for spin-1 particles with characteristics beyond electric charge, such as an anomalous magnetic moment, electrical quadruple moment, and so on.

In [29], within the general method of Gel'fand–Yaglom [3], a relativistic generalized system of first-order equations was constructed for a spin-1 particle with two additional characteristics: anomalous magnetic moment and polarizability. The primary derivation of the generalized equation for a spin-1 particle with two characteristics in addition to electric charge is a separate and rather involved task; therefore, in the present paper we started from the known result of previous work. In fact, this approach is based on the use of an extended set of irreducible representations of the proper Lorentz group to produce more general and complicated equations for a particle with a fixed value of spin. First, the model was developed for a free particle, and a system of spinor equations was obtained; then it was transformed into tensor form. In tensor form, the presence of external electromagnetic fields was taken into account. After eliminating the accessory variables of the complete wave function, the generalized Proca system of 10 equations was derived; it contains two additional interaction terms, which are interpreted as corresponding to the anomalous magnetic moment and polarizability.

In [30], this equation was solved in the presence of a uniform magnetic field.

In the present paper, we considered a situation in which both fields, magnetic and electric, were presented. After separating the variables, we obtained a system of 10 first-order partial differential equations for 10 functions $f_A(r, z)$. To resolve this complicated problem, we took into account the specific symmetry in the structure of the derived system. Accordingly, the complete wave function, consisting of 10 variables $f_A(r, z)$, $A = 1, \dots, 10$ is decomposed into the sum of three projective constituents. The dependence of each component on the polar coordinate is determined by only one function, $F_i(r)$, $i = 1, 2, 3$ which are constructed in terms of confluent hypergeometric functions. In this process, a quantization rule arises due to the presence of a magnetic field.

After that, we derive a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved, and as the result, we obtain three independent solutions.

We can readily verify that, when polarizability parameter vanishes and the electric field is absent, the known results for the energy spectra of a vector particle with an anomalous magnetic moment in the presence of an external uniform magnetic field are recovered.

2. Matrix Equation in Minkowski Space

We start with the following tensor equations (let $D_a = \partial_a + ieA_a$)

$$D^b \Phi_{ab} + e\mu F_{ab} \Phi^b + e\sigma D_a (F^{cd} \Phi_{cd}) - M\Phi_a = 0, \quad D_a \Phi_b - D_b \Phi_a - M\Phi_{ab} = 0; \quad (1)$$

which can be compared with the ordinary Proca system:

$$D^b \Phi_{ab} - M\Phi_a = 0, \quad D_a \Phi_b - D_b \Phi_a - M\Phi_{ab} = 0. \quad (2)$$

In Equation (1), we can see two additional interaction terms, proportional to parameters μ (anomalous magnetic moment) and σ (polarizability); in [29,30], it was proved that both parameters μ, σ are imaginary: $\mu \Rightarrow i\mu, \sigma \Rightarrow i\sigma$, we will take this into account later on. Below, we apply the 10-dimensional column:

$$\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) = (H_1; H_2).$$

Let us recall the matrix form of the Proca system when $\mu = 0, \sigma = 0$. The first equation gives $K^a D_a H_2 - M H_1 = 0$, where

$$K^0 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{vmatrix}, K^1 = \begin{vmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{vmatrix},$$

$$K^2 = \begin{vmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{vmatrix}, K^3 = \begin{vmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

The second equation in (2) leads to $D_a L^a H_1 - M H_2 = 0$, where

$$L^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}, L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, L^2 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}, L^3 = \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ -1 & 0 & 0 & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{vmatrix}.$$

Thus, the system of equations for the ordinary spin-1 particle is presented in block form as

$$K^a D_a H_2 - M H_1 = 0, \quad L^a D_a H_1 - M H_2 = 0.$$

Let us detail the first additional term in (1) (considering identities: $F_{01}\Phi^1 = -E^1\Phi_1$, $F_{12}\Phi^2 = B^3\Phi_2$, and so on)

$$e\mu F_{ab}\Phi^b = e\mu \begin{vmatrix} 0 & -E^1 & -E^2 & -E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{vmatrix} \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix},$$

when allowing for the structure of six Lorentzian generators for a vector field

$$j^{23} = S_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, j^{31} = S_2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, j^{12} = S_3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$j^{01} = T_1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, j^{02} = T_2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, j^{03} = T_3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

we obtain a shorter presentation

$$e\mu F_{\alpha\beta}\Phi^\beta \implies -e\mu (\mathbf{S}\mathbf{B} + \mathbf{T}\mathbf{E})H_1.$$

The second additional term in (1) is

$$e\sigma D_a(F^{cd}\Phi_{cd}) = 2e\sigma D_a \begin{vmatrix} E^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & E^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & E^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -B^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B^3 \end{vmatrix} \begin{vmatrix} \Phi_{01} \\ \Phi_{02} \\ \Phi_{03} \\ \Phi_{23} \\ \Phi_{31} \\ \Phi_{12} \end{vmatrix}.$$

So, we have a generalized Duffin–Kemmer–Petiau equation in block form

$$\begin{aligned} \left(K^a D_a H_2\right)_c - e\mu \left[(\mathbf{S}\mathbf{B} + \mathbf{T}\mathbf{E})H_1\right]_c + e\sigma D_c(F^{kl}\Phi_{kl}) - M(H_1)_c &= 0, \\ \left(L^a D_a H_1\right)_{[kl]} - M(H_2)_{[kl]} &= 0, \end{aligned} \quad (3)$$

where the symbol c denotes the vector index $c = (0,1,2,3)$; the indices $[kl]$ numerate the independent components of the antisymmetric tensor, $[01], [02], [03], [23], [31], [12]$.

3. Extension to Curved Space–Time Models

In Riemannian space, we start with more complicated equations

$$D_\beta \Phi_\alpha^\beta + e\mu F_{\alpha\beta} \Phi^\beta + e\sigma \hat{\partial}_\alpha (F^{\rho\delta} \Phi_{\rho\delta}) - M\Phi_\alpha = 0, \quad D_\alpha \Phi_\beta - D_\beta \Phi_\alpha - M\Phi_{\alpha\beta} = 0;$$

below, two different derivative symbols will be used: $D_\alpha = \nabla_\alpha + ieA_\alpha$, $\hat{\partial}_\alpha = \partial_\alpha + ieA_\alpha$.

Let us transform these equations to tetrad form (apply the notation $e_{(b)}^\beta (\partial_\beta + ieA_\beta) = \hat{\partial}_{(b)}$). Using Ricci rotation coefficients, we present the above equations as follows

$$\begin{aligned} \hat{\partial}_{(b)} \Phi_c^b - \gamma_{acb} \Phi^{ab} + e_{(b);\beta}^\beta \Phi_c^b + e\mu F_{cb} \Phi^b + e\sigma (\partial_{(c)} F^{ab}) \Phi_{ab} + e\sigma F^{ab} \hat{\partial}_{(c)} \Phi_{ab} - M\Phi_c &= 0, \\ \hat{\partial}_{(c)} \Phi_d - \hat{\partial}_{(d)} \Phi_c + \gamma_{bdc} \Phi^b - \gamma_{bcd} \Phi^b - M\Phi_{cd} &= 0. \end{aligned}$$

We recall the known matrix tetrad form of the equation for an ordinary vector particle

$$\left[\beta^c \left(e_{(c)}^\alpha(x) \frac{\partial}{\partial x^\alpha} + \frac{1}{2} J^{ab} \gamma_{abc}\right) - M\right] \Phi = 0, \quad \Phi = \begin{vmatrix} \Phi_a \\ \Phi_{ab} \end{vmatrix} = \begin{vmatrix} H_1 \\ H_2 \end{vmatrix}; \quad (4)$$

The two additional interactions terms are

$$e\mu F_{\alpha\beta} \Phi^\beta = e\mu (\mathbf{S}\mathbf{B} - \mathbf{T}\mathbf{E}) H_1, \quad e\sigma \hat{\partial}_\alpha (F^{cd} \Phi_{cd}) = e\sigma \begin{vmatrix} \hat{\partial}_{(0)} (F^{cd} \Phi_{cd}) \\ \hat{\partial}_{(1)} (F^{cd} \Phi_{cd}) \\ \hat{\partial}_{(2)} (F^{cd} \Phi_{cd}) \\ \hat{\partial}_{(3)} (F^{cd} \Phi_{cd}) \end{vmatrix}.$$

So, we have the following generalized system of equations

$$\begin{aligned} \left(K^c D_c^{(2)} H_2\right)_n - e\mu \left[(\mathbf{S}\mathbf{B} + \mathbf{T}\mathbf{E})H_1\right]_n + e\sigma e_{(n)}^\alpha \hat{\partial}_\alpha (F^{kl} \Phi_{kl}) - M(H_1)_n &= 0, \\ \left(L^c D_c^{(1)} H_1\right)_{[kl]} - (MH_2)_{[kl]} &= 0. \end{aligned} \quad (5)$$

4. Particle in the Uniform Magnetic and Electric Fields

It is convenient to use the cylindrical coordinates $x^\alpha = (t, r, \phi, z)$. The relevant tetrad, Ricci rotation coefficients, and the uniform magnetic and electric fields are determined as

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad e_{(a)}^\alpha = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad (6)$$

$$\gamma_{122} = \frac{1}{r}, A_0 = -Ez, F_{03} = e_{(0)}^t e_{(3)}^z F_{tz} = E, \hat{\partial}_0 \implies -i\epsilon - ieEz,$$

$$A_\phi = -Br^2/2, \quad F_{r\phi} = -Br, \quad F_{12} = e_{(1)}^r e_{(2)}^\phi F_{r\phi} = -B.$$

Correspondingly, the system of equations takes on the form

$$\left[K^0(\partial_0 - ieEz) + K^1\partial_r + K^2\frac{1}{r}\left(\partial_\phi + \frac{ieBr^2}{2} + j_2^{12}\right) + K^3\partial_z \right] H_2 - MH_1$$

$$-e\mu B j_1^{12} H_1 - e\mu E j_1^{03} H_1 - 2Be\sigma \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} \Phi_{12} + 2Ee\sigma \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} \Phi_{03} = 0,$$

$$\left[L^0(\partial_0 - ieEz) + L^1\partial_r + L^2\frac{1}{r}(\partial_\phi + ieBr^2/2 + j_1^{12}) + L^3\partial_z \right] H_1 - MH_2 = 0.$$

It is more convenient to apply the so-called cyclic basis. It is defined by requirement to have a diagonal generator j_1^{12} for the vector field $H_1 = (\Phi_l)$. The necessary transformation $\tilde{\Phi} = U\Phi$ is

$$U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \tilde{j}^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \end{vmatrix}. \quad (7)$$

Vector and tensor generators are transformed according to the rules

$$\tilde{j}_1^{ab} = U j_1^{ab} U^{-1}, \quad \tilde{j}_2^{ab} = \tilde{j}^{ab} \otimes I + I \otimes \tilde{j}^{ab};$$

so that

$$J_{(2)}^{12} = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \implies \tilde{J}_2^{12} = i \begin{vmatrix} -1 & . & . & . & . & . \\ . & 0 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 0 & . \\ . & . & . & . & . & -1 \end{vmatrix}.$$

We should also transform the Duffin–Kemmer matrices β^a to the cyclic basis. It is convenient to apply the block presentation: $\tilde{H}_1 = C_1 H_1$, ($C_1 = U$), $\tilde{H}_2 = (U \otimes U) H_2 = C_2 H_2$; further, we obtain

$$\begin{vmatrix} 0 & \tilde{K}^a \\ \tilde{L}^a & 0 \end{vmatrix} = \begin{vmatrix} 0 & C_1 K^a C_2^{-1} \\ C_2 L^a C_1^{-1} & 0 \end{vmatrix}.$$

We derive

$$C_1 = U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad C_2 = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix}. \quad (8)$$

Further, we readily find the necessary blocks:

$$\bar{K}^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix}, \quad \bar{K}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{K}^2 = \begin{vmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \bar{K}^3 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

$$\bar{L}^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad \bar{L}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{L}^2 = \begin{vmatrix} -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \quad \bar{L}^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix},$$

and expressions for the necessary generators

$$\bar{J}_1^{12} = i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}, \quad \bar{J}_1^{03} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Considering this, we can transform the above two equations to the cyclic basis; so we obtain

$$\left[\bar{K}^0 (\partial_0 - ieEz) + \bar{K}^1 \partial_r + \bar{K}^2 \frac{1}{r} \left(\partial_\phi + \frac{ieBr^2}{2} + \bar{J}_2^{12} \right) + \bar{K}^3 \partial_z \right] \bar{H}_2 - M \bar{H}_1$$

$$\begin{aligned}
& -e\mu B\tilde{j}_1^{12}\tilde{H}_1 - e\mu E\tilde{j}_1^{03}\tilde{H}_1 - 2Be\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} (-i\tilde{\Phi}_{31}) \\
& + 2Ee\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (\partial_0 - ieEz) \\ \partial_r \\ \frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2}) \\ \partial_z \end{vmatrix} \tilde{\Phi}_{02} = 0,
\end{aligned}$$

so that

$$[\bar{L}^0(\partial_0 - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(\partial_\phi + \frac{ieBr^2}{2} + \tilde{j}_1^{12}) + \bar{L}^3\partial_z]\tilde{H}_1 - M\tilde{H}_2 = 0, \quad (9)$$

$$[\bar{L}^0(\partial_0 - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(\partial_\phi + \tilde{j}_1^{12}) + \bar{L}^3\partial_z]\tilde{H}_1 - M\tilde{H}_2 = 0. \quad (10)$$

Now, let us perform separation of the variables, applying the substitution

$$\tilde{H}_1 = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \tilde{\Phi}_0(r, z) \\ \tilde{\Phi}_1(r, z) \\ \tilde{\Phi}_2(r, z) \\ \tilde{\Phi}_3(r, z) \end{vmatrix}, \tilde{H}_2 = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \tilde{\Phi}_{01}(r, z) \\ \tilde{\Phi}_{02}(r, z) \\ \tilde{\Phi}_{03}(r, z) \\ \tilde{\Phi}_{23}(r, z) \\ \tilde{\Phi}_{31}(r, z) \\ \tilde{\Phi}_{12}(r, z) \end{vmatrix} = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \bar{E}_1(r, z) \\ \bar{E}_2(r, z) \\ \bar{E}_3(r, z) \\ \bar{B}_1(r, z) \\ \bar{B}_2(r, z) \\ \bar{B}_3(r, z) \end{vmatrix},$$

in this way, we obtain (for brevity let us make the change in notations: $eB \Rightarrow B$, $eE \Rightarrow E$)

$$[\bar{K}^0(-i\epsilon - iEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}(im + \frac{iBr^2}{2} + \tilde{j}_2^{12}) + \bar{K}^3\partial_z]\tilde{H}_2 - M\tilde{H}_1$$

$$\begin{aligned}
& -\mu B\tilde{j}_1^{12}\tilde{H}_1 - 2B\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (-i\epsilon - iEz) \\ \partial_r \\ \frac{1}{r}(im + \frac{iBr^2}{2}) \\ \partial_z \end{vmatrix} (-i\bar{B}_2) \\
& -\mu E\tilde{j}_1^{03}\tilde{H}_1 + 2E\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} (-i\epsilon - iEz) \\ \partial_r \\ \frac{1}{r}(im + \frac{iBr^2}{2}) \\ \partial_z \end{vmatrix} \bar{E}_2 = 0,
\end{aligned}$$

$$[\bar{L}^0(-i\epsilon - iEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(im + \frac{iBr^2}{2} + \tilde{j}_1^{12}) + \bar{L}^3\partial_z]\tilde{H}_1 - M\tilde{H}_2 = 0.$$

Further, we obtain the explicit form of 10 equations (for brevity, we will omit the overline symbol). With the use of shortening notations

$$\begin{aligned}
a_m &= \frac{\partial}{\partial r} + \frac{Br}{2} + \frac{m}{r}, & a_{m+1} &= \frac{\partial}{\partial r} + \frac{Br}{2} + \frac{m+1}{r}, \\
b_m &= \frac{\partial}{\partial r} - \frac{Br}{2} - \frac{m}{r}, & b_{m-1} &= \frac{\partial}{\partial r} - \frac{Br}{2} - \frac{m-1}{r},
\end{aligned} \quad (11)$$

these equations read

$$\begin{aligned}
 1 \quad & \frac{1}{\sqrt{2}}b_{m-1}E_1 - \frac{1}{\sqrt{2}}a_{m+1}E_3 - \partial_z E_2 + \mu E\Phi_2 - 2iE\sigma(\epsilon + Ez)E_2 + 2B\sigma(Ez + \epsilon)B_2 = M\Phi_0, \\
 2 \quad & i(\epsilon + Ez)E_1 + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)a_m B_2 - \partial_z B_3 \sqrt{2}E\sigma a_m E_2 + iB\mu\Phi_1 = M\Phi_1, \\
 3 \quad & i(\epsilon + Ez)E_2 - \frac{1}{\sqrt{2}}a_{m+1}B_1 - \frac{1}{\sqrt{2}}b_{m-1}B_3 - \mu E\Phi_0 + 2E\sigma\partial_z E_2 + 2iB\sigma\partial_z B_2 = M\Phi_2, \\
 4 \quad & i(\epsilon + Ez)E_3 + \frac{1}{\sqrt{2}}i(2B\sigma - i)b_m B_2 + \partial_z B_1 + \sqrt{2}E\sigma b_m E_2 - iB\mu\Phi_3 = M\Phi_3, \\
 5 \quad & \frac{1}{\sqrt{2}}a_m\Phi_0 - i(\epsilon + Ez)\Phi_1 = ME_1, \quad 6 \quad -\partial_z\Phi_0 - i(\epsilon + Ez)\Phi_2 = ME_2, \\
 7 \quad & -\frac{1}{\sqrt{2}}b_m\Phi_0 - i(\epsilon + Ez)\Phi_3 = ME_3, \quad 8 \quad -\frac{1}{\sqrt{2}}b_m\Phi_2 + \partial_z\Phi_3 = MB_1, \\
 9 \quad & \frac{1}{\sqrt{2}}b_{m-1}\Phi_1 + \frac{1}{\sqrt{2}}a_{m+1}\Phi_3 = MB_2, \quad 10 \quad -\frac{1}{\sqrt{2}}a_m\Phi_2 - \partial_z\Phi_1 = MB_3.
 \end{aligned}$$

5. Projective Operators Method

To analyze the system of equations, we will use the method of projective operators (following the method of Fedorov and Gronskiy [31]). To this end, we consider the third spin projection $Y = -i\bar{J}^{12}$, and make sure that it satisfies the minimal equation $Y(Y - 1)(Y + 1) = 0$. This minimal equation allows us to introduce three projective operators

$$P_0 = 1 - Y^2, \quad P_{+1} = \frac{1}{2}Y(Y + 1), \quad P_{-1} = \frac{1}{2}Y(Y - 1), \quad (12)$$

with the necessary properties

$$P_0^2 = P_0, \quad P_{+1}^2 = P_{+1}, \quad P_{-1}^2 = P_{-1}, \quad P_0 + P_{+1} + P_{-1} = 1.$$

Accordingly, the complete wave function can be expanded in the sum of three parts

$$\Phi = \Phi_0 + \Phi_{+1} + \Phi_{-1}, \quad \Phi_\sigma = P_\sigma\Phi, \quad \sigma = 0, +1, -1. \quad (13)$$

These components have the following dependence on the variable r (in accordance with the Fedorov–Gronsky method, each projective component should be determined by only one function of the polar coordinate r):

$$\Phi_0 = \begin{vmatrix} \Phi_0(z) \\ 0 \\ \Phi_2(z) \\ 0 \\ 0 \\ \bar{E}_2(z) \\ 0 \\ 0 \\ \bar{B}_2(z) \\ 0 \end{vmatrix} F_1(r), \quad \Phi_{+1} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \Phi_3 \\ 0 \\ 0 \\ \bar{E}_3(z) \\ \bar{B}_1(z) \\ 0 \\ 0 \end{vmatrix} F_2(r), \quad \Phi_{-1} = \begin{vmatrix} 0 \\ \Phi_1(z) \\ 0 \\ 0 \\ \bar{E}_1(z) \\ 0 \\ 0 \\ 0 \\ 0 \\ \bar{B}_3(z) \end{vmatrix} F_3(r); \quad (14)$$

$$\begin{aligned}
 1 \quad & \frac{1}{\sqrt{2}}E_1(z)b_{m-1}F_3(r) - \frac{1}{\sqrt{2}}E_3(z)a_{m+1}F_2(r) - \partial_z E_2(z)F_1(r) \\
 & - \mu E\Phi_2(z)F_1(r) - 2iE\sigma(\epsilon + Ez)E_2(z)F_1(r) \\
 & + 2B\sigma(Ez + \epsilon)B_2(z)F_1(r) = M\Phi_0(z)F_1(r), \\
 2 \quad & i(\epsilon + Ez)E_1(z)F_3(r) + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)B_2(z)a_m F_1(r) - \partial_z B_3(z)F_3(r)
 \end{aligned}$$

$$\begin{aligned}
& -\sqrt{2}E\sigma E_2(z)a_m F_1(r) + iB\mu\Phi_1(z)F_3(r) = M\Phi_1(z)F_3(r), \\
3 \quad & i(\epsilon + Ez)E_2(z)F_1(r) - \frac{1}{\sqrt{2}}B_1(z)a_{m+1}F_2(r) - \frac{1}{\sqrt{2}}B_3(z)b_{m-1}F_3(r) \\
& -\mu E\Phi_0(z)F_1(r) + 2E\sigma\partial_z E_2(z)F_1(r) + 2iB\sigma\partial_z B_2(z)F_1(r) = M\Phi_2(z)F_1(r), \\
4 \quad & i(\epsilon + Ez)E_3(z)F_2(r) + \frac{1}{\sqrt{2}}i(2B\sigma - i)B_2(z)b_m F_1(r) + \partial_z B_1(z)F_2(r) \\
& + \sqrt{2}E\sigma E_2(z)b_m F_1(r) - iB\mu\Phi_3(z)F_2(r) = M\Phi_3(z)F_2(r), \\
5 \quad & \frac{1}{\sqrt{2}}\Phi_0(z)a_m F_1(r) - i(\epsilon + Ez)\Phi_1(z)F_3(r) = ME_1(z)F_3(r), \\
6 \quad & -\partial_z\Phi_0(z)F_1(r) - i(\epsilon + Ez)\Phi_2(z)F_1(r) = ME_2(z)F_1(r), \\
7 \quad & -\frac{1}{\sqrt{2}}\Phi_0(z)b_m F_1(r) - i(\epsilon + Ez)\Phi_3(z)F_2(r) = ME_3(z)F_2(r), \\
8 \quad & -\frac{1}{\sqrt{2}}\Phi_2(z)b_m F_1(r) + \partial_z\Phi_3(z)F_2(r) = MB_1(z)F_2(r), \\
9 \quad & \frac{1}{\sqrt{2}}\Phi_1(z)b_{m-1}F_3(r) + \frac{1}{\sqrt{2}}\Phi_3(z)F_2(r) = MB_2(z)F_1(r), \\
10 \quad & -\frac{1}{\sqrt{2}}\Phi_2(z)a_m F_1(r) - \partial_z\Phi_1(z)F_3(r) = MB_3(z)F_3(r).
\end{aligned}$$

In order to obtain equations in the variable z , we impose the following constraints

$$b_{m-1}F_3 = C_1F_1, \quad a_m F_1 = C_4F_3, \quad a_{m+1}F_2 = C_2F_1, \quad b_m F_1 = C_3F_2, \quad (15)$$

so, we obtain

$$\begin{aligned}
1 \quad & \frac{1}{\sqrt{2}}E_1(z)C_1 - \frac{1}{\sqrt{2}}E_3(z)C_2 - \partial_z E_2(z) \\
& -\mu E\Phi_2(z) - 2iE\sigma(\epsilon + Ez)E_2(z) + 2B\sigma(Ez + \epsilon)B_2(z) = M\Phi_0(z), \\
2 \quad & i(\epsilon + Ez)E_1(z) + \frac{1}{\sqrt{2}}(1 - 2iB\sigma)B_2(z)C_4 - \partial_z B_3(z) \\
& -\sqrt{2}E\sigma E_2(z)C_4 + iB\mu\Phi_1(z) = M\Phi_1(z), \\
3 \quad & +i(\epsilon + Ez)E_2(z) - \frac{1}{\sqrt{2}}B_1(z)C_2 - \frac{1}{\sqrt{2}}B_3(z)C_1 \\
& -\mu E\Phi_0(z) + 2E\sigma\partial_z E_2(z) + 2iB\sigma\partial_z B_2(z) = M\Phi_2(z), \\
4 \quad & i(\epsilon + Ez)E_3(z) + \frac{1}{\sqrt{2}}i(2B\sigma - i)B_2(z)C_3 + \partial_z B_1(z) \\
& + \sqrt{2}E\sigma E_2(z)C_3 - iB\mu\Phi_3(z) = M\Phi_3(z), \\
5 \quad & \frac{1}{\sqrt{2}}\Phi_0(z)C_4 - i(\epsilon + Ez)\Phi_1(z) = ME_1(z), \quad 6 \quad -\partial_z\Phi_0(z) - i(\epsilon + Ez)\Phi_2(z) = ME_2(z), \\
7 \quad & -\frac{1}{\sqrt{2}}\Phi_0(z)C_3 - i(\epsilon + Ez)\Phi_3(z) = ME_3(z), \quad 8 \quad -\frac{1}{\sqrt{2}}\Phi_2(z)C_3 + \partial_z\Phi_3(z) = MB_1(z), \\
9 \quad & \frac{1}{\sqrt{2}}\Phi_1(z)C_1 + \frac{1}{\sqrt{2}}\Phi_3(z)C_2 = MB_2(z), \\
10 \quad & -\frac{1}{\sqrt{2}}\Phi_2(z)C_4 - \partial_z\Phi_1(z) = MB_3(z).
\end{aligned}$$

6. Explicit Form of Three Basic Functions

In the differential constraints

$$b_{m-1}F_3(r) = C_1F_1(r), \quad a_m F_1(r) = C_4F_3(r),$$

$$a_{m+1}F_2(r) = C_2F_1(r), \quad b_m F_1(r) = C_3F_2(r),$$

the parameters in each pair can be chosen to be the same: $C_4 = C_1$, $C_3 = C_2$. So, we obtain the following constraints

$$b_{m-1}F_3(r) = C_1F_1(r), \quad a_mF_1(r) = C_1F_3(r),$$

$$a_{m+1}F_2(r) = C_2F_1(r), \quad b_mF_1(r) = C_2F_2(r);$$

and the resulting second-order equations read

$$(b_{m-1}a_m - C_1^2)F_1 = 0, \quad (a_mb_{m-1} - C_1^2)F_3 = 0,$$

$$(a_{m+1}b_m - C_2^2)F_1 = 0, \quad (b_ma_{m+1} - C_2^2)F_2 = 0.$$

These equations explicitly read

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\right)F_1 = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm - B - C_2^2\right)F_1 = 0,$$

therefore, $C_2^2 = C_1^2 - 2B$ and

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\right)F_3 = 0,$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_2^2\right)F_2 = 0.$$

Thus, we have only three equations and the constraint $C_2^2 = C_1^2 - 2B$:

$$1, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + B - C_1^2\right)F_1 = 0,$$

$$2, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - Bm - C_1^2 + 2B\right)F_2 = 0,$$

$$3, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - Bm - C_1^2\right)F_3 = 0.$$

With the notation $B - C_1^2 = X$, the equations take on the form

$$1, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{m^2}{r^2} - Bm + X\right)F_1 = 0,$$

$$2, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m+1)^2}{r^2} - B(m-1) + X\right)F_2 = 0,$$

$$3, \quad \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{B^2r^2}{4} - \frac{(m-1)^2}{r^2} - B(m+1) + X\right)F_3 = 0.$$

In the variable $x = \frac{Br^2}{2}$, we readily find their solutions

$$1, \quad F_1(x) = x^{+\frac{|m|}{2}} e^{-x/2} F_1(x), \quad F_1(x) = \Phi(-n_1, |m| + 1, x),$$

$$X = 2B\left(\frac{|m| + m}{2} + \frac{1}{2} + n_1\right) > B, \quad n_1 = 0, 1, 2, \dots ;$$

$$2, \quad F_2(x) = x^{+\frac{|m+1|}{2}} e^{-x/2} F_3(x), \quad F_3(x) = \Phi(-n_3, |m+1| + 1, x),$$

$$X = 2B\left(\frac{|m+1| + m - 1}{2} + \frac{1}{2} + n_2\right) > B, \quad n_2 = 0, 1, 2, \dots ;$$

$$3, \quad F_3(x) = x^{+\frac{|m-1|}{2}} e^{-x/2} F_2(x), \quad F_2(x) = \Phi(-n_2, |m-1| + 1, x),$$

$$X = 2B \left(\frac{|m-1| + m + 1}{2} + \frac{1}{2} + n_3 \right) > B, \quad n_3 = 0, 1, 2, \dots$$

In all three cases, the quantity X is the same; below, we apply the variant

$$X = 2BN > 0, \quad N = \frac{|m| + m}{2} + \frac{1}{2} + n, \quad n = 0, 1, 2, \dots, \quad (16)$$

where the parameter N takes half-integer values; note the formulas

$$C_4 = C_1 = i\sqrt{X-B}, \quad C_3 = C_2 = i\sqrt{X+B}. \quad (17)$$

7. Solving Equations in z -Variable

Let us turn to the system in the z -variable, allowing for (17). It is convenient to divide the resulting equations into two groups:

Subsystem *I* (it is algebraic with respect to the variables $\bar{\Phi}_0, \bar{\Phi}_2, \bar{E}_1, \bar{E}_3, \bar{B}_1, \bar{B}_3$)

$$\begin{aligned} 1 \quad & \frac{1}{\sqrt{2}} E_1(z) i\sqrt{X-B} - \frac{1}{\sqrt{2}} E_3(z) i\sqrt{X+B} - \partial_z E_2(z) \\ & - \mu E \Phi_2(z) - 2iE\sigma(\epsilon + Ez)E_2(z) + 2B\sigma(Ez + \epsilon)B_2(z) = M\Phi_0(z), \\ 3 \quad & i(\epsilon + Ez)E_2(z) - \frac{1}{\sqrt{2}} B_1(z) i\sqrt{X+B} - \frac{1}{\sqrt{2}} B_3(z) i\sqrt{X-B} \\ & - \mu E \Phi_0(z) + 2E\sigma\partial_z E_2(z) + 2iB\sigma\partial_z B_2(z) = M\Phi_2(z), \\ 5 \quad & \frac{1}{\sqrt{2}} \Phi_0(z) i\sqrt{X-B} - i(\epsilon + Ez)\Phi_1(z) = ME_1(z), \\ 7 \quad & -\frac{1}{\sqrt{2}} \Phi_0(z) i\sqrt{X+B} - i(\epsilon + Ez)\Phi_3(z) = ME_3(z), \\ 8 \quad & -\frac{1}{\sqrt{2}} \Phi_2(z) i\sqrt{X+B} + \partial_z \Phi_3(z) = MB_1(z), \\ 10 \quad & -\frac{1}{\sqrt{2}} \Phi_2(z) i\sqrt{X-B} - \partial_z \Phi_1(z) = MB_3(z); \end{aligned}$$

Subsystem *II*

$$\begin{aligned} 2 \quad & i(\epsilon + Ez)E_1(z) + \frac{1}{\sqrt{2}} (1 - 2iB\sigma)B_2(z) i\sqrt{X-B} - \partial_z B_3(z) \\ & - \sqrt{2}E\sigma E_2(z) i\sqrt{X-B} + iB\mu\Phi_1(z) = M\Phi_1(z), \\ 4 \quad & i(\epsilon + Ez)E_3(z) + \frac{1}{\sqrt{2}} i(2B\sigma - i)B_2(z) i\sqrt{X+B} + \partial_z B_1(z) \\ & + \sqrt{2}E\sigma E_2(z) i\sqrt{X+B} - iB\mu\Phi_3(z) = M\Phi_3(z), \\ 6 \quad & -\partial_z \Phi_0(z) - i(\epsilon + Ez)\Phi_2(z) = ME_2(z), \\ 9 \quad & \frac{1}{\sqrt{2}} \Phi_1(z) i\sqrt{X-B} + \frac{1}{\sqrt{2}} \Phi_3(z) i\sqrt{X+B} = MB_2(z); \end{aligned}$$

Equation (9) permits eliminating the variable B_2 from the previous three.

Let us resolve the system *I* with respect to the variables $\bar{\Phi}_0, \bar{\Phi}_2, \bar{E}_1, \bar{E}_3, \bar{B}_1, \bar{B}_3$:

$$\Phi_0 = \frac{1}{2 \left((M^2 + X)^2 - M^2 E^2 \mu^2 \right)}$$

$$\begin{aligned}
& \times \left[4BM \left((M^2 + X)(Ez + \epsilon) - i\partial_z ME\mu \right) \sigma B_2 - 2M \left(iE(Ez + \epsilon) \left(M\mu + 2(M^2 + X)\sigma \right) \right. \right. \\
& \left. \left. + \partial_z (M^2 + 2E^2\mu\sigma M + X) \right) E_2 + \sqrt{2} \left((M^2 + X)(Ez + \epsilon) - i\partial_z ME\mu \right) \left(\sqrt{X-B}\Phi_1 - \sqrt{B+X}\Phi_3 \right) \right], \\
\Phi_2 = & \frac{1}{2 \left((M^2 + X)^2 - M^2 E^2 \mu^2 \right)} \left[i \left(4BM \left(\partial_z (M^2 + X) + iME(Ez + \epsilon)\mu \right) \sigma B_2 \right. \right. \\
& \left. \left. + 2M \left(EzM^2 + \epsilon M^2 - i\partial_z E\mu M + EXz + X\epsilon + 2E \left(ME(Ez + \epsilon)\mu \right. \right. \right. \right. \\
& \left. \left. \left. - i\partial_z (M^2 + X) \right) \sigma \right) E_2 + \sqrt{2} \left(\partial_z (M^2 + X) + iME(Ez + \epsilon)\mu \right) \left(\sqrt{X-B}\Phi_1 - \sqrt{B+X}\Phi_3 \right) \right], \\
E_1 = & -\frac{1}{2M \left(M^2 - E\mu M + X \right) \left(M^2 + E\mu M + X \right)} \\
& \times \left[iE\sqrt{X^2 - B^2}z\Phi_3 M^2 + i\sqrt{X^2 - B^2}\epsilon\Phi_3 M^2 \right. \\
& \left. - 2i\sqrt{2}B\sqrt{X-B} \left((M^2 + X)(Ez + \epsilon) - i\partial_z ME\mu \right) \sigma B_2 M \right. \\
& \left. + \sqrt{2}\sqrt{X-B} \left(i\partial_z (M^2 + 2E^2\mu\sigma M + X) - E(Ez + \epsilon) \left(M\mu + 2(M^2 + X)\sigma \right) \right) E_2 M \right. \\
& \left. + \partial_z E\sqrt{X-B}\sqrt{B+X}\Phi_3 M + i \left(-2M^2 E^2 (Ez + \epsilon)\mu^2 \right. \right. \\
& \left. \left. + i\partial_z ME(X-B)\mu + (M^2 + X)(2M^2 + B + X)(Ez + \epsilon) \right) \Phi_1 \right. \\
& \left. + iEX\sqrt{X^2 - B^2}z\Phi_3 + iX\sqrt{X^2 - B^2}\epsilon\Phi_3 \right], \\
E_3 = & -\frac{1}{2M \left(M^2 - E\mu M + X \right) \left(M^2 + E\mu M + X \right)} \\
& \times \left[iE\sqrt{X^2 - B^2}z\Phi_1 M^2 + i\sqrt{X^2 - B^2}\epsilon\Phi_1 M^2 + 2i\sqrt{2}B\sqrt{B+X} \right. \\
& \times \left((M^2 + X)(Ez + \epsilon) - i\partial_z ME\mu \right) \sigma B_2 M + \sqrt{2}\sqrt{B+X} \left(E(Ez + \epsilon) \left(M\mu + 2(M^2 + X)\sigma \right) \right. \\
& \left. \left. - i\partial_z (M^2 + 2E^2\mu\sigma M + X) \right) E_2 M + \partial_z E\sqrt{X^2 - B^2}\mu\Phi_1 M \right. \\
& \left. + iEX\sqrt{X^2 - B^2}z\Phi_1 + iX\sqrt{X^2 - B^2}\epsilon\Phi_1 - iB \left((M^2 + X)(Ez + \epsilon) - i\partial_z ME\mu \right) \Phi_3 \right. \\
& \left. + i \left(-2M^2 E^2 (Ez + \epsilon)\mu^2 + i\partial_z MEX\mu + (M^2 + X)(2M^2 + X)(Ez + \epsilon) \right) \Phi_3 \right], \\
B_1 = & \frac{1}{2M \left(M^2 - E\mu M + X \right) \left(M^2 + E\mu M + X \right)} \\
& \times \left[\partial_z \sqrt{X^2 - B^2}\Phi_1 M^2 + 2\sqrt{2}B\sqrt{B+X} \left(\partial_z (M^2 + X) + iME(Ez + \epsilon)\mu \right) \sigma B_2 M \right. \\
& \left. + \sqrt{2}\sqrt{B+X} \left(EzM^2 + \epsilon M^2 - i\partial_z E\mu M + EXz + X\epsilon \right. \right. \\
& \left. \left. + 2E \left(ME(Ez + \epsilon)\mu - i\partial_z (M^2 + X) \right) \sigma \right) E_2 M + iE^2 \sqrt{X^2 - B^2}z\mu\Phi_1 M \right. \\
& \left. + iE\sqrt{X^2 - B^2}\epsilon\mu\Phi_1 M + \partial_z X\sqrt{X^2 - B^2}\Phi_1 - B \left(\partial_z (M^2 + X) + iME(Ez + \epsilon)\mu \right) \Phi_3 \right. \\
& \left. + \left(-2\partial_z M^2 E^2 \mu^2 - iMEX(Ez + \epsilon)\mu + \partial_z (M^2 + X)(2M^2 + X) \right) \Phi_3 \right], \\
B_3 = & -\frac{1}{2M \left(M^2 - E\mu M + X \right) \left(M^2 + E\mu M + X \right)} \\
& \times \left[\partial_z \sqrt{X^2 - B^2}\Phi_3 M^2 - 2\sqrt{2}B\sqrt{X-B} \left(\partial_z (M^2 + X) + iME(Ez + \epsilon)\mu \right) \sigma B_2 M \right. \\
& \left. - \sqrt{2}\sqrt{X-B} \left(EzM^2 + \epsilon M^2 - i\partial_z E\mu M + EXz + X\epsilon + 2E \left(ME(Ez + \epsilon)\mu - i\partial_z (M^2 + X) \right) \sigma \right) E_2 M \right.
\end{aligned}$$

$$+iE^2\sqrt{X-B}\sqrt{B+\bar{X}}z\mu\Phi_3M+iE\sqrt{X-B}\sqrt{B+\bar{X}}\epsilon\mu\Phi_3M \\ +\left(-2\partial_zM^2E^2\mu^2+iME(B-X)(Ez+\epsilon)\mu+\partial_z\left(M^2+X\right)\left(2M^2+B+X\right)\right)\Phi_1+\partial_zX\sqrt{X^2-B^2}\Phi_3\Big].$$

Now, substitute these expressions in equations of the group II. This results in

1

$$\left(-\frac{\sqrt{2}B\partial_z^2\sigma\sqrt{X-B}(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2}+\frac{\sqrt{X-B}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\right) \\ \times\left((2B\sigma+i)(M^2-\mu ME+X)(M^2+\mu ME+X)-2B\sigma(M^2+X)(Ez+\epsilon)^2-2iB\mu M\sigma E^2\right)B_2 \\ +\left(\frac{i\partial_z^2E\sqrt{X-B}(2\sigma(M^2+X)+\mu M)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-\frac{iE\sqrt{X-B}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\right) \\ \times(-i(M^2+2\mu M\sigma E^2+X)+2\sigma(M^2-\mu ME+X)(M^2+\mu ME+X) \\ -((Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)))E_2 \\ +\left(\frac{\partial_z^2\sqrt{X^2-B^2}(M^2+X)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}+\frac{\sqrt{X^2-B^2}((M^2+X)(Ez+\epsilon)^2+i\mu ME^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\right)\Phi_3 \\ +\left(\frac{\partial_z^2((M^2+X)(B+2M^2+X)-2\mu^2M^2E^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}+\frac{1}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\right) \\ \times\left[2iB\mu M(M^2-\mu ME+X)(M^2+\mu ME+X)+(M^2+X)(B+2M^2+X)(Ez+\epsilon)^2\right. \\ \left.+i\mu ME^2(B-X)+2\mu^2M^4E^2-2\mu^2M^2E^2(Ez+\epsilon)^2-2M^2(M^2+X)^2\right]\Phi_1=0,$$

2

$$\left(\frac{\sqrt{2}B\partial_z^2\sigma\sqrt{B+\bar{X}}(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2}\right. \\ \left.+\frac{\sqrt{B+\bar{X}}}{\sqrt{2}}\left(\frac{2B\sigma(\mu^2M^2E^2-(M^2+X)(M^2-(Ez+\epsilon)^2+X)+i\mu ME^2)}{(M^2+X)^2-\mu^2M^2E^2}+i\right)\right)B_2 \\ +\left(\frac{E\sqrt{B+\bar{X}}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}(2i\sigma(M^2-\mu ME+X)(M^2+\mu ME+X) \right. \\ \left.-i(Ez+\epsilon)^2(2\sigma(M^2+X)+\mu M)+M^2+2\mu M\sigma E^2+X) \right. \\ \left.-\frac{i\partial_z^2E\sqrt{B+\bar{X}}(2\sigma(M^2+X)+\mu M)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\right)E_2 \\ +\left(\frac{\partial_z^2\sqrt{X^2-B^2}(M^2+X)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}+\frac{\sqrt{X^2-B^2}((M^2+X)(Ez+\epsilon)^2+i\mu ME^2)}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\right)\Phi_1 \\ +\left(\frac{1}{4M}\partial_z^2\left(4-\frac{2(B+X)(M^2+X)}{(M^2+X)^2-\mu^2M^2E^2}\right)+\frac{1}{2M(M^2-\mu ME+X)(M^2+\mu ME+X)}\right) \\ \times\left[2iB\mu^3M^3E^2-i\mu M(B(2(M^2+X)^2+E^2)+E^2X)+(M^2+X)(-(B-X)(Ez+\epsilon)^2 \right. \\ \left.-2M^4+2M^2((Ez+\epsilon)^2-X))+2\mu^2M^2E^2(M-Ez-\epsilon)(M+Ez+\epsilon)\right]\Phi_3=0,$$

3

$$\left(\frac{2iB\partial_z^2\mu M^2\sigma E}{(M^2+X)^2-\mu^2M^2E^2}-\frac{2BM\sigma E(M^2-i\mu M(Ez+\epsilon)^2+X)}{(M^2+X)^2-\mu^2M^2E^2}\right)B_2 \\ +\left(\frac{i\partial_z^2\mu ME\sqrt{X-B}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-\frac{E\sqrt{X-B}(M^2-i\mu M(Ez+\epsilon)^2+X)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\right)\Phi_1 \\ +\left(\frac{E\sqrt{B+\bar{X}}(M^2-i\mu M(Ez+\epsilon)^2+X)}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}-\frac{i\partial_z^2\mu ME\sqrt{B+\bar{X}}}{\sqrt{2}((M^2+X)^2-\mu^2M^2E^2)}\right)\Phi_3 \\ +\left(\frac{1}{(M^2+X)^2-\mu^2M^2E^2}\partial_z^2M(M^2+2\mu M\sigma E^2+X) \right. \\ \left.+\frac{M}{(M^2+X)^2-\mu^2M^2E^2}(-M^4+M^2(E^2(\mu^2+2i\sigma+z^2)+2Ez\epsilon-2X+\epsilon^2)\right)$$

$$+\mu ME^2(2\sigma(Ez + \epsilon)^2 + i) + X(2i\sigma E^2 + (Ez + \epsilon)^2 - X))E_2 = 0,$$

4

$$\frac{i\Phi_1\sqrt{X-B}}{\sqrt{2}} + \frac{i\Phi_3\sqrt{B+X}}{\sqrt{2}} + B_2(-M) = 0.$$

With the help of the fourth equation, we can eliminate the variable B_2 from the three remaining equations (let us change the notations $\Phi_1 = G, \Phi_3 = H, E_2 = F$); in order to remove the fractions, we multiply each equation by

$$2M(M^2 + X - ME\mu)(M^2 + X + ME\mu)((M^2 + X)^2 - M^2E^2\mu^2),$$

so, we obtain the following three equations

1

$$\begin{aligned} & \left[i\sqrt{2}\partial_z^2 ME\sqrt{X-B}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(2\sigma(M^2 + X) + \mu M) \right. \\ & - i\sqrt{2}ME\sqrt{X-B}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(-i(M^2 + 2\mu M\sigma E^2 + X) \\ & + 2\sigma(M^2 - \mu ME + X)(M^2 + \mu ME + X) - ((Ez + \epsilon)^2(2\sigma(M^2 + X) + \mu M))) \left. \right] F \\ & + \left[\partial_z^2(M^2 - \mu ME + X)(M^2 + \mu ME + X)(2iB\sigma(B - X)(M^2 + X) \right. \\ & + BM^2 + BX + 2M^4 - 2\mu^2 M^2 E^2 + 3M^2 X + X^2) + (M^2 - \mu ME + X) \\ & \times (M^2 + \mu ME + X)(i(X - B)((2B\sigma + i)(M^2 - \mu ME + X)(M^2 + \mu ME + X) \\ & - 2B\sigma(M^2 + X)(Ez + \epsilon)^2 - 2iB\mu M\sigma E^2) + 2iB\mu M(M^2 - \mu ME + X) \\ & \times (M^2 + \mu ME + X) + (M^2 + X)(B + 2M^2 + X)(Ez + \epsilon)^2 + i\mu ME^2(B - X) \\ & + 2\mu^2 M^4 E^2 - 2\mu^2 M^2 E^2(Ez + \epsilon)^2 - 2M^2(M^2 + X)^2) \left. \right] G \\ & + \left[\partial_z^2(1 - 2iB\sigma)\sqrt{X^2 - B^2}(M^2 + X)(M^2 - \mu ME + X)(M^2 + \mu ME + X) \right. \\ & + i(2B\sigma + i)\sqrt{X^2 - B^2}((M^2 + X)^2 - \mu^2 M^2 E^2)(\mu^2(-M^2)E^2 + (M^2 + X) \\ & \times (M^2 - (Ez + \epsilon)^2 + X) - i\mu ME^2) \left. \right] H = 0, \end{aligned}$$

2

$$\begin{aligned} & \left[\sqrt{2}ME\sqrt{B+X}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(2i\sigma(M^2 - \mu ME + X)(M^2 + \mu ME + X) \right. \\ & - i(Ez + \epsilon)^2(2\sigma(M^2 + X) + \mu M) + M^2 + 2\mu M\sigma E^2 + X) \\ & - i\sqrt{2}\partial_z^2 ME\sqrt{B+X}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(2\sigma(M^2 + X) + \mu M) \left. \right] F \\ & + \left[\partial_z^2(1 + 2iB\sigma)\sqrt{X^2 - B^2}(M^2 + X)(M^2 - \mu ME + X)(M^2 + \mu ME + X) \right. \\ & - i(2B\sigma - i)\sqrt{X^2 - B^2}((M^2 + X)^2 \\ & - \mu^2 M^2 E^2)(\mu^2(-M^2)E^2 + (M^2 + X)(M^2 - (Ez + \epsilon)^2 + X) - i\mu ME^2) \left. \right] G \\ & + \left[\partial_z^2(M^2 - \mu ME + X)(M^2 + \mu ME + X)(2iB\sigma(B + X)(M^2 + X) \right. \\ & - BM^2 - BX + 2M^4 - 2\mu^2 M^2 E^2 + 3M^2 X + X^2) + (M^2 - \mu ME + X) \\ & \times (M^2 + \mu ME + X)(2iB\mu^3 M^3 E^2 - i\mu M(B(2(M^2 + X)^2 + E^2) + E^2 X) + \\ & + i(B + X)(i((M^2 + X)^2 - \mu^2 M^2 E^2) - 2B\sigma(\mu^2(-M^2)E^2 \\ & + (M^2 + X)(M^2 - (Ez + \epsilon)^2 + X) - i\mu ME^2)) \\ & + (M^2 + X)(-(B - X)(Ez + \epsilon)^2 - 2M^4 + 2M^2((Ez + \epsilon)^2 - X)) \\ & + 2\mu^2 M^2 E^2(M - Ez - \epsilon)(M + Ez + \epsilon)) \left. \right] H = 0, \end{aligned}$$

3

$$\begin{aligned}
& \left[2\partial_z^2 M^2 (M^2 - \mu ME + X)(M^2 + \mu ME + X)(M^2 + 2\mu M\sigma E^2 + X) \right. \\
& \quad + 2M^2 (M^2 - \mu ME + X)(M^2 + \mu ME + X) \\
& \quad \times (-M^4 + M^2(E^2(\mu^2 + 2i\sigma + z^2) + 2Ez\epsilon - 2X + \epsilon^2) \\
& \quad \left. + \mu ME^2(2\sigma(Ez + \epsilon)^2 + i) + X(2i\sigma E^2 + (Ez + \epsilon)^2 - X)) \right] F \\
& + \left[-\sqrt{2}\partial_z^2 \mu M^2 E(2B\sigma - i)\sqrt{X-B}(M^2 - \mu ME + X)(M^2 + \mu ME + X) \right. \\
& \quad \left. - i\sqrt{2}ME(2B\sigma - i)\sqrt{X-B}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(M^2 - i\mu M(Ez + \epsilon)^2 + X) \right] G \\
& + \left[\sqrt{2}ME(1 - 2iB\sigma)\sqrt{B+X}(M^2 - \mu ME + X)(M^2 + \mu ME + X)(M^2 - i\mu M(Ez + \epsilon)^2 + X) \right. \\
& \quad \left. - \sqrt{2}\partial_z^2 \mu M^2 E(2B\sigma + i)\sqrt{B+X}(M^2 - \mu ME + X)(M^2 + \mu ME + X) \right] H = 0,
\end{aligned}$$

Let us write the last system in symbolical form

$$\begin{aligned}
1 \quad & a_1 F'' + b_1 F + c_1 G'' + d_1 G + l_1 H'' + n_1 H = 0, \\
2 \quad & a_2 F'' + b_2 F + c_2 G'' + d_2 G + l_2 H'' + n_2 H = 0, \\
3 \quad & a_3 F'' + b_3 F + c_3 G'' + d_3 G + l_3 H'' + n_3 H = 0.
\end{aligned}$$

We will combine equations

$$(1) \cdot \alpha + (2) \cdot \beta + (3) \cdot \gamma = 0;$$

in this way, we obtain the following equation

$$\begin{aligned}
& (\alpha a_1 + \beta a_2 + \gamma a_3)F'' + (\alpha b_1 + \beta b_2 + \gamma b_3)F \\
& (\alpha c_1 + \beta c_2 + \gamma c_3)G'' + (\alpha d_1 + \beta d_2 + \gamma d_3)G \\
& (\alpha l_1 + \beta l_2 + \gamma l_3)H'' + (\alpha n_1 + \beta n_2 + \gamma n_3)H = 0.
\end{aligned}$$

We will distinguish three different cases:

$$\begin{aligned}
& \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 = 1, \\
I \quad & \alpha_1 c_1 + \beta_1 c_2 + \gamma_1 c_3 = 0, \\
& \alpha_1 l_1 + \beta_1 l_2 + \gamma_1 l_3 = 0; \\
II \quad & \alpha_2 a_1 + \beta_2 a_2 + \gamma_2 a_3 = 0, \quad \alpha_3 a_1 + \beta_3 a_2 + \gamma_3 a_3 = 0, \\
& \alpha_2 c_1 + \beta_2 c_2 + \gamma_2 c_3 = 1, \quad III \quad \alpha_3 c_1 + \beta_3 c_2 + \gamma_3 c_3 = 0, \\
& \alpha_2 l_1 + \beta_2 l_2 + \gamma_2 l_3 = 0; \quad \alpha_3 l_1 + \beta_3 l_2 + \gamma_3 l_3 = 1;
\end{aligned}$$

the corresponding three solutions have the form

$$\begin{aligned}
& \alpha_1 = \frac{\mu E(2B\sigma - i)\sqrt{X-B}}{2\sqrt{2}\left((M^2 + X)^2 - \mu^2 M^2 E^2\right)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
I \quad & \beta_1 = \frac{i\mu E(1 - 2iB\sigma)\sqrt{B+X}}{2\sqrt{2}\left((M^2 + X)^2 - \mu^2 M^2 E^2\right)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
& \gamma_1 = \frac{M^2(M^2 - \mu^2 E^2 + X) + 2iB^2\sigma(M^2 + X)}{2M^2(M^2 - \mu ME + X)^2(M^2 + \mu ME + X)^2(2iB^2\sigma + M^2 + 2\mu M\sigma E^2)};
\end{aligned}$$

$$\begin{aligned}
& \alpha_2 = \frac{2iB^2\sigma + B(-1+2i\sigma X) + 2M(M+2\mu\sigma E^2) + X}{4((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
II \quad & \beta_2 = \frac{i(2B\sigma + i)\sqrt{X-B}\sqrt{B+X}}{4((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
& \gamma_2 = -\frac{iE\sqrt{X-B}(2\sigma(M^2+X) + \mu M)}{2\sqrt{2}M((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}; \\
& \alpha_3 = \frac{i(-2B\sigma + i)\sqrt{X-B}\sqrt{B+X}}{4((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
III \quad & \beta_3 = \frac{2iB^2\sigma - 2iB\sigma X + B + 2M(M+2\mu\sigma E^2) + X}{4((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}, \\
& \gamma_3 = \frac{iE\sqrt{B+X}(2\sigma(M^2+X) + \mu M)}{2\sqrt{2}M((M^2+X)^2 - \mu^2 M^2 E^2)^2 (2iB^2\sigma + M^2 + 2\mu M\sigma E^2)}.
\end{aligned}$$

So, the above equations can be written

$$\begin{aligned}
(1) \quad & F'' + (\alpha_1 b_1 + \beta_1 b_2 + \gamma_1 b_3)F \\
& + (\alpha_1 d_1 + \beta_1 d_2 + \gamma_1 d_3)G + (\alpha_1 n_1 + \beta_1 n_2 + \gamma_1 n_3)H = 0, \\
(2) \quad & G'' + (\alpha_2 b_1 + \beta_2 b_2 + \gamma_2 b_3)F \\
& + (\alpha_2 d_1 + \beta_2 d_2 + \gamma_2 d_3)G + (\alpha_2 n_1 + \beta_2 n_2 + \gamma_2 n_3)H = 0, \\
(3) \quad & H'' + (\alpha_3 b_1 + \beta_3 b_2 + \gamma_3 b_3)F \\
& + (\alpha_3 d_1 + \beta_3 d_2 + \gamma_3 d_3)G + (\alpha_3 n_1 + \beta_3 n_2 + \gamma_3 n_3)H = 0.
\end{aligned}$$

Let us make the necessary change $\sigma \Rightarrow i\sigma, \mu \Rightarrow i\mu$. So, we obtain (also it is convenient to apply the designation $(Ez + \epsilon) = \Sigma$):

$$\begin{aligned}
(1) \quad & F'' + \Sigma^2 F \\
& + \frac{-M^2(-2B^2\sigma + W^2(\mu^2 + 2\sigma) + X) + \mu MW^2(2\sigma(2B^2\sigma + X) - 1) + 2B^2\sigma(2\sigma W^2 + X) - M^4}{M^2 - 2\sigma(B^2 + \mu MW^2)} F \\
& + \frac{W(2B\sigma - 1)\sqrt{X-B}(M(-B\mu + \mu M(B\mu + M) + M) - 2B^2\sigma(\mu M + 1))}{\sqrt{2}(M^3 - 2M\sigma(B^2 + \mu MW^2))} G + \\
& + \frac{W(2B\sigma + 1)\sqrt{B+X}(M(B\mu + \mu M(M - B\mu) + M) - 2B^2\sigma(\mu M + 1))}{\sqrt{2}(M^3 - 2M\sigma(B^2 + \mu MW^2))} H = 0, \\
(2) \quad & G'' + \Sigma^2 G + \frac{1}{2M^2 - 4\sigma(B^2 + \mu MW^2)} \\
& \times \left[2B^3\sigma(\mu M - 1) + B^2(2\sigma(2M^2 + \mu MX + X) + \mu M - 4\sigma^2 W^2(\mu M + 1) - 1) \right. \\
& + B(-(\mu M - 1)(2M^2 + X) + 2\sigma W^2(\mu M(2\mu M - 1) + 1) + 4\sigma^2 W^2 X(\mu M + 1)) \\
& \left. + 2\sigma W^2(2\mu M^3 + X(\mu M - 1)) - 2M^2(M^2 + X) \right] G - \\
& - \frac{MW\sqrt{X-B}(\mu M + 1)(4\sigma^2 W^2 + 1)}{\sqrt{2}(M^2 - 2\sigma(B^2 + \mu MW^2))} F \\
& + \frac{(2B\sigma + 1)\sqrt{X-B}\sqrt{B+X}(B(\mu M - 1) - 2\sigma W^2(\mu M + 1))}{4\sigma(B^2 + \mu MW^2) - 2M^2} H = 0, \\
(3) \quad & H'' + \Sigma^2 H \\
& + \frac{1}{2M^2 - 4\sigma(B^2 + \mu MW^2)} \left[-2B^3\sigma(\mu M - 1) + B^2(2\sigma(2M^2 + \mu MX + X) + \mu M \right. \\
& \left. - 4\sigma^2 W^2(\mu M + 1) - 1) + B((\mu M - 1)(2M^2 + X) + 2\sigma W^2(\mu M(1 - 2\mu M) - 1)) \right]
\end{aligned}$$

$$\begin{aligned}
& -4\sigma^2 W^2 X(\mu M + 1)) + 2\sigma W^2(2\mu M^3 + X(\mu M - 1)) - 2M^2(M^2 + X)]H \\
& + \frac{MW\sqrt{B+X}(\mu M + 1)(4\sigma^2 W^2 + 1)}{\sqrt{2}(M^2 - 2\sigma(B^2 + \mu MW^2))}F \\
& + \frac{(2B\sigma - 1)\sqrt{X-B}\sqrt{B+X}(B(\mu M - 1) + 2\sigma W^2(\mu M + 1))}{4\sigma(B^2 + \mu MW^2) - 2M^2}G = 0.
\end{aligned}$$

Let us present the system in matrix form

$$\Delta = \frac{d^2}{dz^2} + \Sigma^2(z), \Delta \begin{vmatrix} F \\ G \\ H \end{vmatrix} = A \begin{vmatrix} F \\ G \\ H \end{vmatrix}, \quad \Delta \Psi(z) = A \Psi(z), A = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Now, we will find the transformation that diagonalizes the system

$$\bar{\Psi} = S\Psi, \quad \Delta \bar{\Psi}(z) = \bar{A}\bar{\Psi}(z), \quad \bar{A} = SAS^{-1} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix}, \quad S = \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix},$$

we must find solutions for equation $SA = \bar{A}S$; explicitly, it reads

$$\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix};$$

with the following three similar subsystems

$$\begin{cases} (A_1 - \lambda_1)s_{11} + A_2s_{12} + A_3s_{13} = 0 \\ B_1s_{11} + (B_2 - \lambda_1)s_{12} + B_3s_{13} = 0 \\ C_1s_{11} + C_2s_{12} + (C_3 - \lambda_1)s_{13} = 0 \end{cases},$$

$$\begin{cases} (A_1 - \lambda_2)s_{21} + A_2s_{22} + A_3s_{23} = 0 \\ B_1s_{21} + (B_2 - \lambda_2)s_{22} + B_3s_{23} = 0 \\ C_1s_{21} + C_2s_{22} + (C_3 - \lambda_2)s_{23} = 0 \end{cases},$$

$$\begin{cases} (A_1 - \lambda_3)s_{31} + A_2s_{32} + A_3s_{33} = 0 \\ B_1s_{31} + (B_2 - \lambda_3)s_{32} + B_3s_{33} = 0 \\ C_1s_{31} + C_2s_{32} + (C_3 - \lambda_3)s_{33} = 0 \end{cases},$$

or differently

$$\begin{vmatrix} (A_1 - \lambda) & A_2 & A_3 \\ B_1 & (B_2 - \lambda) & B_3 \\ C_1 & C_2 & (C_3 - \lambda) \end{vmatrix} \begin{vmatrix} s_{i1} \\ s_{i2} \\ s_{i3} \end{vmatrix} = 0, \quad i = 1, 2, 3. \quad (18)$$

From vanishing the determinant

$$\det \begin{vmatrix} (A_1 - \lambda) & A_2 & A_3 \\ B_1 & (B_2 - \lambda) & B_3 \\ C_1 & C_2 & (C_3 - \lambda) \end{vmatrix} = 0$$

we derive the cubic equation

$$\begin{aligned}
& \lambda^3 - \lambda^2(A_1 + B_2 + C_3) + \\
& + \lambda(A_2B_1 - A_1B_2 + A_3C_1 - A_1C_3 + B_3C_2 - B_2C_3) \\
& + A_3B_2C_1 - A_2B_3C_1 - A_3B_1C_2 + A_1B_3C_2 + A_2B_1C_3 - A_1B_2C_3 = 0.
\end{aligned}$$

Let us write down equations with solutions that determine the elements of the matrix S :

$$\begin{aligned} (A_1 - \lambda_1)s_{11} + A_2s_{12} + A_3 &= 0 \\ B_1s_{11} + (B_2 - \lambda_1)s_{12} + B_3 &= 0, \quad \text{assuming } s_{13} = 1; \\ (A_1 - \lambda_2)s_{21} + A_2s_{22} + A_3 &= 0 \\ B_1s_{21} + (B_2 - \lambda_2)s_{22} + B_3 &= 0, \quad \text{assuming } s_{23} = 1; \\ (A_1 - \lambda_3)s_{31} + A_2s_{32} + A_3 &= 0 \\ B_1s_{31} + (B_2 - \lambda_3)s_{32} + B_3 &= 0, \quad \text{assuming } s_{33} = 1. \end{aligned}$$

After performing this transformation, we obtain three separate equations

$$\begin{aligned} \left(\frac{d^2}{dz^2} + (Ez + \epsilon)^2 - \lambda_1 \right) \bar{F} &= 0, \quad \left(\frac{d^2}{dz^2} + (Ez + \epsilon)^2 - \lambda_2 \right) \bar{G} = 0, \\ \left(\frac{d^2}{dz^2} + (Ez + \epsilon)^2 - \lambda_3 \right) \bar{H} &= 0. \end{aligned}$$

These equations have the same structure as a scalar particle in the uniform electric field

$$\left(\frac{d^2}{dz^2} + (Ez + \epsilon)^2 - \lambda \right) \Phi(z) = 0. \quad (19)$$

We can transform Equation (19) to the new variable (assuming that $E > 0$)

$$Z = i \frac{(Ez + \epsilon)^2}{E}, \quad \Lambda = \frac{\lambda}{4E}, \quad (20)$$

then, we obtain the confluent hypergeometric equation

$$\left(\frac{d^2}{dZ^2} + \frac{1/2}{Z} \frac{d}{dZ} - \frac{1}{4} + \frac{i\Lambda}{Z} \right) \Phi(Z) = 0; \quad (21)$$

its solutions were given in [32].

8. Conclusions

We studied a generalized Duffin–Kemmer–Petiau equation for spin-1 particles with two additional characteristics besides electric charge, namely anomalous magnetic moment and polarizability, in the presence of both external uniform magnetic and electric fields.

After separating the variables, we obtain a system of 10 first-order partial differential equations for 10 functions $f_i(r, z)$.

To describe the r -dependence of the 10 functions $f_A(r, z)$, $A = 1, \dots, 10$, we applied the method of Fedorov–Gronskiy [31]. Thus, the complete 10-component wave function is decomposed into the sum of three projective constituents. The dependence of each component on the polar coordinate r is determined by only one corresponding function, $F_i(r)$, $i = 1, 2, 3$.

These three basic functions are constructed in terms of confluent hypergeometric functions, and in this process a quantization rule arises due to the presence of a magnetic field.

After that, we derived a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system was solved using the elimination method and through special linear combinations of the involved functions.

As the result, we obtain three separate second-order differential equations, whose solutions can be constructed in the terms of confluent hypergeometric functions.

Due to the spin value of $S = 1$, we should expect in advance the existence of three types of solutions, $2S + 1 = 3$; these are precisely the ones found as the main result. Further,

the constructed solutions depend on a quantized parameter arising from the presence of a magnetic field, so the radiation spectra of such a particle depend on the magnitude of the external magnetic field. Moreover, the wave functions and spectra depend on two additional characteristics of the particle.

These solutions may be helpful for experimental testing of the intrinsic structure of vector bosons.

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