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Spin 1 Particle with Two Additional Electromagnetic Characteristics in the Presence of a Uniform Electric Field

A. V. Ivashkevich,[1,*] A. M. Kuzmich,[2,†] P. O. Sachenok,[3,‡] and E. M. Ovsiyuk[3,§]

[1]*B. I. Stepanov Institute of Physics of NAS of Belarus, 68 Nezavisimosti Ave., 220072 Minsk, BELARUS*

[2]*Brest State A. Pushkin University, 21 Kosmonavtov Blvd., 224016 Brest, BELARUS*

[3]*Mozyr State Pedagogical University named after I. P. Shamyakin, 28 Studencheskaya Str, 247760 Mozyr, BELARUS*

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In the paper, we study a generalized Duffin – Kemmer equation for spin 1 particle with two characteristics, anomalous magnetic moment and polarizability in the presence of an external uniform electric field. This approach is extended to space-time models with a pseudo-Riemannian structure, within the tetrad method. We specify the basic equations to the cylindrical coordinates, after separating the variables, we get the system of 10 first order partial differential equations for 10 functions $f_i(r, z)$. To describe the r-dependence of these functions, we apply the method by Fedorov – Gronskiy; in this approach, the complete 10-component wave function is decomposed into the sum of three projective constituents, dependence of each component on the polar coordinate is determined by the only functions $F_i(r), i = 1, 2, 3$; the last are constructed in terms of the Bessel functions. After that we derive a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved with the use of eliminating method and of special linear combining of the involved functions; as a result we find three independent solutions for the last system. The types of solutions for a vector particle with two additional electromagnetic characteristics in the presence of an external uniform electric field are investigated.

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1. Introduction

Within the general method by Gel'fand – Yaglom [1], for a spin 1 particle in [2] it was constructed a relativistic generalized system of the first order equations for a particle with two additional characteristics, anomalous magnetic moment and polarizability (a number of relevant papers see in [3]—[14]). First, the model was developed for a free particle, and the system of spinor equations was obtained; then it was transformed to tensor form. In tensor form, the presence of external electromagnetic fields was taken into account. After eliminating the accessory variables of the complete wave function, it was derived the minimal system of 10 equations, it contains two additional interaction terms which are interpreted as related to the anomalous magnetic moment and polarizability.

In the present paper, we extend this equation to space-time models with pseudo-Riemannian structure within the tetrad method, and specify this generalized Daffin – Kemmer – Petiau equation in cylindrical coordinates and tetrad, taking into account the presence of the external uniform electric field.

After separating the variables, we get the

*E-mail: ivashkevich.alina@yandex.by

†E-mail: miss.nastya.01@list.ru

‡E-mail: polinasacenok@gmail.com

§E-mail: e.ovsiyuk@mail.ru

system of 10 first order partial differential equations for 10 functions $f_i(r,z)$. To describe the r-dependence of 10 functions $f_A(r,z)$, $A = 1,...,10$, we apply the method by Fedorov – Gronskiy [15]; in this approach, the complete 10-component wave function is decomposed into the sum of three projective constituents, dependence of each component on the polar coordinate is determined by only functions $F_i(r)$, $i = 1,2,3$; they are constructed in terms of Bessel functions. After that we derive a system of 10 ordinary differential equations for 10 functions $f_A(z)$. This system is solved with the use of the elimination method and with the help of special linear combining of the involved functions. As a result we find three independent solutions for the last system. Thus, in this paper, the types of solutions for a vector particle with two additional electromagnetic characteristics in presence of external uniform electric field are found.

2. Matrix equation in Minkowski space

We start [2] with the following tensor equations (let $D_a = \partial_a + ieA_a$)

$$\begin{gathered} D^b\Phi_{ab} + e\mu F_{ab}\Phi^b \\ +e\sigma D_a(F^{cd}\Phi_{cd}) - M\Phi_a = 0, \\ D_a\Phi_b - D_b\Phi_a - M\Phi_{ab} = 0; \end{gathered} \tag{1}$$

they may be compared with the ordinary Proca system

$$\begin{gathered} D^b\Phi_{ab} - M\Phi_a = 0, \\ D_a\Phi_b - D_b\Phi_a - M\Phi_{ab} = 0. \end{gathered} \tag{2}$$

In (1), we can see two additional interaction terms, proportional to parameters μ (anomalous magnetic moment) and σ (polarizability); in the paper [16], it was proved that parameters μ, σ are imaginary: $\mu \Longrightarrow i\mu, \sigma \Longrightarrow i\sigma$ (we will take this into account later on).

We use the 10-dimensional column:

$$\Phi = (\Phi_0, \Phi_1, \Phi_2, \Phi_3; \Phi_{01}, \Phi_{02}, \Phi_{03}, \Phi_{23}, \Phi_{31}, \Phi_{12}) = (H_1; H_2).$$

Let us recall the matrix form of the Proca system when $\mu = 0, \sigma = 0$. The first equation gives $K^a D_a H_2 - MH_1 = 0$, where

$$K^0 = \begin{vmatrix} . & . & . & . & . & . \\ -1 & . & . & . & . & . \\ . & -1 & . & . & . & . \\ . & . & -1 & . & . & . \end{vmatrix}, K^1 = \begin{vmatrix} -1 & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & -1 & . \end{vmatrix},$$

$$K^2 = \begin{vmatrix} . & -1 & . & . & . & . \\ . & . & . & . & . & -1 \\ . & . & . & . & . & . \\ . & . & . & +1 & . & . \end{vmatrix}, K^3 = \begin{vmatrix} . & . & -1 & . & . & . \\ . & . & . & . & +1 & . \\ . & . & . & -1 & . & . \\ . & . & . & . & . & . \end{vmatrix}.$$

The second equation in (2) leads to $D_a L^a H_1 - MH_2 = 0$, where

$$L^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, L^1 = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

$$L^2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, L^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

Thus, the system of equations for the ordinary spin 1 particle is presented in the block form as follows

$$\begin{gathered} K^a D_a H_2 - MH_1 = 0, \\ L^a D_a H_1 - MH_2 = 0. \end{gathered} \tag{3}$$

Let us detail the first additional term in (1) (taking in mind identities: $F_{01}\Phi^1 = -F_{01}\Phi_1 = -E^1\Phi_1$, $F_{12}\Phi^2 = -F_{12}\Phi_2 = B^3\Phi_2$, and so on)

$$e\mu F_{ab}\Phi^b = \begin{vmatrix} 0 & -E^1 & -E^2 & -E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{vmatrix} \begin{vmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{vmatrix},$$

whence allowing for the structure of six Lorentzian generators for vector field

$$j^{12} = S_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad j^{31} = S_2 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad j^{23} = S_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

$$j^{01} = T_1 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad j^{02} = T_2 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}, \quad j^{03} = T_3 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

we obtain the more short presentation for this interaction term

$$e\mu F_{\alpha\beta}\Phi^{\beta} \quad \Longrightarrow \quad -e\mu\Big(\mathbf{S}\,\mathbf{B} + \mathbf{T}\,\mathbf{E}\Big)H_1.$$

The second additional term in (1) is

$$e\sigma D_a(F^{cd}\Phi_{cd}) = 2e\sigma D_a \begin{vmatrix} E^1 & 0 & 0 & 0 & 0 & 0 \\ 0 & E^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & E^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B^1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -B^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B^3 \end{vmatrix} \begin{vmatrix} \Phi_{01} \\ \Phi_{02} \\ \Phi_{03} \\ \Phi_{23} \\ \Phi_{31} \\ \Phi_{12} \end{vmatrix}.$$

So we have a generalized Duffin – Kemmer – Petiau equation in the block form

$$\Big(K^a D_a H_2\Big)_c - e\mu\Big[(\mathbf{S}\,\mathbf{B} + \mathbf{T}\,\mathbf{E})H_1\Big]_c + e\sigma D_c(F^{kl}\Phi_{kl}) - M(H_1)_c = 0, \Big(L^a D_a H_1\Big)_{[kl]} - M(H_2)_{[kl]} = 0.$$

3. Extension to curved space-time models

In Riemannian space-time, we start with the tensor covariant equations

$$D_\beta \Phi_\alpha{}^{\beta} + e\mu F_{\alpha\beta}\Phi^\beta + e\sigma\hat{\partial}_\alpha(F^{\rho\delta}\Phi_{\rho\delta}) - M\Phi_\alpha = 0, \quad D_\alpha\Phi_\beta - D_\beta\Phi_\alpha - M\Phi_{\alpha\beta} = 0; \tag{4}$$

below two different derivative symbols will be used: $D_\alpha = \nabla_\alpha + ieA_\alpha$ and $\hat{\partial}_\alpha = \partial_\alpha + ieA_\alpha$. Let us transform these equations to tetrad form. First, we obtain

$$D_\beta\Big(e_{(a)\alpha}e^{\beta}_{(b)}\Phi^{ab}\Big) + e\mu\, e^{(a)}_{\alpha}F_{(a)(b)}\Phi^b$$
$$+e\sigma\,\hat{\partial}_\alpha(F^{ab}\Phi_{ab}) - M\Phi_\alpha = 0, D_\alpha e^{(b)}_{\beta}\Phi_b - D_\beta e^{(a)}_{\alpha}\Phi_a - Me_{(a)\alpha}e_{(b)\beta}\Phi^{ab} = 0;$$

whence it follows

$$e_{(a)\alpha}e^{\beta}_{(b)}\hat{\partial}_{\beta}\Phi^{ab}+(\nabla_{\beta}e_{(a)\alpha})e^{\beta}_{(b)}\Phi^{ab}+e_{(a)\alpha}(\nabla_{\beta}e^{\beta}_{(b)})\Phi^{ab}+e\mu\ e^{(a)}_{\alpha}F_{(a)(b)}\Phi^{b}$$
$$+e\sigma(\partial_{\alpha}F^{ab})\Phi_{ab}+e\sigma F^{ab}\hat{\partial}_{\alpha}\Phi_{ab}-M\Phi_{\alpha}=0,$$

$$e^{(b)}_{\beta}\hat{\partial}_{\alpha}\Phi_{b}-e^{(b)}_{\alpha}\hat{\partial}_{\beta}\Phi_{b}+(\nabla_{\alpha}e^{(b)}_{\beta})\Phi_{b}-(\nabla_{\beta}e^{(b)}_{\alpha})\Phi_{b}-Me_{(a)\alpha}e_{(b)\beta}\Phi^{ab}=0.$$

Multiplying the first equation by $e^{\alpha}_{(c)}$:

$$e^{\alpha}_{(c)}e_{(a)\alpha}e^{\beta}_{(b)}\hat{\partial}_{\beta}\Phi^{ab}+(\nabla_{\beta}e_{(a)\alpha})e^{\alpha}_{(c)}e^{\beta}_{(b)}\Phi^{ab}+e^{\alpha}_{(c)}e_{(a)\alpha}(\nabla_{\beta}e^{\beta}_{(b)})\Phi^{ab} \tag{5}$$
$$+e\mu\ e^{\alpha}_{(c)}e^{(a)}_{\alpha}F_{(a)(b)}\Phi^{b}+e\sigma\ e^{\alpha}_{(c)}(\partial_{\alpha}F^{ab})\Phi_{ab}+e\sigma F^{ab}e^{\alpha}_{(c)}\hat{\partial}_{\alpha}\Phi_{ab}-Me^{\alpha}_{(c)}\Phi_{\alpha}=0,$$

we get (applying the shortening notation $e^{\beta}_{(b)}(\partial_{\beta}+ieA_{\beta})=\hat{\partial}_{(b)}$)

$$\hat{\partial}_{(b)}\Phi_{c}{}^{b}+e_{(a)\alpha;\beta}e^{\alpha}_{(c)}e^{\beta}_{(b)}\Phi^{ab}+e^{\beta}_{(b);\beta}\Phi_{c}{}^{b}+e\mu F_{cb}\Phi^{b}+e\sigma(\partial_{(c)}F^{ab})\Phi_{ab}+e\sigma\ F^{ab}\hat{\partial}_{(c)}\Phi_{ab}-M\Phi_{c}=0.$$

The second equation leads to

$$\hat{\partial}_{(c)}\Phi_{d}-\hat{\partial}_{(d)}\Phi_{c}+e_{(b)\beta;\alpha}e^{\beta}_{(d)}e^{\alpha}_{(c)}\Phi^{b}-e_{(b)\alpha;\beta}e^{\alpha}_{(c)}e^{\beta}_{(d)}\Phi^{b}-M\Phi_{cd}=0.$$

Taking into account the definition for Ricci rotation coefficients $\gamma_{abc}=-e_{(a)\rho;\sigma}e^{\rho}_{(b)}e^{\sigma}_{(c)}$, $e^{\alpha}_{(a);\alpha}=\gamma_{ba}{}^{b}$, we can rewrite the above equations in a shorter form

$$\hat{\partial}_{(b)}\Phi_{c}{}^{b}-\gamma_{acb}\Phi^{ab}+e^{\beta}_{(b);\beta}\Phi_{c}{}^{b}+e\mu F_{cb}\Phi^{b}+e\sigma(\partial_{(c)}F^{ab})\Phi_{ab}+e\sigma F^{ab}\hat{\partial}_{(c)}\Phi_{ab}-M\Phi_{c}=0,$$
$$\hat{\partial}_{(c)}\Phi_{d}-\hat{\partial}_{(d)}\Phi_{c}+\gamma_{bdc}\Phi^{b}-\gamma_{bcd}\Phi^{b}-M\Phi_{cd}=0.$$

We should recall the known matrix tetrad form of the ordinary vector particle

$$\Big[\beta^{c}\Big(e^{\alpha}_{(c)}(x)\frac{\partial}{\partial x^{\alpha}}+\frac{1}{2}J^{ab}\gamma_{abc}\Big)-M\Big]\Phi=0,\qquad \Phi=\left|\begin{array}{c}\Phi_{a}\\ \Phi_{ab}\end{array}\right|=\left|\begin{array}{c}H_{1}\\ H_{2}\end{array}\right|; \tag{6}$$

two additional interactions terms are

$$e\mu\ F_{\alpha\beta}\Phi^{\beta}=e\mu\Big(\mathbf{S}\,\mathbf{B}-\mathbf{T}\,\mathbf{E}\Big)H_{1},\qquad e\sigma\hat{\partial}_{\alpha}(F^{cd}\Phi_{cd})=e\sigma\left|\begin{array}{c}\hat{\partial}_{(0)}(F^{cd}\Phi_{cd})\\ \hat{\partial}_{(1)}(F^{cd}\Phi_{cd})\\ \hat{\partial}_{(2)}(F^{cd}\Phi_{cd})\\ \hat{\partial}_{(3)}(F^{cd}\Phi_{cd})\end{array}\right|. \tag{7}$$

Thus, we have the following generalized system of equations

$$\Big(K^{c}D^{(2)}_{c}H_{2}\Big)_{n}-e\mu\Big[(\mathbf{SB}+\mathbf{TE})H_{1}\Big]_{n}+e\sigma e^{\alpha}_{(n)}\hat{\partial}_{\alpha}(F^{kl}\Phi_{kl})-M(H_{1})_{n}=0,$$
$$\Big(L^{c}D^{(1)}_{c}H_{1}\Big)_{[kl]}-(MH_{2})_{[kl]}=0.$$

4. Particle in the electric field

Cylindrical coordinates $x^{\alpha}=(t,r,\phi,z)$, the relevant tetrad, Ricci rotation coefficients, and the uniform electric field are determined by relations

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, g_{\alpha\beta} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \; e^{\alpha}_{(a)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$\gamma_{122} = -\gamma_{212} = +\gamma_{21}{}^2 = e^{\alpha}_{(1);\alpha} = \frac{1}{r}, A_0 = -Ez, \; F_{03} = e^t_{(0)} e^z_{(3)} F_{tz} = +E, \hat{\partial}_0 \Longrightarrow -i\epsilon - ieEz.$$

Correspondingly, the system of equations takes on the form

$$\begin{gathered} \Big[K^0(\partial_0 - ieEz) + K^1 \partial_r + K^2 \frac{1}{r}(\partial_\phi + j_2^{12}) + K^3 \partial_z \Big] H_2 \\ -e\mu E j_1^{03} H_1 + 2Ee\sigma \begin{vmatrix} \partial_0 - ieEz \\ \partial_r \\ \frac{1}{r}\partial_\phi \\ \partial_z \end{vmatrix} E_3 - MH_1 = 0, \\ \Big[L^0(\partial_0 - ieEz) + L^1 \partial_r + L^2 \frac{1}{r}(\partial_\phi + j_1^{12}) + L^3 \partial_z \Big] - MH_2 = 0. \end{gathered} \tag{8}$$

5. Cyclic basis

It is more convenient to apply the so called cyclic basis. It is defined by requirement to have a diagonal generator j_1^{12} for the vector field $H_1 = (\Phi_l)$. The needed transformation $\bar{\Phi} = U\Phi$ is determined by the matrix U

$$U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}, \bar{j}^{12} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +i \end{vmatrix}.$$

Correspondingly, vector and tensor generators transform according to the rules

$$\bar{J}_1^{ab} = U j^{ab} U^{-1}, \; \bar{J}_2^{ab} = \bar{j}^{ab} \otimes I + I \otimes \bar{j}^{ab}. \tag{9}$$

Let us find a 6-dimensional presentation for tensor generators

$$J_{(2)}^{12} = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix}. \tag{10}$$

We should transform the Duffin – Kemmer matrices β^a to the cyclic basis as well; it is convenient to apply the block presentation: $\bar{H}_1 = C_1 H_1$, $(C_1 = U)$, $\bar{H}_2 = (U \otimes U) H_2 = C_2 H_2$; further we obtain

$$\begin{vmatrix} 0 & \bar{K}^a \\ \bar{L}^a & 0 \end{vmatrix} = \begin{vmatrix} 0 & C_1 K^a C_2^{-1} \\ C_2 L^a C_1^{-1} & 0 \end{vmatrix}.$$

Taking in mind the formulas

$$C_1 = U = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix},$$

$$C_1^{-1} = U^{-1} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

we derive explicit form of the matrix C_2, $U \otimes U \Longrightarrow C_2, \bar{H}_2 = C_2 H_2$

$$C_2 = \begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$C_2^{-1} = \begin{vmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & -i & 0 \end{vmatrix}.$$

Further, we readily find the needed blocks:

$$\bar{K}^0 = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{vmatrix},$$

$$\bar{K}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{K}^2 = \begin{vmatrix} \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{K}^3 = \begin{vmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix},$$

$$\bar{L}^1 = \begin{vmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{L}^2 = \begin{vmatrix} -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 \end{vmatrix},$$

$$\bar{L}^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \bar{L}^0 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

and also expressions for the needed generators

$$\bar{J}_1^{12} = i \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \end{vmatrix}, \quad \bar{j}_1^{03} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\bar{J}_2^{12} = i \begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{vmatrix}.$$

Now let us transform the above system to the cyclic basis. The first equation leads to

$$\Big[C_1 K^0 C_2^{-1} C_2 (\partial_0 - ieEz) + (C_1 K^1 C_2^{-1} C_2 \partial_r$$
$$+ C_1 K^2 C_2^{-1} C_2 \frac{1}{r} \partial_\phi + j_2^{12} + C_1 K^3 C_2^{-1} C_2 \partial_z \Big] C_2^{-1} C_2 H_2 - M C_1 H_1$$

$$+e\mu E(C_1 j_1^{03} C_1^{-1}) C_1 H_1 - 2Ee\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} \partial_0 - ieEz \\ \partial_r \\ 1/r\,\partial_\phi \\ \partial_z \end{vmatrix} \bar{\Phi}_{02} = 0;$$

whence allowing for the identity

$$\begin{vmatrix} -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ \Phi_{03} \\ 0 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \bar{\Phi}_{02} = \Phi_{03} \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix},$$

the first equation is transformed to the form

$$\Big[\bar{K}^0(-i\epsilon - ieEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}(\partial_\phi + \bar{j}_2^{12}) + \bar{K}^3\partial_z\Big]\bar{H}_2 + e\mu E\bar{j}_1^{03}\bar{H}_1$$
$$+2Ee\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} \partial_0 - ieEz \\ \partial_r \\ 1/r\,\partial_\phi \\ \partial_z \end{vmatrix} \bar{\Phi}_{02} - M\bar{H}_1 = 0, \tag{11}$$

The second equation transforms as follows

$$\Big[\bar{L}^0(\partial_0 - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(\partial_\phi + \bar{j}_1^{12}) + \bar{L}^3\partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0. \tag{12}$$

Now let us perform the separation of the variables in cyclic basis. Taking in mind the substitution

$$\bar{H}_1 = e^{-i\epsilon t}e^{im\phi}\begin{vmatrix} \bar{\Phi}_0(r,z) \\ \bar{\Phi}_1(r,z) \\ \bar{\Phi}_2(r,z) \\ \bar{\Phi}_3(r,z) \end{vmatrix}, \quad \bar{H}_2 = e^{-i\epsilon t}e^{im\phi}\begin{vmatrix} \bar{E}_1(r,z) \\ \bar{E}_2(r,z) \\ \bar{E}_3(r,z) \\ \bar{B}_1(r,z) \\ \bar{B}_2(r,z) \\ \bar{B}_3(r,z) \end{vmatrix}, \tag{13}$$

we get

$$\Big[\bar{K}^0(-i\epsilon - ieEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}(im + \bar{j}_2^{12}) + \bar{K}^3\partial_z\Big]\bar{H}_2 - M\bar{H}_1$$
$$+e\mu E\bar{j}_1^{03}\bar{H}_1 + 2Ee\sigma \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix} \begin{vmatrix} -i\epsilon - ieEz \\ \partial_r \\ im/r \\ \partial_z \end{vmatrix} \bar{\Phi}_{02} = 0,$$

$$\Big[\bar{L}^0(-i\epsilon - ieEz) + \bar{L}^1\partial_r + \bar{L}^2\frac{1}{r}(im + \bar{j}_1^{12}) + \bar{L}^3\partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0.$$

Let us specify separately two additional terms:

$$+e\mu E\bar{j}_1^{03}\bar{H}_1(r) = +e\mu E\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}\begin{vmatrix} \bar{\Phi}_0 \\ \bar{\Phi}_1 \\ \bar{\Phi}_2 \\ \bar{\Phi}_3 \end{vmatrix} = \begin{vmatrix} +e\mu E\bar{\Phi}_2 \\ 0 \\ +e\mu E\bar{\Phi}_0 \\ 0 \end{vmatrix},$$

$$+2Ee\sigma\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{vmatrix}\begin{vmatrix} -i\epsilon - ieEz \\ \partial_r \\ im/r \\ \partial_z \end{vmatrix}\bar{\Phi}_{02} = \begin{vmatrix} 2\,Ee\sigma\,(-i\epsilon - ieEz) \\ -\sqrt{2}Ee\sigma\,(\partial_r + m/r) \\ 2\,Ee\sigma\,\partial_z \\ \sqrt{2}Ee\sigma\,(\partial_r - m/r) \end{vmatrix}\bar{\Phi}_{02}\,.$$

so we arrive at

$$\Big[\bar{K}^0(-i\epsilon - ieEz) + \bar{K}^1\partial_r + \bar{K}^2\frac{1}{r}(im + \bar{j}_2^{12}) + \bar{K}^3\partial_z\Big]\bar{H}_2 - M\bar{H}_1$$
$$+\begin{vmatrix} e\mu E\bar{\Phi}_2 \\ 0 \\ +e\mu E\bar{\Phi}_0 \\ 0 \end{vmatrix} + \begin{vmatrix} 2\,Ee\sigma\,(-i\epsilon - ieEz)\bar{E}_2 \\ -\sqrt{2}Ee\sigma\,(\partial_r + m/r)\bar{E}_2 \\ 2\,Ee\sigma\,\partial_z\bar{E}_2 \\ \sqrt{2}Ee\sigma\,(\partial_r - m/r)\bar{E}_2 \end{vmatrix} = 0, \qquad (14)$$
$$\bar{L}^c\Big[(-i\epsilon - ieEz) + \partial_r + \frac{1}{r}(im + \bar{j}_1^{12}) + \partial_z\Big]\bar{H}_1 - M\bar{H}_2 = 0.$$

Whence we obtain the following 10 equations

$$1 \quad \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m-1}{r}\Big)\bar{E}_1(r,z) - \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m+1}{r}\Big)\bar{E}_3(r,z) - \partial_z\bar{E}_2$$
$$+e\mu\,E\bar{\Phi}_2(r,z) - 2\,iEe\sigma\,(\epsilon + eEz)\,\bar{E}_2(r,z) = M\bar{\Phi}_0(r,z)\,,$$

$$2 \quad i\,(\epsilon + eEz)\,\bar{E}_1(r,z) + \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)\bar{B}_2(r,z) - \partial_z\bar{B}_3(r,z) - \sqrt{2}Ee\sigma\,\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)\bar{E}_2(r,z) = M\bar{\Phi}_1(r,z),$$

$$3 \quad +i\,(\epsilon + eEz)\,\bar{E}_2(r,z) - \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m+1}{r}\Big)\bar{B}_1(r,z) - \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m-1}{r}\Big)\bar{B}_3(r,z)$$
$$+e\mu\,E\bar{\Phi}_0(r,z) + 2\,Ee\sigma\,\partial_z\bar{E}_2(r,z) = M\bar{\Phi}_2(r,z)\,,$$

$$4 \quad i\,(\epsilon + eEz)\,\bar{E}_3(r,z) + \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)\bar{B}_2(r,z) + \partial_z\bar{B}_1(r,z) + \sqrt{2}\,Ee\sigma\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)\bar{E}_2(r,z) = M\bar{\Phi}_3(r,z),$$

$$5 \quad \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)\,\bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_1(r,z) = M\bar{E}_1(r,z)\,,$$

$$6 \quad -\partial_z\bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_2(r,z) = M\bar{E}_2(r,z)\,,$$

$$7 \quad -\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)\bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_3(r,z) = M\bar{E}_3(r,z)\,,$$

$$8 \quad -\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m}{r}\Big)\bar{\Phi}_2(r,z) + \partial_z\bar{\Phi}_3(r,z) = M\bar{B}_1(r,z)\,,$$

$$9 \quad \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} - \frac{m-1}{r}\Big)\bar{\Phi}_1(r,z) + \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m+1}{r}\Big)\bar{\Phi}_3(r,z) = M\bar{B}_2(r,z)\,,$$

$$10 \quad -\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial r} + \frac{m}{r}\Big)\bar{\Phi}_2(r,z) - \partial_z\bar{\Phi}_1(r,z) = M\bar{B}_3(r,z)\,.$$

With the use of the shortening notations

$$a_m = \frac{\partial}{\partial r} + \frac{m}{r},\ a_{m+1} = \frac{\partial}{\partial r} + \frac{m+1}{r},$$
$$b_m = \frac{\partial}{\partial r} - \frac{m}{r},\ b_{m-1} = \frac{\partial}{\partial r} - \frac{m-1}{r}, \tag{15}$$

the above 10 equations read as follows

$$1\ \ \frac{1}{\sqrt{2}} b_{m-1}\bar{E}_1(r,z) - \frac{1}{\sqrt{2}} a_{m+1}\bar{E}_3(r,z) - \partial_z \bar{E}_2 + e\mu\, E\bar{\Phi}_2(r,z) - 2\, iEe\sigma\,(\epsilon + eEz)\,\bar{E}_2(r,z) = M\bar{\Phi}_0(r,z)\,,$$

$$2\ \ i\,(\epsilon + eEz)\,\bar{E}_1(r,z) + \frac{1}{\sqrt{2}} a_m \bar{B}_2(r,z) - \partial_z \bar{B}_3(r,z) - \sqrt{2} Ee\sigma\, a_m \bar{E}_2(r,z) = M\bar{\Phi}_1(r,z)\,,$$

$$3\ \ i\,(\epsilon + eEz)\,\bar{E}_2(r,z) - \frac{1}{\sqrt{2}} a_{m+1}\bar{B}_1(r,z) - \frac{1}{\sqrt{2}} b_{m-1}\bar{B}_3(r,z) + e\mu\, E\bar{\Phi}_0(r,z) + 2\, Ee\sigma\, \partial_z \bar{E}_2(r,z) = M\bar{\Phi}_2(r,z)\,,$$

$$4\ \ i\,(\epsilon + eEz)\,\bar{E}_3(r,z) + \frac{1}{\sqrt{2}} b_m \bar{B}_2(r,z) + \partial_z \bar{B}_1(r,z) + \sqrt{2}\, Ee\sigma b_m \bar{E}_2(r,z) = M\bar{\Phi}_3(r,z)\,,$$

$$5\ \ \frac{1}{\sqrt{2}} a_m\, \bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_1(r,z) = M\bar{E}_1(r,z)\,,$$

$$6\ \ -\partial_z \bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_2(r,z) = M\bar{E}_2(r,z)\,,$$

$$7\ \ -\frac{1}{\sqrt{2}} b_m \bar{\Phi}_0(r,z) - i\,(\epsilon + eEz)\,\bar{\Phi}_3(r,z) = M\bar{E}_3(r,z)\,,$$

$$8\ \ -\frac{1}{\sqrt{2}} b_m \bar{\Phi}_2(r,z) + \partial_z \bar{\Phi}_3(r,z) = M\bar{B}_1(r,z)\,,$$

$$9\ \ \frac{1}{\sqrt{2}} b_{m-1}\bar{\Phi}_1(r,z) + \frac{1}{\sqrt{2}} a_{m+1}\bar{\Phi}_3(r,z) = M\bar{B}_2(r,z)\,,$$

$$10\ \ -\frac{1}{\sqrt{2}} a_m \bar{\Phi}_2(r,z) - \partial_z \bar{\Phi}_1(r,z) = M\bar{B}_3(r,z)\,.$$

6. Projective operators and the Fedorov – Gronskiy method

To analyze the system of equations, we will use the method of projective operators. To this end, we consider the third spin projection $Y = -i\bar{J}^{12}$, and make sure that it satisfies the minimal equation $Y(Y-1)(Y+1) = 0$. This minimal equation allows us to introduce three projective operators

$$P_0 = 1 - Y^2, P_{+1} = \frac{Y(Y+1)}{2}, P_{-1} = \frac{Y(Y-1)}{2},$$

with the needed properties

$$P_0^2 = P_0,\ P_{+1}^2 = P_{+1},\ P_{-1}^2 = P_{-1}.$$

Accordingly, the complete wave function can be expanded into the sum of three parts

$$\bar{\Phi} = \bar{\Phi}_0 + \bar{\Phi}_{+1} + \bar{\Phi}_{-1}, \bar{\Phi}_\sigma = P_\sigma \bar{\Phi}, \sigma = 0, +1, -1.$$

These components have the following dependence on the variable r (in accordance with the Fedorov – Gronsky method [15], each projective component should be determined by only one function of r):

$$\bar{\Phi}_0(r,z) = \begin{vmatrix} \bar{\Phi}_0(z) \\ 0 \\ \bar{\Phi}_2(z) \\ 0 \\ 0 \\ \bar{E}_2(z) \\ 0 \\ 0 \\ \bar{B}_2(z) \\ 0 \end{vmatrix} F_1(r),$$

$$\bar{\Phi}_{+1}(r) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ \bar{\Phi}_3 \\ 0 \\ 0 \\ \bar{E}_3(z) \\ \bar{B}_1(z) \\ 0 \\ 0 \end{vmatrix} F_2(r),$$

$$\Phi_{-1}(r,z)=\left|\begin{array}{c}0\\ \bar{\Phi}_1(z)\\ 0\\ 0\\ \bar{E}_1(z)\\ 0\\ 0\\ 0\\ 0\\ \bar{B}_3(z)\end{array}\right|F_3(r)$$

the columns are made up of yet unknown numerical coefficients. Taking this into account, we can rewrite the previous systems of equations differently

$$1)\quad \frac{1}{\sqrt{2}}\bar{E}_1(z)b_{m-1}F_3(r)-\frac{1}{\sqrt{2}}\bar{E}_3(z)a_{m+1}F_2(r) -\partial_z\bar{E}_2(z)F_1(r)+e\mu\,E\bar{\Phi}_2(z)F_1(r) -2\,iEe\sigma\,(\epsilon+eEz)\,\bar{E}_2(z)F_1(r)=M\bar{\Phi}_0(z)F_1(r)\,,$$

$$2)\quad i\,(\epsilon+eEz)\,\bar{E}_1(z)F_3(r)+\frac{1}{\sqrt{2}}\bar{B}_2(z)a_mF_1(r) -\partial_z\bar{B}_3(z)F_3(r)-\sqrt{2}Ee\sigma\,\bar{E}_2(z)a_mF_1(r) =M\bar{\Phi}_1(z)F_3(r)\,,$$

$$3)\quad i\,(\epsilon+eEz)\,\bar{E}_2(z)F_1(r)-\frac{1}{\sqrt{2}}\bar{B}_1(z)a_{m+1}F_2(r) -\frac{1}{\sqrt{2}}\bar{B}_3(z)b_{m-1}F_3(r)+e\mu\,E\bar{\Phi}_0(z)F_1(r) +2\,Ee\sigma\,\partial_z\bar{E}_2(z)F_1(r)=M\bar{\Phi}_2(z)F_1(r)\,,$$

$$4)\quad i\,(\epsilon+eEz)\,\bar{E}_3(z)(z)F_2(r)+\frac{1}{\sqrt{2}}\bar{B}_2(z)b_mF_1(r) +\partial_z\bar{B}_1(z)F_2(r)+\sqrt{2}\,Ee\sigma b_m\bar{E}_2(z)F_1(r) =M\bar{\Phi}_3(z)F_2(r)\,,$$

$$5)\quad \frac{1}{\sqrt{2}}\,\bar{\Phi}_0(z)a_mF_1(r)-i\,(\epsilon+eEz)\,\bar{\Phi}_1(z)F_3(r) =M\bar{E}_1(z)F_3(r)\,,$$

$$6)\quad -\partial_z\bar{\Phi}_0(z)F_1(r)-i\,(\epsilon+eEz)\,\bar{\Phi}_2(z)F_1(r) =M\bar{E}_2(z)F_1(r)\,,$$

$$7)\quad -\frac{1}{\sqrt{2}}\bar{\Phi}_0(z)b_mF_1(r) -i\,(\epsilon+eEz)\,\bar{\Phi}_3(z)F_2(r)=M\bar{E}_3(z)F_2(r)\,,$$

$$8)\quad -\frac{1}{\sqrt{2}}\bar{\Phi}_2(z)b_mF_1(r)+\partial_z\bar{\Phi}_3(z)F_2(r) =M\bar{B}_1(z)F_2(r)\,,$$

$$9)\quad \frac{1}{\sqrt{2}}\bar{\Phi}_1(z)b_{m-1}F_3(r)+\frac{1}{\sqrt{2}}\bar{\Phi}_3(z)a_{m+1}F_2(r) =M\bar{B}_2(z)F_1(r)\,,$$

$$10)\quad -\frac{1}{\sqrt{2}}\bar{\Phi}_2(z)a_mF_1(r)-\partial_z\bar{\Phi}_1(z)F_3(r) =M\bar{B}_3(z)F_3(r)\,.$$

In order to obtain differential equations in the variable z, we should impose the following constraints

$$\begin{aligned} &b_{m-1}F_3=C_1F_1,\quad a_mF_1=C_4F_3,\\ &a_{m+1}F_2=C_2F_1,\quad b_mF_1=C_3F_2, \end{aligned}\tag{16}$$

in this way we arrive at

$$\frac{1}{\sqrt{2}}C_1\bar{E}_1(z)-\frac{1}{\sqrt{2}}C_2\bar{E}_3(z)-\partial_z\bar{E}_2(z) +e\mu E\bar{\Phi}_2(z)-2iEe\sigma(\epsilon+eEz)\bar{E}_2(z)=M\bar{\Phi}_0(z),$$

$$+i(\epsilon+eEz)\bar{E}_1(z)+\frac{1}{\sqrt{2}}C_4\bar{B}_2(z)-\partial_z\bar{B}_3(z) -\sqrt{2}Ee\sigma\,C_4\bar{E}_2(z)=M\bar{\Phi}_1(z)\,,$$

$$+i(\epsilon+eEz)\bar{E}_2(z)-\frac{1}{\sqrt{2}}C_2\bar{B}_1(z)-\frac{1}{\sqrt{2}}C_1\bar{B}_3(z) +e\mu\,E\bar{\Phi}_0(z)+2\,Ee\sigma\,\partial_z\bar{E}_2(z)=M\bar{\Phi}_2(z)\,,$$

$$+i\,(\epsilon+eEz)\,\bar{E}_3(z)(z)+\frac{1}{\sqrt{2}}C_3\bar{B}_2(z) +\partial_z\bar{B}_1(z)+\sqrt{2}\,Ee\sigma C_3\bar{E}_2(z)=M\bar{\Phi}_3(z)\,;$$

$$\frac{1}{\sqrt{2}}\,C_4\bar{\Phi}_0(z)-i\,(\epsilon+eEz)\,\bar{\Phi}_1(z)=M\bar{E}_1(z)\,,$$

$$-\partial_z\bar{\Phi}_0(z)-i\,(\epsilon+eEz)\,\bar{\Phi}_2(z)=M\bar{E}_2(z)\,,$$

$$-\frac{1}{\sqrt{2}}C_3\bar{\Phi}_0(z)-i\,(\epsilon+eEz)\,\bar{\Phi}_3(z)=M\bar{E}_3(z)\,,$$

$$-\frac{1}{\sqrt{2}}C_3\bar{\Phi}_2(z)+\partial_z\bar{\Phi}_3(z)=M\bar{B}_1(z)\,,$$

$$\frac{1}{\sqrt{2}}C_1\bar{\Phi}_1(z)+\frac{1}{\sqrt{2}}C_2\bar{\Phi}_3(z)=M\bar{B}_2(z)\,,$$

$$-\frac{1}{\sqrt{2}}C_4\bar{\Phi}_2(z)-\partial_z\bar{\Phi}_1(z)=M\bar{B}_3(z)\,.$$

7. Explicit form of three basic functions

From the differential constraints

$$b_{m-1}F_3 = C_1 F_1, \quad a_m F_1 = C_4 F_3, \qquad a_{m+1}F_2 = C_2 F_1, \quad b_m F_1 = C_3 F_2, \tag{17}$$

it is obvious that the parameters in each pair can be chosen the same: $C_4 = C_1, C_3 = C_2$. Therefore, we obtain the second-order equations

$$(b_{m-1}a_m - C_1^2)F_1 = 0, \ (a_m b_{m-1} - C_1^2)F_3 = 0,$$
$$(a_{m+1}b_m - C_2^2)F_1 = 0, \ (b_m a_{m+1} - C_2^2)F_2 = 0.$$

Further we obtain the explicit form of the second-order equations

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C_1^2\Big)F_1 = 0,$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C_2^2\Big)F_1 = 0 \tag{18}$$
$$\Longrightarrow \quad C_2^2 = C_1^2 = C^2;$$

and

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - C_2^2\Big)F_2 = 0,$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - C_2^2\Big)F_3 = 0.$$

Thus, we have only three different equations (and the constraint $C_2^2 = C_1^2 = C^2$):

$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - C^2\Big)F_1 = 0,$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m+1)^2}{r^2} - C^2\Big)F_2 = 0, \tag{19}$$
$$\Big(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{(m-1)^2}{r^2} - C^2\Big)F_3 = 0.$$

In the variable $x = iCr$, they take the Bessel form

$$\Big(\frac{d^2}{d^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{m^2}{x^2}\Big)F_1 = 0, \qquad F_1(x) \sim J_{\pm m}(x),$$

$$\Big(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{(m+1)^2}{x^2}\Big)F_2 = 0, \qquad F_2(x) \sim J_{\pm(m+1)}(x),$$

$$\Big(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + 1 - \frac{(m-1)^2}{x^2}\Big)F_3 = 0, \qquad F_3(x) \sim J_{\pm(m-1)}(x).$$

8. Solving equations in z-variable

Taking in mind the identities $C_1 = C_2 = C_3 = C_4 = C$, we get the following system in z-variable:

$$\frac{1}{\sqrt{2}}C\bar{E}_1(z) - \frac{1}{\sqrt{2}}C\bar{E}_3(z) - \partial_z \bar{E}_2(z) + e\mu\, E\bar{\Phi}_2(z) - 2\,iEe\sigma(\epsilon + eEz)\bar{E}_2(z) = M\bar{\Phi}_0(z)\,,$$

$$i(\epsilon + eEz)\bar{E}_1(z) + \frac{1}{\sqrt{2}}C\bar{B}_2(z) - \partial_z\bar{B}_3(z) - \sqrt{2}Ee\sigma\, C\bar{E}_2(z) = M\bar{\Phi}_1(z)\,,$$

$$i(\epsilon + eEz)\bar{E}_2(z) - \frac{1}{\sqrt{2}}C\bar{B}_1(z) - \frac{1}{\sqrt{2}}C\bar{B}_3(z) + e\mu\, E\bar{\Phi}_0(z) + 2\, Ee\sigma\, \partial_z\bar{E}_2(z) = M\bar{\Phi}_2(z)\,,$$

$$i\,(\epsilon + eEz)\,\bar{E}_3(z)(z) + \frac{1}{\sqrt{2}}C\bar{B}_2(z) + \partial_z\bar{B}_1(z) + \sqrt{2}\, Ee\sigma C\bar{E}_2(z) = M\bar{\Phi}_3(z)\,,$$

$$\frac{1}{\sqrt{2}}\, C\bar{\Phi}_0(z) - i\,(\epsilon + eEz)\,\bar{\Phi}_1(z) = M\bar{E}_1(z)\,,$$

$$-\partial_z\bar{\Phi}_0(z) - i\,(\epsilon + eEz)\,\bar{\Phi}_2(z) = M\bar{E}_2(z)\,,$$

$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_0(z) - i\,(\epsilon + eEz)\,\bar{\Phi}_3(z) = M\bar{E}_3(z)\,,$$

$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_2(z) + \partial_z\bar{\Phi}_3(z) = M\bar{B}_1(z)\,,$$

$$\frac{1}{\sqrt{2}}C\bar{\Phi}_1(z) + \frac{1}{\sqrt{2}}C\bar{\Phi}_3(z) = M\bar{B}_2(z)\,,$$

$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_2(z) - \partial_z\bar{\Phi}_1(z) = M\bar{B}_3(z)\,.$$

For shortness, we change the notations:

$$eE \Longrightarrow E, \ eE\mu \Longrightarrow E\mu, \ eE\sigma \Longrightarrow E\sigma, \qquad \epsilon + Ez = W(z), \quad W' = E.$$

Let us divide equations into two groups:

I

$$\frac{1}{\sqrt{2}}C\bar{E}_1(z)-\frac{1}{\sqrt{2}}C\bar{E}_3(z)-\partial_z\bar{E}_2(z)$$
$$+\mu\,E\bar{\Phi}_2(z)-2\,iE\sigma W(z)\bar{E}_2(z)=M\bar{\Phi}_0(z)\,,$$
$$iW(z)\bar{E}_2(z)-\frac{1}{\sqrt{2}}C\bar{B}_1(z)-\frac{1}{\sqrt{2}}C\bar{B}_3(z)$$
$$+\mu\,E\bar{\Phi}_0(z)+2\,E\sigma\,\partial_z\bar{E}_2(z)=M\bar{\Phi}_2(z)\,,$$
$$\frac{1}{\sqrt{2}}\,C\bar{\Phi}_0(z)-iW(z)\bar{\Phi}_1(z)=M\bar{E}_1(z)\,,$$
$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_0(z)-iW(z)\bar{\Phi}_3(z)=M\bar{E}_3(z)\,,$$
$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_2(z)+\partial_z\bar{\Phi}_3(z)=M\bar{B}_1(z)\,,$$
$$-\frac{1}{\sqrt{2}}C\bar{\Phi}_2(z)-\partial_z\bar{\Phi}_1(z)=M\bar{B}_3(z)\,;$$

II

$$iW(z)\bar{E}_1(z)+\frac{1}{\sqrt{2}}C\bar{B}_2(z)-\partial_z\bar{B}_3(z)$$
$$-\sqrt{2}E\sigma\,C\bar{E}_2(z)=M\bar{\Phi}_1(z)\,,$$
$$iW(z)\bar{E}_3(z)+\frac{1}{\sqrt{2}}C\bar{B}_2(z)+\partial_z\bar{B}_1(z)$$
$$+\sqrt{2}\,E\sigma C\bar{E}_2(z)=M\bar{\Phi}_3(z)\,,$$
$$-\partial_z\bar{\Phi}_0(z)-iW(z)\bar{\Phi}_2(z)=M\bar{E}_2(z)\,,$$
$$\frac{1}{\sqrt{2}}C\bar{\Phi}_1(z)+\frac{1}{\sqrt{2}}C\bar{\Phi}_3(z)=M\bar{B}_2(z)\,.$$

The system I can be solved with respect to six variables $\bar{\Phi}_0,\bar{\Phi}_2,\bar{E}_1,\bar{E}_3,\bar{B}_1,\bar{B}_3$; so we obtain

$$\Phi_0=\frac{1}{2\big(\big(C^2-M^2\big)^2-M^2E^2\mu^2\big)}\Big[2M\big((\partial_z+2iE(Ez+\epsilon)\sigma)C^2+M\big(iE(Ez+\epsilon)(\mu-2M\sigma)$$
$$-\partial_z\big(M-2E^2\mu\sigma\big)\big)\big)E_2+i\sqrt{2}C((C^2-M^2)(Ez+\epsilon)-i\partial_zME\mu)\big(\Phi_1-\Phi_3\big)\Big],$$

$$\Phi_2=-\frac{1}{2\big(\big(C^2-M^2\big)^2-M^2E^2\mu^2\big)}\Big[2iM\big((Ez+\epsilon-2i\partial_zE\sigma)C^2-M(MEz+M\epsilon+i\partial_zE\mu)$$
$$+2ME(i\partial_zM+E(Ez+\epsilon)\mu)\sigma\big)E_2+\sqrt{2}C(\partial_z(C^2-M^2)+iME(Ez+\epsilon)\mu)\big(\Phi_1-\Phi_3\big)\Big],$$

$$E_1=\frac{1}{2\big(M\big(C^2-M^2\big)^2-M^3E^2\mu^2\big)}\Big[\sqrt{2}CM\big((\partial_z+2iE(Ez+\epsilon)\sigma)C^2$$
$$+M\big(iE(Ez+\epsilon)(\mu-2M\sigma)-\partial_z\big(M-2E^2\mu\sigma\big)\big)\big)E_2$$
$$-i\big(((C^2-M^2)(Ez+\epsilon)-i\partial_zME\mu)\Phi_3C^2+\big(i\partial_zME\mu C^2-2M^2E^2(Ez+\epsilon)\mu^2+\big(C^4-3M^2C^2+2M^4\big)(Ez+\epsilon)\big)\Phi_1\big)\Big],$$

$$E_3=-\frac{1}{2\big(M\big(C^2-M^2\big)^2-M^3E^2\mu^2\big)}\Big[\sqrt{2}CM\big((\partial_z+2iE(Ez+\epsilon)\sigma)C^2$$
$$+M\big(iE(Ez+\epsilon)(\mu-2M\sigma)-\partial_z\big(M-2E^2\mu\sigma\big)\big)\big)E_2+i\big(((C^2-M^2)(Ez+\epsilon)$$
$$-i\partial_zME\mu)\Phi_1C^2+\big(i\partial_zME\mu C^2-2M^2E^2(Ez+\epsilon)\mu^2+\big(C^4-3M^2C^2+2M^4\big)(Ez+\epsilon)\big)\Phi_3\big)\Big],$$

$$B_1=\frac{1}{2M\big(C^2-M^2\big)^2-2M^3E^2\mu^2}\Big[(\partial_z(C-M)(C+M)+iME(Ez+\epsilon)\mu)\Phi_1C^2$$
$$+i\sqrt{2}M\big((Ez+\epsilon-2i\partial_zE\sigma)C^2-M(MEz+M\epsilon+i\partial_zE\mu)+2ME(i\partial_zM+E(Ez+\epsilon)\mu)\sigma\big)CE_2$$
$$+\big(\partial_zC^4+M(-3\partial_zM-iE(Ez+\epsilon)\mu)C^2+2\partial_zM^2(M-E\mu)(M+E\mu)\big)\Phi_3\Big],$$

$$B_3 = -\frac{1}{2M\big(C^2-M^2\big)^2 - 2M^3E^2\mu^2}\Big[\big((\partial_z(C^2-M^2)+iME(Ez+\epsilon)\mu)\Phi_3 C^2\big)$$
$$+i\sqrt{2}M\big((Ez+\epsilon-2i\partial_z E\sigma)C^2 - M(MEz+M\epsilon+i\partial_z E\mu)+2ME(i\partial_z M+E(Ez+\epsilon)\mu)\sigma\big)CE_2$$
$$+\big(-\partial_z C^4 + M(3\partial_z M + iE(Ez+\epsilon)\mu)C^2 - 2\partial_z M^2(M-E\mu)(M+E\mu)\big)\Phi_1\Big].$$

Now, we substitute these expressions into four remaining equations; in this way, we get four equations for the variables $\bar{\Phi}_1, \bar{\Phi}_3, \bar{E}_2, \bar{B}_2$:

$$1)\qquad \frac{B_2 C}{\sqrt{2}} + E_2\Big(-\frac{C\partial_z^2 E\big(2C^2\sigma + M(\mu-2M\sigma)\big)}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}$$
$$+\frac{1}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}\Big)\Big[CE\big(-2\sigma\big((C^2-M^2)\big(C^2-(M-Ez-\epsilon)(M+Ez+\epsilon)\big)$$
$$+\mu^2\big(-M^2\big)E^2+i\mu ME^2\big)-i(C^2-M^2)+\mu(-M)(Ez+\epsilon)^2\big)\Big]$$
$$+\Phi_3\Big(\frac{C^2\partial_z^2(C^2-M^2)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}+\frac{C^2\big((C^2-M^2)(Ez+\epsilon)^2+i\mu ME^2\big)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}\Big)$$
$$+\Phi_1\Big(\frac{\partial_z^2\big(C^4-3C^2M^2+2M^2(M-\mu E)(M+\mu E)\big)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}$$
$$+\frac{1}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}\Big[C^4\big((Ez+\epsilon)^2-2M^2\big)+C^2M\big(4M^3-3M(Ez+\epsilon)^2-i\mu E^2\big)$$
$$-2M^2(M-\mu E)(M+\mu E)(M-Ez-\epsilon)(M+Ez+\epsilon)\Big]\Big)=0,$$

$$2)\qquad \frac{B_2 C}{\sqrt{2}} + \Phi_1\Big(\frac{C^2\partial_z^2(C^2-M^2)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}+\frac{C^2\big((C^2-M^2)(Ez+\epsilon)^2+i\mu ME^2\big)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}\Big)$$
$$+E_2\Big(\frac{C\partial_z^2 E\big(2C^2\sigma+M(\mu-2M\sigma)\big)}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}$$
$$+\frac{1}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}\Big[CE\big(2C^4\sigma+C^2\big(-4M^2\sigma+2\sigma(Ez+\epsilon)^2+i\big)$$
$$+M\big(2M^3\sigma+M\big(-2\sigma\big(\mu^2E^2+(Ez+\epsilon)^2\big)-i\big)+\mu\big((Ez+\epsilon)^2+2i\sigma E^2\big)\big)\big)\Big]\Big)$$
$$+\Phi_3\Big(\frac{\partial_z^2\big(C^4-3C^2M^2+2M^2(M-\mu E)(M+\mu E)\big)}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}$$
$$+\frac{1}{2M\big(C^2-M^2\big)^2-2\mu^2M^3E^2}\Big[C^4\big((Ez+\epsilon)^2-2M^2\big)+C^2M\big(4M^3-3M(Ez+\epsilon)^2-i\mu E^2\big)$$
$$-2M^2(M-\mu E)(M+\mu E)(M-Ez-\epsilon)(M+Ez+\epsilon)\Big]\Big)=0,$$

$$3)\qquad E_2\Big(\frac{\partial_z^2 M\big(M\big(M-2\mu\sigma E^2\big)-C^2\big)}{\big(C^2-M^2\big)^2-\mu^2M^2E^2}$$

$$+\frac{1}{\big(C^2-M^2\big)^2-\mu^2M^2E^2}\Big[M\big(2i\sigma E^2\big(-C^2+M\big(M+i\mu(Ez+\epsilon)^2\big)\big)$$

$$-(C^2-M^2)\big(C^2-(M-Ez-\epsilon)(M+Ez+\epsilon)\big)+\mu^2M^2E^2-i\mu ME^2\big)\Big]\Big)$$

$$+\Phi_1\Big(-\frac{C\partial_z^2\mu ME}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}+\frac{CE\big(\mu(-M)(Ez+\epsilon)^2-i(C-M)(C+M)\big)}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}\Big)$$

$$\Phi_3\Big(\frac{C\partial_z^2\mu ME}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}+\frac{CE\big(\mu M(Ez+\epsilon)^2+i(C-M)(C+M)\big)}{\sqrt{2}\big(\big(C^2-M^2\big)^2-\mu^2M^2E^2\big)}\Big)=0,$$

$$4)\qquad B_2(-M)+\frac{C\Phi_1}{\sqrt{2}}+\frac{C\Phi_3}{\sqrt{2}}=0.$$

Eliminating the variable B_2, we obtain the system of three equations (let us change the notations, $G=\Phi_1,\ H=\Phi_3,\ F=E_2$) :

$$1)\qquad \Big(-\sqrt{2}C\partial_z^2ME\big(2C^2\sigma+M(\mu-2M\sigma)\big)$$

$$+\sqrt{2}CME\big(-2\sigma\big((C^2-M^2)\big(C^2-M^2+(Ez+\epsilon)^2\big)+\mu^2\big(-M^2\big)E^2+i\mu ME^2\big)$$

$$-i(C^2-M^2)+\mu(-M)(Ez+\epsilon)^2\big)\big)+G\big(\mu^2M^2E^2\big(2(M-Ez-\epsilon)(M+Ez+\epsilon)-C^2\big)$$

$$-iC^2\mu ME^2+\partial_z^2\big(C^4-3C^2M^2+2M^2(M-\mu E)(M+\mu E)\big)$$

$$+\big(C^4-3C^2M^2+2M^4\big)\big(C^2-(M-Ez-\epsilon)(M+Ez+\epsilon)\big)\Big)F$$

$$+\Big(C^6+C^4\big((Ez+\epsilon)^2-2M^2\big)+C^2\partial_z^2(C^2-M^2)+C^2M\big(M^3-\mu^2ME^2-M(Ez+\epsilon)^2+i\mu E^2\big)\Big)=0,$$

$$2)\qquad \Big(\sqrt{2}C\partial_z^2ME\big(2C^2\sigma+M(\mu-2M\sigma)\big)$$

$$+\sqrt{2}CME\big(2C^4\sigma+C^2\big(-4M^2\sigma+2\sigma(Ez+\epsilon)^2+i\big)+M\big(2M^3\sigma$$

$$+M\big(-2\sigma\big(\mu^2E^2+(Ez+\epsilon)^2\big)-i\big)+\mu\big((Ez+\epsilon)^2+2i\sigma E^2\big)\big)\big)\Big)F$$

$$+\Big(\mu^2M^2E^2\big(2(M-Ez-\epsilon)(M+Ez+\epsilon)-C^2\big)-iC^2\mu ME^2$$

$$+\partial_z^2\big(C^4-3C^2M^2+2M^2(M-\mu E)(M+\mu E)\big)+\big(C^4-3C^2M^2+2M^4\big)\big(C^2-(M-Ez-\epsilon)(M+Ez+\epsilon)\big)\Big)H$$

$$+\Big(C^6+C^4\big((Ez+\epsilon)^2-2M^2\big)+C^2\partial_z^2(C^2-M^2)+C^2M\big(M^3-\mu^2ME^2-M(Ez+\epsilon)^2+i\mu E^2\big)\Big)G=0,$$

$$3)\qquad \Big(2\partial_z^2M^2\big(M\big(M-2\mu\sigma E^2\big)-C^2\big)$$

$$+2M^2\big(-(C^2-M^2)\big(C^2-M^2+(Ez+\epsilon)^2\big)+2i\sigma E^2\big(-C^2+M\big(M+i\mu(Ez+\epsilon)^2\big)\big)+\mu^2M^2E^2-i\mu ME^2\big)\Big)F$$

$$+\Big(-\sqrt{2}C\partial_z^2\mu M^2E+\sqrt{2}CME\big(\mu(-M)(Ez+\epsilon)^2-i(C^2-M^2)\big)\Big)H$$

$$+\Big(\sqrt{2}C\partial_z^2\mu M^2E+\sqrt{2}CME\big(\mu M(Ez+\epsilon)^2+i(C^2-M^2)\big)\Big)G=0,$$

Let us sum and subtract the first two equations as

$$1+2)\qquad \Big((C^2-M^2)^2-\mu^2M^2E^2\big)\big(C^2+\partial_z^2-(M^2-(Ez+\epsilon)^2)\Big)(G+H)=0,$$

$$1-2)\qquad M\Big(-2\sqrt{2}C^5\sigma E-\sqrt{2}C^3E\big(2\sigma\big(\partial_z^2-2M^2+(Ez+\epsilon)^2\big)+i\big)$$
$$-\sqrt{2}CME\big(M\big(-2\sigma\big(\partial_z^2+\mu^2E^2+(Ez+\epsilon)^2\big)-i\big)+\mu\big(\partial_z^2+2i\sigma E^2+(Ez+\epsilon)^2\big)+2M^3\sigma\big)\Big)F$$
$$+M\Big(C^4M-C^2\big(\partial_z^2M-2M^3+M(Ez+\epsilon)^2+i\mu E^2\big)$$
$$+M(M-\mu E)(M+\mu E)\big(\partial_z^2-M^2+(Ez+\epsilon)^2\big)\Big)(H-G)=0,$$

$$3)\qquad M\Big(i\sqrt{2}CME\big(M+i\mu\big(\partial_z^2+(Ez+\epsilon)^2\big)\big)-i\sqrt{2}C^3E\Big)(G-H)$$
$$+M\Big(-2C^4M-2C^2M\big(\partial_z^2-2M^2+2i\sigma E^2+(Ez+\epsilon)^2\big)+$$
$$+2M^2\big(\partial_z^2\big(M-2\mu\sigma E^2\big)-M^3+M\big(E^2\big(\mu^2+2i\sigma+z^2\big)+2Ez\epsilon+\epsilon^2\big)+\mu E^2\big(-2\sigma(Ez+\epsilon)^2-i\big)\big)\Big)F=0.$$

It is convenient to introduce new notations

$$G+H=V,\quad G-H=U;\tag{20}$$

then we get

$$1)\qquad \Big(\partial_z^2+(Ez+\epsilon)^2+C^2-M^2\Big)V(z)=0,\tag{21}$$

$$2)\qquad \Big(-\sqrt{2}C\partial_z^2ME\big(2C^2\sigma+M(\mu-2M\sigma)\big)$$
$$+\sqrt{2}CME\big(-2\sigma\big((C^2-M^2)\big(C^2-(M-(Ez+\epsilon)^2)\big)$$
$$+\mu^2\big(-M^2\big)E^2+i\mu ME^2\big)-iC^2+M\big(-\mu(Ez+\epsilon)^2+iM\big)\big)\Big)F$$
$$+\Big(\partial_z^2M^2\big(-C^2+M^2-\mu^2E^2\big)+M\big(C^4(-M)+C^2\big(2M^3-M(Ez+\epsilon)^2-i\mu E^2\big)$$
$$-M(M-\mu E)(M+\mu E)\big(M^2-(Ez+\epsilon)^2\big)\big)\Big)U=0,$$

$$3)\qquad \Big(-\sqrt{2}C\partial_z^2\mu M^2E+\sqrt{2}CME\big(M\big(-\mu(Ez+\epsilon)^2+iM\big)-iC^2\big)\Big)U$$
$$+\Big(2\partial_z^2M^2\big(M\big(M-2\mu\sigma E^2\big)-C^2\big)+2M^2\big(-C^4+C^2\big(2M^2-2i\sigma E^2-(Ez+\epsilon)^2\big)$$
$$+M\big(-M^3+M\big(E^2\big(\mu^2+2i\sigma+z^2\big)+2Ez\epsilon+\epsilon^2\big)+\mu E^2\big(-2\sigma(Ez+\epsilon)^2-i\big)\big)\big)\Big)F=0.$$

Let us consider the system of two linked second and third equations as

$$2)\quad A_2U''+B_2(z)U+K_2F''+L_2(z)F=0,\quad 3)\quad A_3U''+B_3(z)U+K_3F''+L_3(z)F=0.$$

Let us combine last two equations as follows. Multiplying equation marked as "2)"with β and multiplying equation marked as "3)"with γ and add these together we get

$$(\beta A_2 + \gamma A_3)U'' + (\beta B_2 + \gamma B_3)U + (\beta K_2 + \gamma K_3)F'' + (\beta L_2 + \gamma L_3)F = 0.$$

Let us consider two different variant:
the first one

$$1) \qquad (\beta A_2 + \gamma A_3)^{=1}U'' + (\beta B_2 + \gamma B_3)U + (\beta K_2 + \gamma K_3)^{=0}F'' + (\beta L_2 + \gamma L_3)F = 0 \quad \Longrightarrow$$

$$\beta A_2 + \gamma A_3 = 1 \quad \beta K_2 + \gamma K_3 = 0,$$

solution of the linear system is

$$\beta = \frac{M\left(M - 2\mu\sigma W^2\right) - C^2}{M\left(\left(C^2 - M^2\right)^2 - \mu^2 M^2 W^2\right)\left(M - 2\mu\sigma W^2\right)},$$
$$\gamma = \frac{CW\left(2C^2\sigma + M(\mu - 2M\sigma)\right)}{\sqrt{2}M^2\left(\left(C^2 - M^2\right)^2 - \mu^2 M^2 W^2\right)\left(M - 2\mu\sigma W^2\right)},$$

so that

$$1) \qquad U'' + \Big(\frac{C^2\left(M^2 - 2i\sigma E^2\right)}{M\left(M - 2\mu\sigma E^2\right)} - M^2 + (Ez + \epsilon)^2\Big)U + \frac{\sqrt{2}CE(\mu M - i)\left(4\sigma^2 E^2 - 1\right)}{M - 2\mu\sigma E^2}F = 0$$

the second one is

$$2) \qquad + (\beta B_2 + \gamma B_3)U + (\beta K_2 + \gamma K_3)^{=1}F'' + (\beta L_2 + \gamma L_3)F = 0 \quad \Longrightarrow$$

$$\beta K_2 + \gamma K_3 = 1, \quad \beta A_2 + \gamma A_3 = 0,$$

solution of the linear system is

$$\beta = \frac{C\mu E}{\sqrt{2}M\left(\left(C^2 - M^2\right)^2 - \mu^2 M^2 E^2\right)\left(M - 2\mu\sigma E^2\right)},$$
$$\gamma = -\frac{C^2 - M^2 + \mu^2 E^2}{2M\left(\left(C^2 - M^2\right)^2 - \mu^2 M^2 E^2\right)\left(M - 2\mu\sigma E^2\right)},$$

so that

$$2) \qquad F'' + \frac{C^2\left(M - 2\mu\sigma E^2\right) - M^3 + M\left(E^2\left(\mu^2 + 2i\sigma + z^2\right) + 2Ez\epsilon + \epsilon^2\right) + \mu E^2\left(-2\sigma(Ez + \epsilon)^2 - i\right)}{M - 2\mu\sigma E^2}F$$
$$+ \frac{CE(-\mu M + i)}{\sqrt{2}\left(M - 2\mu\sigma E^2\right)}U = 0$$

Thus, after this transformation we arrive at the following much more simple equations

$$U'' + \Big(\frac{C^2\left(M^2 - 2i\sigma E^2\right)}{M\left(M - 2\mu\sigma E^2\right)} - M^2 + (Ez+\epsilon)^2\Big)U + \frac{\sqrt{2}CE(\mu M - i)\left(4\sigma^2E^2 - 1\right)}{M - 2\mu\sigma E^2}F = 0,$$

$$F'' + \frac{\left(C^2\left(M - 2\mu\sigma E^2\right) - M^3 + M\left(E^2\left(\mu^2 + 2i\sigma + z^2\right) + 2Ez\epsilon + \epsilon^2\right) + \mu E^2\left(-2\sigma(Ez+\epsilon)^2 - i\right)\right)}{M - 2\mu\sigma E^2}F$$
$$+ \frac{CE(-\mu M + i)}{\sqrt{2}\left(M - 2\mu\sigma E^2\right)}U = 0.$$

They can be reduced to more convenient form

$$\begin{aligned} &F'' + \Big(C^2 + (Ez+\epsilon)^2\Big)F + \frac{[\mu(M\mu - i) + 2i\sigma M]E^2 - M^3}{M - 2\mu\sigma E^2}F + \frac{CE(-\mu M + i)}{\sqrt{2}\left(M - 2\mu\sigma E^2\right)}U = 0, \\ &U'' + \Big(-M^2 + (Ez+\epsilon)^2\Big)U + \frac{C^2\left(M^2 - 2i\sigma E^2\right)}{M\left(M - 2\mu\sigma E^2\right)}U + \frac{\sqrt{2}CE(\mu M - i)(4\sigma^2E^2 - 1)}{M - 2\mu\sigma E^2}F = 0. \end{aligned} \tag{22}$$

For additional checking, let us fix the physical dimensions of the quantities

$$[M] = \frac{1}{L}, \quad [C] = \frac{1}{L}, \quad [\epsilon] = \frac{1}{L},$$
$$[E] = \frac{1}{L^2}, \quad [\mu] = L, \quad [\sigma] = L^2.$$

Let us present the system in matrix form

$$\Delta = \frac{d^2}{dz^2} + (\epsilon + Ez)^2, \ \Delta \begin{vmatrix} F \\ U \end{vmatrix} = T \begin{vmatrix} F \\ U \end{vmatrix},$$
$$\Delta\Psi(z) = T\Psi(z), \ T = \begin{vmatrix} A & K \\ L & B \end{vmatrix}.$$

Now we will find the transformations which diagonalizes the system

$$\Psi' = S\Psi, \quad \Delta\bar{\Psi}(z) = \bar{T}\bar{\Psi}(z),$$
$$\bar{T} = STS^{-1} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix}, \ S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}$$

so that

$$ST = \bar{T}S,$$
$$\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix} \begin{vmatrix} A & K \\ L & B \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix};$$

whence follow two similar subsystems

$$\begin{cases} (A - \lambda_1)s_{11} + Ls_{12} = 0 \\ Ks_{11} + (B - \lambda_1)s_{12} = 0 \end{cases},$$
$$\begin{cases} (A - \lambda_2)s_{21} + Ls_{22} = 0 \\ Ks_{21} + (B - \lambda_2)s_{22} = 0 \end{cases}.$$

From vanishing the determinant we get

$$\lambda^2 - (A+B)\lambda + AB - LK = 0,$$
$$\lambda = \frac{A+B}{2} \pm \sqrt{\frac{(A+B)^2}{4} - (AB - LK)}. \tag{23}$$

Explicitly, the involved parameters read

$$\frac{A+B}{2} = \frac{M\left(-2C^2M + 2M^3 - \mu^2ME^2 + i\mu E^2\right) + 2\sigma E^2(C^2 - M^2)(\mu M + i)}{2M\left(M - 2\mu\sigma E^2\right)},$$

$$\frac{(A+B)^2}{4} = \frac{\left(M\left(2C^2M - 2M^3 + \mu^2ME^2 - i\mu E^2\right) - 2\sigma E^2(C^2 - M^2)(\mu M + i)\right)^2}{4M^2\left(M - 2\mu\sigma E^2\right)^2},$$

$$-(AB - LK) =$$
$$= -\frac{C^4M^2 - 2C^2M^4 - 2i\sigma E^2\left(C^2 - M^2\right)^2 + iC^2\mu M E^2 + C^2E^2 + M^6 - \mu^2M^4E^2 + i\mu M^3E^2}{M\left(M - 2\mu\sigma E^2\right)}.$$

Solutions of eq. (23) have the form

$$\lambda = \frac{1}{2M(M - 2\mu\sigma E^2)}$$
$$\times\Big[\ 2C^2\Big(- M^2 + \mu M\sigma E^2 + i\sigma E^2\Big) + 2M^4 + E^2\Big(- 2\mu M^3\sigma - \mu^2M^2 - 2iM^2\sigma + i\mu M\Big)$$
$$\pm\sqrt{E^2(\mu M - i)^2\Big(4C^4\sigma^2E^2 + 4C^2M\left(-2M\sigma^2E^2 + M - \mu\sigma E^2\right) + M^2E^2(\mu - 2M\sigma)^2\Big)}\ \Big]. \qquad (24)$$

Let us take into account that two parameters are imaginary, $\sigma \Rightarrow i\sigma,\ \mu \Rightarrow i\mu$; then we get

$$\lambda = \frac{1}{2M\left(M + 2\mu\sigma E^2\right)}$$
$$\times\Big[2M^2\left(M^2 - C^2\right) + E^2\Big(2\sigma(M^2 - C^2)(\mu M + 1) + \mu M(\mu M - 1)\Big)$$
$$\pm\sqrt{E^2(\mu M - 1)^2\Big(4C^4\sigma^2E^2 - 4C^2M\left(2M\sigma^2E^2 + M + \mu\sigma E^2\right) + M^2E^2(\mu - 2M\sigma)^2\Big)}\Big] \qquad (25)$$

If $C = M$, we obtain

$$\lambda = \frac{1}{2M\left(M + 2\mu\sigma E^2\right)}\Big[\mu M E^2(\mu M - 1) \pm \sqrt{-M^2E^2(\mu M - 1)^2\left(4M\left(M + 2\mu\sigma E^2\right) - \mu^2E^2\right)}\Big]; \qquad (26)$$

correspondingly we have two separate equations

$$\Big(\frac{d^2}{dz^2} + (\epsilon + Ez)^2 - \lambda_1\Big)\bar{F} = 0, \quad \Big(\frac{d^2}{dz^2} + (\epsilon + Ez)^2 - \lambda_2\Big)\bar{U} = 0. \qquad (27)$$

Let us impose an additional restriction $A + B = 0$, this permits us to get fixed parameter C:

$$C = \pm\frac{i\sqrt{M}\sqrt{2M^3 - \mu^2ME^2 - 2M\sigma E^2(\mu M + i) + i\mu E^2}}{\sqrt{-2M^2 + 2\sigma E^2(\mu M + i)}};$$

let $\sigma \Rightarrow i\sigma,\ \mu \Rightarrow i\mu$, then

$$C = \pm\frac{i\sqrt{M}\sqrt{2M^3 + \mu^2ME^2 + 2M\sigma E^2(\mu M + 1) - \mu E^2}}{\sqrt{2}\sqrt{-M^2 - \sigma E^2(\mu M + 1)}}$$

and we get two roots different only in sign:

$$\lambda(M, E, \mu, \sigma) = \pm\frac{i(1 - \mu M)E}{2\sqrt{M + 2\mu\sigma E^2}\ \Big(M^2 + \sigma E^2(\mu M + 1)\Big)}$$
$$\times\Big[\ 2\sigma E^2\Big(4M^3 + \mu M^4 - \mu E^2\Big) + M^2\Big(4M^3 + \mu^2ME^2 - 2\mu E^2\Big) + 4M\sigma^2E^4\Big(1 + \mu M\Big)^2\ \Big]^{1/2}. \qquad (28)$$

correspondingly we have two separate equations

$$\Big(\frac{d^2}{dz^2} + (\epsilon + Ez)^2 + \lambda\Big)\bar{F} = 0, \quad \Big(\frac{d^2}{dz^2} + (\epsilon + Ez)^2 - \lambda\Big)\bar{U} = 0. \qquad (29)$$

9. Solving the differential equation

The derived equation has the same structure as for a scalar particle in the uniform electric field (we transform the equation to the new variable, assuming that $E > 0$)

$$\Big(\frac{d^2}{dz^2} + (Ez+\epsilon)^2 - m^2\Big)\Phi(z) = 0, \quad Z = i\frac{(Ez+\epsilon)^2}{E}, \quad \Lambda = \frac{m^2}{4E}; \tag{30}$$

then we get the confluent hypergeometric equation [17]

$$\Big(\frac{d^2}{dZ^2} + \frac{1/2}{Z}\frac{d}{dZ} - \frac{1}{4} + \frac{i\Lambda}{Z}\Big)\Phi(Z) = 0\,. \tag{31}$$

This equation has two singular points. The point $Z = 0$ is regular, behavior of solutions near this point is given by the formulas $Z \to 0,\ \Phi(Z) = Z^A,\ A = 0, 1/2$. The point $Z = \infty$ is irregular point of the rank 2. Indeed, in the inverse variable $y = Z^{-1}$ we get the equation

$$\Big(\frac{d^2}{dy^2} + \frac{3}{2y}\frac{d}{dy} - \frac{1}{4y^4} + \frac{i\Lambda}{y^3}\Big)\Phi = 0\,. \tag{32}$$

Asymptotic of solutions at $y \to 0$ should have the structure $\Phi = y^C e^{D/y}$. Further we arrive at

$$\begin{gathered} D^2 - \frac{1}{4} = 0, \\ -2CD + 2D - \frac{3}{2}D + i\Lambda = 0\,; \\ D_1 = +\frac{1}{2}, \quad C_1 = \frac{1}{4} + i\Lambda\,; \\ D_2 = -\frac{1}{2}, \quad C_2 = \frac{1}{4} - i\Lambda\,. \end{gathered} \tag{33}$$

Therefore, in infinity there are possible two behaviors

$$Z \to \infty,$$

$$\Phi = Z^{-C}e^{DZ} = \begin{cases} Z^{-C_1}e^{D_1 Z} = Z^{-1/4-i\Lambda}e^{+Z/2}, \\ Z^{-C_2}e^{D_2 Z} = Z^{-1/4+i\Lambda}e^{-Z/2}, \end{cases}$$

where (we use the main branch of the logarithmic function)

$$\begin{gathered} Z = i\frac{(\epsilon+Ez)^2}{E} = iZ_0,\ Z_0 > 0\,, \\ e^{\pm Z/2} = e^{\pm iZ_0/2}\,, \\ Z^{-1/4\mp i\Lambda} = \Big(e^{\ln iZ_0}\Big)^{-1/4\mp i\Lambda} \\ = \Big(e^{\ln Z_0 + i\pi/2}\Big)^{-1/4\mp i\Lambda}\,. \end{gathered} \tag{34}$$

Let us find solutions in the whole region of the variable Z. To this end, we apply the substitution $\Phi(Z) = Z^A\, e^{BZ}\, f(Z)$; taking in mind the constraints $A = 0\,, 1/2\,,\ B = -1/2$ we get the equation

$$\Big[Z\,\frac{d^2}{dZ^2} + (2A + 1/2 - Z)\,\frac{d}{dZ} - (A + 1/4 - i\Lambda)\Big] f(Z) = 0\,,$$

which can be recognized as the confluent hypergeometric equation with parameters

$$a = A + 1/4 - i\Lambda\,, \quad c = 2A + 1/2\,, \qquad f(Z) = Z^A\, e^{-Z/2}\, F(a, c; Z)\,.$$

Without loss of generality we can use the value $A = 0$:

$$A = 0,\ a = 1/4 - i\Lambda\,,\ c = +1/2\,, \qquad \Phi(Z) = e^{-Z/2}\, f(Z)\,. \tag{35}$$

The confluent hypergeometric equation has different sets of linearly independent solutions.

First, consider the following ones

$$Y_1(Z) = F(a,c;Z) = e^Z F(c-a,c;-Z),$$
$$Y_2(Z) = Z^{1-c} F(a-c+1, 2-c; Z) \tag{36}$$
$$= Z^{1-c} e^Z F(1-a, 2-c; -Z) \; .$$

They lead to the complete functions

$$\Phi_1 = e^{-Z/2} F(a,c;Z) = e^{+Z/2} F(c-a,c;-Z),$$
$$\Phi_2 = e^{-Z/2} Z^{1-c} F(a-c+1, 2-c; Z)$$
$$= Z^{1-c} e^{+Z/2} F(1-a, 2-c; -Z) \; .$$

Taking in mind the identities

$$c = \frac{1}{2},\; a = \frac{1}{4} - i\Lambda,\; c - a = \frac{1}{4} + i\Lambda = a^*,$$
$$c = c^* = \frac{1}{2},\; Z^* = -Z \; ,$$
$$a - c + 1 = \frac{3}{4} - i\Lambda = (1-a)^*,$$
$$(2-c) = (2-c)^* = \frac{3}{2} \; ,$$

we can conclude that the solution $\Phi_1(Z)$ is given by the real-valued function, the second $\Phi_2(Z)$ has a definite symmetry under complex conjugation:

$$\Phi_1(Z) = +[\Phi_1(Z)]^*,\; \Phi_2(Z) = i[\Phi_2(Z)]^*. \tag{37}$$

This property of the function $\Phi_2(Z)$ may be presented differently when using other normalization

$$\bar{\Phi}_2(Z) = \frac{1-i}{\sqrt{2}} \Phi_2(Z)$$
$$= \Big(\frac{1-i}{\sqrt{2}} \Phi_2(Z) \Big)^* = (\bar{\Phi}_2(Z))^* \; .$$

At small Z the above solutions behave

$$Y_1(Z) \approx 1,$$
$$Y_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\frac{i}{eE}}\; (\epsilon + eEz) \; ;$$
$$\Phi_1(Z) \approx 1,$$
$$\Phi_2(Z) \approx \sqrt{Z} = \sqrt{iZ_0} = \sqrt{\frac{i}{eE}}\; (\epsilon + eEz) \; .$$

At large $Z = iZ_0, Z_0 \to +\infty$ we can apply the asymptotic formula (see in [17])

$$F(a,c,Z) = \Big(\frac{\Gamma(c)}{\Gamma(c-a)} (-Z)^{-a} + ... \Big) + \Big(\frac{\Gamma(c)}{\Gamma(a)} e^Z Z^{a-c} + ... \Big) \; . \tag{38}$$

Taking into account identities

$$(-Z)^{-a} = (-iZ_0)^{-1/4+i\Lambda} = \Big(e^{\ln Z_0 - i\pi/2} \Big)^{-1/4+i\Lambda}$$
$$= e^{-(-1/4+i\Lambda)i\pi/2}\; e^{(-1/4+i\Lambda)\ln Z_0} \; ,$$
$$Z^{a-c} = (iZ_0)^{-1/4-i\Lambda} = \Big(e^{\ln Z_0 + i\pi/2} \Big)^{-1/4-i\Lambda}$$
$$= e^{+(-1/4-i\Lambda)i\pi/2}\; e^{(-1/4-i\Lambda)\ln Z_0} \; ,$$
$$\frac{\Gamma(c)}{\Gamma(c-a)} = \frac{\Gamma(1/2)}{\Gamma(1/4+i\Lambda)},\; \frac{\Gamma(c)}{\Gamma(a)} = \frac{\Gamma(1/2)}{\Gamma(1/4-i\Lambda)}$$

we find the following behavior of the solutions

$$Y_1(Z) = F(a,c,Z) = e^{iZ_0/2} \Big\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\Lambda)}\; e^{-(-1/4+i\Lambda)i\pi/2}\; e^{(-1/4+i\Lambda)\ln Z_0} e^{-iZ_0/2} + \frac{\Gamma(1/2)}{\Gamma(1/4-i\Lambda)} e^{+(-1/4-i\Lambda)i\pi/2}\; e^{(-1/4-i\Lambda)\ln Z_0} e^{+iZ_0/2} \Big\} . \tag{39}$$

Whence after transition to the variable $\Phi_1(Z)$ we get

$$\Phi_1(Z) = \Big\{ \frac{\Gamma(1/2)}{\Gamma(1/4+i\Lambda)} \; e^{-(-1/4+i\Lambda)i\pi/2} \; e^{(-1/4+i\Lambda)\ln Z_0} e^{-iZ_0/2}$$
$$+ \frac{\Gamma(1/2)}{\Gamma(1/4-i\Lambda)} e^{+(-1/4-i\Lambda)i\pi/2} \; e^{(-1/4-i\Lambda)\ln Z_0} e^{+iZ_0/2} \Big\}, \qquad (40)$$

where we can see the sum of two conjugate terms. Similarly, we can examine behavior in infinity of the second solution $F(a-c+1, 2-c; Z)$:

$$F(a-c+1, 2-c, Z) = \frac{\Gamma(2-c)}{\Gamma(1-a)}(-Z)^{-a+c-1} + \frac{\Gamma(2-c)}{\Gamma(a-c+1)} e^Z Z^{a-1} \; .$$

Whence taking into account the identities

$$(-Z)^{-a+c-1} = (-iZ_0)^{-3/4+i\Lambda}$$
$$= \left(e^{\ln Z_0 - i\pi/2} \right)^{-3/4+i\Lambda} = e^{-(-3/4+i\Lambda)i\pi/2} \; e^{(-3/4+i\Lambda)\ln Z_0} \; ,$$
$$Z^{a-1} = \left(e^{\ln Z_0 + i\pi/2} \right)^{-3/4-i\Lambda} = e^{+(-3/4-i\Lambda)i\pi/2} \; e^{(-3/4-i\Lambda)\ln Z_0} \; ,$$
$$\frac{\Gamma(2-c)}{\Gamma(1-a)} = \frac{\Gamma(3/2)}{\Gamma(3/4+i\Lambda)}, \quad \frac{\Gamma(2-c)}{\Gamma(a-c+1)} = \frac{\Gamma(3/2)}{\Gamma(3/4-i\Lambda)},$$

we find the following behavior in infinity

$$F(a-c+1, 2-c, Z) = e^{iZ_0/2}$$

$$\times \Big\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\Lambda)} \; e^{-(-3/4+i\Lambda)i\pi/2} \; e^{(-3/4+i\Lambda)\ln Z_0} e^{-iZ_0/2}$$
$$+ \frac{\Gamma(3/2)}{\Gamma(3/4-i\Lambda)} e^{+(-3/4-i\Lambda)i\pi/2} \; e^{(-3/4-i\Lambda)\ln Z_0} e^{+iZ_0/2} \Big\} . \qquad (41)$$

Whence for the function $\Phi_2(Z)$ we derive (allowing for $\sqrt{Z} = e^{(1/2)(\ln Z_0 + i\pi/2)}$)

$$\Phi_2(Z) = \sqrt{Z} F(a-c+1, 2-c, Z)$$

$$= e^{i\pi/4} \Big\{ \frac{\Gamma(3/2)}{\Gamma(3/4+i\Lambda)} \; e^{-(-3/4+i\Lambda)i\pi/2} \; e^{(-1/4+i\Lambda)\ln Z_0} e^{-iZ_0/2}$$

$$+ \frac{\Gamma(3/2)}{\Gamma(3/4-i\Lambda)} e^{+(-3/4-i\Lambda)i\pi/2} \; e^{(-1/4-i\Lambda)\ln Z_0} e^{+iZ_0/2} \Big\}. \qquad (42)$$

It is possible to construct two solutions which at infinity behave as conjugate functions. To this end, we should use other pair of independent solutions (see in [17])

$$Y_5(Z) = \Psi(a, c; Z), Y_7(Z) = e^Z \Psi(c-a, c; -Z) \; .$$

Two pairs $\{Y_5, Y_7\}$ and $\{Y_1, Y_2\}$ are related by Kummer formulas (see in [17])

$$Y_5 = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} Y_1 + \frac{\Gamma(c-1)}{\Gamma(a)} Y_2 \,, \qquad Y_7 = \frac{\Gamma(1-c)}{\Gamma(1-a)} Y_1 - \frac{\Gamma(c-1)}{\Gamma(c-a)} e^{i\pi c} Y_2 \,. \tag{43}$$

Whence we can derive asymptotic relations in the region $\mid Z \mid \to \infty$

$$Y_5(Z) = (iZ_0)^{-1/4+i\Lambda} = \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\Lambda} \,,$$

$$Y_7(Z) = e^Z \Psi(c-a, c; -Z) = e^Z (-iZ_0)^{a-c}$$

$$= e^{iZ_0} (-iZ_0)^{-1/4-i\Lambda} = e^{iZ_0} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\Lambda} .$$

The last formulas after translating them to variables $\Phi(Z)$ take on the form

$$\Phi_5(Z) = e^{-Z/2}\; Y_5(Z) = e^{-iZ_0/2} \left(e^{\ln Z_0 + i\pi/2}\right)^{-1/4+i\Lambda} \,, \qquad \Phi_7(Z) = e^{-Z/2}\; Y_7(Z) = e^{+iZ_0/2} \left(e^{\ln Z_0 - i\pi/2}\right)^{-1/4-i\Lambda} . \tag{44}$$

These functions are conjugate to each other, they are presented in the combinations (40) and (42).

10. Conclusion

The generalized Duffin – Kemmer equation for a spin 1 particle with anomalous magnetic moment and polarizability is studied in presence of external uniform electric field. After separating the variables, we derive the system of 10 first order differential equations in polar coordinate. To resolve this system, we apply the method by Fedorov – Gronskiy. Within this approach, the complete 10-component wave function is decomposed in three projective constituents, dependence of each on the polar coordinates is determined by only one function. We find expressions for this basic function $F_i(r)$ in terms of Bessel functions. After this, the system of 10 equations in z-variable is obtained. Three independent solutions of this system are found.

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