Nonlinear Phenomena in Complex Systems, vol. **28**, no. 3 (2025), pp. 248 - 260

Electromagnetic Field in the Newman-Unti-Tamburino Spacetime

N. G. Krylova* and V. M. Red'kov†
B. I. Stepanov Institute of Physics of NAS of Belarus,
68 Nezavisimosti Ave., 220072 Minsk, BELARUS‡
(Received 23 May, 2025)

Using the conventional tetrad method by Tetrode-Weyl-Fock-Ivanenko, we specify the Maxwell equations for Newman-Unti-Tamburino (NUT) spacetime. We apply the covariant Majorana–Oppenheimer matrix presentation of the Maxwell theory. Separation of the variables is performed, and the equations for angular and radial components are solved in terms of hypergeometric and confluent Heun functions respectively. We find the NUT-charge dependent quantization rule for the angular separation constant. Behavior of the radial components with structure of outgoing and ingoing waves is studied near the outer event horizon, and we demonstrate that the probability of particle-antiparticle production on the outer event horizon decreases with the increase of the NUT charge; the expression of temperature for the Hawking radiation of the photons coincides with that for the fermions production on the horizon. The effective constitutive relations, generated by metric structure of NUT spacetime, are derived; it is shown that the existence of the NUT charge leads to entanglement of electric and magnetic field components in these relations.

PACS numbers: 02.30.Gp, 02.40.Ky, 03.65Ge, 04.62.+v
Keywords: quantum mechanics, Maxwell theory, Majorana–Oppenheimer formalism, Riemannian geometry, Newman-Unti-Tamburino spacetime, Hawking radiation, constitutive relations
DOI: https://doi.org/10.5281/zenodo.17237014

1. Introduction

The Newman-Unti-Tamburino (NUT) metric is an axially symmetric vacuum solution of Einstein equations with two parameters, the black hole mass and the NUT parameter. The NUT spacetime is generalization of the Schwarzschild one, due to the presence the NUT parameter (or NUT charge) [1–3]. NUT parameter is understood as a gravito-magnetic charge, or as gravito-magnetic monopole, or magnetic (gravitomagnetic) mass [3, 4].

For NUT spacetime the singularities of the Misner string type arise. This leads to the difficulties in thermodynamical analysis and, as a consequence, in physical interpretation of the NUT parameter [5]. Mostly, the NUT parameter is interpreted as a linear source of a pure angular momentum [6] or the twist parameter of the surrounding vacuum spacetime or electromagnetic (EM) universe in the presence of the EM field [7].

The existence of the Misner string as well as the non-vanishing $g_{t\phi}$ components in metric tensor, lead to the spacetime areas where T-symmetry is broken and circular time-like (null) geodesics exist. By this reason, sometimes NUT spacetime is considered as nonphysical one. However, in [8] it was shown that geodesics of the freely falling observer are not closed time-like ones, so the NUT spacetime can be geodesically complete, without causal pathologies. Currently, the black holes with NUT parameter are considered as one of the most intriguing cosmological objects.

As shown in [9], the black hole with NUT charge has the smaller Hawking temperature,

*E-mail: `nina-kr@tut.by`
†E-mail: `v.redkov@dragon.bas-net.by`
‡Also at Belarusian State Agrarian Technical University, 99 Nezavisimosti Ave., Minsk, BELARUS

and pure gravitomagnetic monopoles without ordinary mass (if exist) may not be decayed due to the Hawking radiation by now.

In [8] it was demonstrated that the supercritically charged black holes with NUT parameter belong to traversable wormhole solutions. Besides, the NUT black holes may exhibit a twist in the lensing pattern [10], and an asymmetry of black hole shadow or the Lense-Thirring effect [3, 11].

Classical equations of motion in NUT spacetimes were extensively studied [12–14]. However, the papers on the quantum-mechanical problems of the particles in the background of NUT spacetimes are few. In [15], within the Newman-Penrose formalism, the Maxwell equations have been studied in Taub-NUT background which has singularity at whole axes $\theta = 0, \pi$. After separating the variables, solutions of the angular equations were constructed in terms of Jacobi polynomials. The radial equations (see equations (28a) and (44a) in [15]) were transformed respectively to equations with hypergeometric and Heun's structure on the left-hand side, while the right-hand sides include terms of different order in frequency ω. Only approximate solutions with the zeroth order in frequency ω in the right-hand side have been found in terms of the hypergeometric and Heun functions.

The goal of the present paper is to study the electromagnetic field in background of original NUT spacetime which has singularity only at semiaxes $\theta = \pi$. We apply the conventional tetrad method developed by Tetrode-Weyl-Fock-Ivanenko in [16–19], and covariant Majorana–Oppenheimer matrix presentation of the Maxwell theory [20–22], in the background of the original NUT spacetime. We have solved the angular and radial equations in terms of hypergeometric and Heun functions, respectively.

2. Ricci rotation coefficients for NUT space

NUT-metric is determined by the line element

$$ds^2 = \Phi \left(dt + 4a \sin^2(\theta/2) d\phi\right)^2 - \frac{dr^2}{\Phi} - \left(a^2 + r^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (2.1)$$

$$\Phi = 1 - \frac{r_g r + 2a^2}{r^2 + a^2} = \frac{\Delta}{\rho^2},$$

where $t$is a time coordinate, t, θ, ϕ are spherical coordinates, $r_g = 2M$ is a Schwarzschild horizon of black hole with mass M, a is a NUT parameter, $\rho = r^2 + a^2$, $\Delta = r^2 + r_g r - a^2$.

The corresponding metric tensor is non-diagonal

$$g_{\alpha\beta} = \begin{vmatrix} \Phi & 0 & 0 & 2a\Phi\left(1-\cos\theta\right) \\ 0 & -\frac{1}{\Phi} & 0 & 0 \\ 0 & 0 & -a^2 - r^2 & 0 \\ 2a\Phi\left(1-\cos\theta\right) & 0 & 0 & 4a^2\Phi\left(1-\cos\theta\right)^2 - \left(a^2+r^2\right)\sin^2\theta \end{vmatrix}.$$

We chose the following tetrad

$$e_{(a)\alpha}(x) = \begin{vmatrix} \sqrt{\Phi} & 0 & 0 & 2a\sqrt{\Phi}\left(1-\cos\theta\right) \\ 0 & \frac{1}{\sqrt{\Phi}} & 0 & 0 \\ 0 & 0 & \sqrt{a^2+r^2} & 0 \\ 0 & 0 & 0 & \sqrt{a^2+r^2}\sin\theta \end{vmatrix}. \qquad (2.2)$$

Applying the known formulas [23]

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab}),$$

$$\lambda_{abc} = \left(\frac{\partial e_{(a)\alpha}}{\partial x^\beta} - \frac{\partial e_{(a)\beta}}{\partial x^\alpha}\right) e^\alpha_{(b)} e^\beta_{(c)},$$

we find the relevant Ricci rotation coefficients

$$\gamma_{ab0} = \begin{vmatrix} 0 & \frac{\Phi'}{2\sqrt{\Phi}} & 0 & 0 \\ -\frac{\Phi'}{2\sqrt{\Phi}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a\sqrt{\Phi}}{a^2+r^2} \\ 0 & 0 & -\frac{a\sqrt{\Phi}}{a^2+r^2} & 0 \end{vmatrix},$$

$$\gamma_{ab1} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix},$$

$$\gamma_{ab2} = \begin{vmatrix} 0 & 0 & 0 & \frac{a\sqrt{\Phi}}{a^2+r^2} \\ 0 & 0 & \frac{r\sqrt{\Phi}}{a^2+r^2} & 0 \\ 0 & -\frac{r\sqrt{\Phi}}{a^2+r^2} & 0 & 0 \\ -\frac{a\sqrt{\Phi}}{a^2+r^2} & 0 & 0 & 0 \end{vmatrix}, \qquad (2.3)$$

$$\gamma_{ab3} = \begin{vmatrix} 0 & 0 & -\frac{a\sqrt{\Phi}}{a^2+r^2} & 0 \\ 0 & 0 & 0 & \frac{r\sqrt{\Phi}}{a^2+r^2} \\ \frac{a\sqrt{\Phi}}{a^2+r^2} & 0 & 0 & \frac{1}{\tan\theta\sqrt{a^2+r^2}} \\ 0 & -\frac{r\sqrt{\Phi}}{a^2+r^2} & -\frac{1}{\tan\theta\sqrt{a^2+r^2}} & 0 \end{vmatrix}.$$

3. Maxwell equations, separating the variables

It is convenient to apply the matrix complex Silberstein – Majorana – Oppenheimer formalism, so the covariant matrix Maxwell equation reads (for more detail see [20–22, 24])

$$\alpha^a \Big(e^{\beta}_{(a)} \frac{\partial}{\partial x^{\beta}} + \frac{1}{2} j^{mn} \gamma_{mna} \Big) \Psi = 0, \qquad \Psi = \begin{vmatrix} 0 \\ \mathbf{E} + ic\mathbf{B} \end{vmatrix}, \qquad (3.1)$$

where $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields vectors, generators j^{mn} of the complex vector representation of orthogonal group $SO(3.C)$ equal

$$j^{23} = s_1, \quad j^{01} = is_1, \quad j^{31} = s_2, \quad j^{02} = is_2, \quad j^{12} = s_3, \quad j^{03} = is_3.$$

In the cyclic basis, the matrix s read

$$s_1 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{vmatrix}, \quad s_2 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ 0 & 0 & -i & 0 \end{vmatrix},$$

$$s_3 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

Taking into account expression for the tetrad (2.2) and the Ricci rotation coefficients (2.3), the Maxwell matrix equation (3.1) is obtained in the following form

$$\Big[\Big(\alpha^0 \frac{\rho}{\sqrt{\Delta}} + \alpha^3 \frac{2a}{\rho} \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \Big) \frac{\partial}{\partial t} - \alpha^1 \frac{\sqrt{\Delta}}{\rho} \frac{\partial}{\partial r}$$
$$-i\, \alpha^0 s_1 \Big(\frac{\Delta'}{2\sqrt{\Delta}\rho} - \frac{(r+ia)\sqrt{\Delta}}{\rho^3} \Big) + \Big(\alpha^3 s_2 - \alpha^2 s_3 \Big) \frac{(r+ia)\sqrt{\Delta}}{\rho^3} - \frac{1}{\rho} \Sigma_{\theta,\phi} \Big] \Psi = 0, \qquad (3.2)$$

$$\Sigma_{\theta,\phi} = \alpha^2 \frac{\partial}{\partial \theta} + \alpha^3 \Big(\frac{1}{\sin\theta} \frac{\partial}{\phi} + s_1 \frac{1}{\tan\theta} \Big), \quad \alpha^0 = -i, \alpha^1 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{vmatrix},$$

$$\alpha^2 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ -1 & 0 & -i & 0 \end{vmatrix}, \quad \alpha^3 = \frac{1}{\sqrt{2}}\begin{vmatrix} 0 & -i & 0 & -i \\ -i & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -i & 0 & 1 & 0 \end{vmatrix}.$$

As the NUT-metric does not depend on the time and angle ϕ, we should search wave functions in the form

$$\Psi = e^{-i\omega t}e^{im\phi}\begin{vmatrix} 0 \\ R_1(r)T_1(\theta) \\ R_2(r)T_2(\theta) \\ R_3(r)T_3(\theta) \end{vmatrix}. \tag{3.3}$$

Substituting the last in the equation (3.2), we get

$$R_2T_2\Big(2a\omega\tan\frac{\theta}{2} + m\csc\theta\ \Big) + R_1T_1\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} + \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \sqrt{2}\sqrt{\Delta}T_1R_1' + R_2T_2' = 0, \tag{3.4}$$

$$\begin{gathered} R_1T_1\Big(2a\omega\tan\frac{\theta}{2} - \cot\theta + m\csc\theta\Big) + R_3T_3\Big(2a\omega\tan\frac{\theta}{2} + \cot\theta + m\csc\theta\Big) \\ + \frac{2\sqrt{2}\sqrt{\Delta}R_2T_2(r+ia)}{\rho^2} + \sqrt{2}\sqrt{\Delta}T_2R_2' - R_1T_1' + R_3T_3' = 0, \end{gathered} \tag{3.5}$$

$$R_2T_2\Big(2a\omega\tan\frac{\theta}{2} + m\csc\theta\Big) + R_3T_3\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} - \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \sqrt{2}\sqrt{\Delta}T_3R_3' - R_2T_2' = 0, \tag{3.6}$$

$$\begin{gathered} -R_1T_1\Big(2a\omega\tan\frac{\theta}{2} - \cot\theta + m\csc\theta\Big) + R_3T_3\Big(2a\omega\tan\frac{\theta}{2} + \cot\theta + m\csc(\theta)\Big) \\ + \frac{i\sqrt{2}\rho^2R_2T_2\omega}{\sqrt{\Delta}} + R_1T_1' + R_3T_3' = 0. \end{gathered} \tag{3.7}$$

With the use of simple algebraic calculation we separate the variables.

The radial equations read (we introduce the separation constants $\xi_1, \xi_2, \xi_3, \xi_4$)

$$\sqrt{2}\sqrt{\Delta}R_1' + R_1\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} + \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \xi_1R_2 = 0, \tag{3.8}$$

$$\sqrt{2}\sqrt{\Delta}R_3' + R_3\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} - \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \xi_2R_2 = 0, \tag{3.9}$$

$$\frac{\sqrt{2}}{2}\sqrt{\Delta}R_2' + \sqrt{2}R_2\Big(\frac{\sqrt{\Delta}(r+ia)}{\rho^2} + \frac{i\rho^2\omega}{2\sqrt{\Delta}}\Big) + \xi_3R_3 = 0, \tag{3.10}$$

$$\frac{\sqrt{2}}{2}\sqrt{\Delta}R_2' + \sqrt{2}R_2\Big(\frac{\sqrt{\Delta}(r+ia)}{\rho^2} - \frac{i\rho^2\omega}{2\sqrt{\Delta}}\Big) - \xi_4R_1 = 0; \tag{3.11}$$

and for the angular components:

$$T_2' + T_2\Big(2a\omega\tan\frac{\theta}{2} + m\csc\theta\Big) - \xi_1 T_1 = 0, \quad T_3' + T_3\Big(2a\omega\tan\frac{\theta}{2} + \cot\theta + m\csc\theta\Big) - \xi_3 T_2 = 0, \quad (3.12)$$

$$T_1' - T_1\Big(2a\omega\tan\frac{\theta}{2} - \cot\theta + m\csc\theta\Big) - \xi_4 T_2 = 0, \quad T_2' - T_2\Big(2a\omega\tan\frac{\theta}{2} + m\csc(\theta)\ Big) + \xi_2 T_3 = 0. \quad (3.13)$$

Without loss of generality won can take $\xi_1 = -\xi_4 = \Lambda_1$, $\xi_2 = \xi_3 = \Lambda_2$. Then the last equations take the form

$$\sqrt{2}\sqrt{\Delta}R_1' + R_1\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} + \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \Lambda_1 R_2 = 0,$$

$$\frac{\sqrt{2}}{2}\sqrt{\Delta}R_2' + \sqrt{2}R_2\Big(\frac{\sqrt{\Delta}(r+ia)}{\rho^2} - \frac{i\rho^2\omega}{2\sqrt{\Delta}}\Big) + \Lambda_1 R_1 = 0, \quad (3.14)$$

$$\sqrt{2}\sqrt{\Delta}R_3' + R_3\Big(\frac{\Delta'}{\sqrt{2}\sqrt{\Delta}} - \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}}\Big) + \Lambda_2 R_2 = 0,$$

$$\frac{\sqrt{2}}{2}\sqrt{\Delta}R_2' + \sqrt{2}R_2\Big(\frac{\sqrt{\Delta}(r+ia)}{\rho^2} + \frac{i\rho^2\omega}{2\sqrt{\Delta}}\Big) + \Lambda_2 R_3 = 0; \quad (3.15)$$

and

$$T_2' + T_2\left(2a\omega\tan\frac{\theta}{2} + m\csc\theta\right) - \Lambda_1 T_1 = 0,$$

$$T_1' - T_1\left(2a\omega\tan\frac{\theta}{2} - \cot\theta + m\csc\theta\right) + \Lambda_1 T_2 = 0, \quad (3.16)$$

$$T_3' + T_3\left(2a\omega\tan\frac{\theta}{2} + \cot\theta + m\csc\theta\right) - \Lambda_2 T_2 = 0,$$

$$T_2' - T_2\left(2a\omega\tan\frac{\theta}{2} + m\csc\theta\right) + \Lambda_2 T_3 = 0. \quad (3.17)$$

Three of four equations in both these systems are independent. The fourth equation can be expressed as a combination of the last three if the following condition performed:

$$4a\omega - \Lambda_2^2 + \Lambda_1^2 = 0. \quad (3.18)$$

4. Angular equations solution

Expressing T_2 from the first equation in the system (3.16) and substituting it into the second one, we get the second-order equation for the function T_1:

$$T_1'' + \cot\theta T_1' + \Big(\Lambda_1^2 + 4a^2\omega^2 - \frac{2a\omega(1+2m+4a\omega)}{1+\cos\theta} - \frac{(1+m^2)}{\sin^2\theta} + 2\cot\theta\Big(\frac{m}{\sin\theta} + a\omega\tan\frac{\theta}{2}\Big)\Big)T_1 = 0. \quad (4.1)$$

In the same way, one get the equation for the function T_2:

$$T_2'' + \cot\theta T_2' + \Big(\Lambda_1^2 + 2a\omega + 4a^2\omega^2 - \frac{4a\omega(m+2a\omega)}{1+\cos\theta} - \frac{m^2}{\sin^2\theta}\Big)T_2 = 0. \quad (4.2)$$

Introducing the new variable $z = \sin^2\frac{\theta}{2}$, one transform the equations (4.1)-(4.2) to the form

$$(1-z)zT_1'' + (1-2z)T_1' + \Big(\Lambda_1^2 + 2a\omega(2a\omega+1) - \frac{(4a\omega+m+1)^2}{4z} + \frac{(m-1)^2}{4(z-1)}\Big)T_1 = 0, \quad (4.3)$$

$$(1-z)zT_2'' + (1-2z)T_2' + \big(\Lambda_1^2 + +2a\omega(2a\omega+1) - \frac{2a\omega(2a\omega+m)}{z} + \frac{m^2}{4(z-1)z}\Big) T_2 = 0. \quad (4.4)$$

We search the solution with the structure $T_1 = z^A(z-1)^B G_1$, $T_2 = z^C(z-1)^D G_2$; substituting the last in the equations (4.3)-(4.4), one get

$$(1-z)zG_1'' + (1+2A-2z(A+B+1))G_1' + \big(2a\omega(2a\omega+1) - (A+B)(A+B+1) + \Lambda_1^2\big)G_1 = 0,$$

$$(1-z)zG_2'' + (1+2C-2z(C+D+1))G_2' + \left(2a\omega(2a\omega+1) - (C+D)(C+D+1) + \Lambda_1^2\right)G_2 = 0,$$

here

$$A = \pm\frac{1}{2}(1+m+4a\omega),\ B = \pm\frac{1-m}{2},$$

$$C = \pm\frac{1}{2}(m+4a\omega),\ D = \pm\frac{m}{2}.$$

The equations for G_1, G_2 have the structure of hypergeometric type

$$z(1-z)G'' + [c-(a+b+1)z]G' - abG = 0,$$

$$G = {}_2F_1(a,b,c;z);$$

then the general form of solutions is (K_1, K_2 stand for some numerical constants):

$$T_1 = K_1\, z^A(z-1)^B G(a_1,b_1,c_1;z),$$

$$T_2 = K_2\, z^C(z-1)^D G(a_2,b_2,c_2;z),$$

$$a_1,\, b_1 = \frac{1}{2} + A + B \pm \frac{1}{2}\sqrt{(1+4a\omega)^2 + 4\Lambda_1^2},$$

$$c_1 = 1+2A;$$

$$a_2,\, b_2 = \frac{1}{2} + C + D \pm \frac{1}{2}\sqrt{(1+4a\omega)^2 + 4\Lambda_1^2},$$

$$c_2 = 1+2C.$$

The cases of positive and negative values of number m should be considered separately:

$$\underline{m>0,}\quad A = \frac{1}{2}\Big(1+m+4a\omega\Big),\ B = \frac{m-1}{2},$$

$$C = \frac{1}{2}\Big(m+4a\omega\Big),\ D = \frac{m}{2},$$

$$c_1 = 2+m+4a\omega,$$
$$a_1 = \frac{1}{2}\Big(1+2m+4a\omega - \frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big),$$

$$b_1 = \frac{1}{2}\Big(1+2m+4a\omega + \frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big),$$

$$c_2 = c_1 - 1,\ a_2 = a_1,\ b_2 = b_1;$$

$$\underline{m<0,}\quad A = -\frac{1}{2}\Big(1+m+4a\omega\Big),\ B = -\frac{m-1}{2},$$

$$C = -\frac{1}{2}\Big(m+4a\omega\Big),\ D = -\frac{m}{2},$$

$$c_1 = 1-m-4a\omega,$$

$$a_1 = \frac{1}{2}\Big(1-2m-4a\omega - \frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big),$$

$$b_1 = \frac{1}{2}\Big(1-2m-4a\omega + \frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big),$$

$$c_2 = c_1 - 1,\ a_2 = a_1,\ b_2 = b_1.$$

We introduce the quantization rule in usual way by imposing the condition that the power series for the hypergeometric function are terminated, namely (b takes on non-positive integer values):

$$\underline{m>0,}\quad b_1 = \frac{1}{2}\Big(1+2m+4a\omega + \frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big) = -n_1 \Rightarrow \\ \Lambda_1^2 = (m+n_1)(1+m+n_1+4a\omega) = N_1(N_1+1+4a\omega); \tag{4.5}$$

$$\underline{m<0,}\quad b_1=\frac{1}{2}\Big(1-2m-4a\omega+\frac{1}{2}\sqrt{(1+4a\omega)^2+4\Lambda_1^2}\Big)=-n_2\Rightarrow \\ \Lambda_1^2=(1-m+n_2)(-m+n_2-4a\omega)=N_2(N_2-1-4a\omega); \tag{4.6}$$

in the second case, the constraint $n_2-m-4a\omega-m>0$ has to performed; so that in both cases $\Lambda^2>0$.

5. Solving the radial equations

In the system (3.14) we apply the following substitutions $R_1=F/\sqrt{\Delta}$, $R_2=\sqrt{2}G/\sqrt{\Delta}$, so we obtain

$$F'+\frac{i\rho^2\omega}{\Delta}F+\frac{\Lambda_1}{\sqrt{\Delta}}G=0,$$

$$G'-\Big(\frac{i\rho^2\omega}{\Delta}+\frac{\Delta'}{2\Delta}-\frac{2(r+ia)}{\rho^2}\Big)G+\frac{\Lambda_1}{\sqrt{\Delta}}F=0. \tag{5.1}$$

Eliminating the function G, one derives the second-order equation for the function F (taking into account the explicit expressions for Δ, ρ and introducing r_1 and r_2 as the roots of equation $\Delta=0$: $r_1=1/2(r_g-\sqrt{r_g^2+4a^2})$, $r_2=1/2(r_g+\sqrt{r_g^2+4a^2})$):

$$F''+\frac{2}{r-ia}F'+\Big(\omega^2+\frac{-\Lambda_1^2-2a\omega+2ir_1\omega+2r_1^2\omega^2}{(r-r_1)(r_1-r_2)} \\ -\frac{-\Lambda_1^2-2a\omega+2ir_2\omega+2r_2^2\omega^2}{(r-r_2)(r_1-r_2)}-\frac{ir_1\omega-r_1^2\omega^2}{(r-r_1)^2}-\frac{ir_2\omega-r_2^2\omega^2}{(r-r_2)^2}\Big)F=0. \tag{5.2}$$

In the same way, we get the second-order equation for the function G:

$$G''+\frac{2}{r-ia}G'+\Big(\omega^2+\frac{-\Lambda_1^2-2a\omega+2r_1^2\omega^2}{(r-r_1)(r_1-r_2)}+\frac{2ia+r_1}{2r_1(r-r_1)(r_1-r_2)}-\frac{-\Lambda_1^2-2a\omega+2r_2^2\omega^2}{(r-r_2)(r_1-r_2)} \\ -\frac{2ia+r_2}{2r_2(r-r_2)(r_1-r_2)}+\frac{1+4r_1^2\omega^2}{4(r-r_1)^2}+\frac{1+4r_2^2\omega^2}{4(r-r_2)^2}-\frac{i}{a(r-ia)}-\frac{2}{(r-ia)^2}\Big)G=0. \tag{5.3}$$

Solutions of the equation (5.2) are searched in the form

$$F=\frac{1}{(r-ia)}(r-r_1)^{\alpha}(r-r_2)^{\beta}e^{-\gamma r}f;$$

substituting the last into (5.2) leads to

$$f''-\Big(2\gamma-\frac{2\alpha}{r-r_1}-\frac{2\beta}{r-r_2}\Big)f'+\Big(-\frac{-\Lambda_1^2+2\alpha\beta+2\alpha\gamma\,(r_2-r_1)+2ir_1\omega+2r_1^2\omega^2-2a\omega}{(r-r_1)\,(r_2-r_1)} \\ +\frac{-\Lambda_1^2+2\alpha\beta-2\beta\gamma\,(r_2-r_1)+2ir_2\omega+2r_2^2\omega^2-2a\omega}{(r-r_2)\,(r_2-r_1)}\Big)f=0 \tag{5.4}$$

at $\gamma=\pm i\omega$; $\alpha=-ir_1\omega, 1+ir_1\omega$; $\beta=-ir_2\omega, 1+ir_2\omega$. In a new variable $v=\frac{r-r_1}{r_2-r_1}$ the equation (5.4) is transformed to the confluent

Heun equation:

$$f'' + \Big(-2\gamma(r_2 - r_1) + \frac{2\alpha}{v} + \frac{2\beta}{v-1} \Big) f' + \Big(-\frac{A}{v} + \frac{B}{v-1} \Big) f = 0, \tag{5.5}$$

$$A = -\Lambda_1^2 + 2\alpha\beta + 2\alpha\gamma\,(r_2 - r_1) + 2ir_1\omega + 2r_1^2\omega^2 - 2a\omega,$$

$$B = -\Lambda_1^2 + 2\alpha\beta - 2\beta\gamma\,(r_2 - r_1) + 2ir_2\omega + 2r_2^2\omega^2 - 2a\omega.$$

The solution of the equation (5.5) can be written as follows

$$f = C_1 \mathrm{HeunC}\left[-A, B - A, 2\alpha, 2\beta, 2\gamma\,(r_1 - r_2)\,, v\right]$$

$$+ C_2 v^{1-2\alpha} \mathrm{HeunC}\left[-A_1, B - A_1 - 2(2\alpha - 1)\beta, 2 - 2\alpha, 2\beta, 2\gamma\,(r_1 - r_2)\,, v\right],$$

here $A_1 = A - 2(2\alpha - 1)\left(\beta + \gamma\,(r_2 - r_1)\right)$. Then, the original function R_1 has the form

$$R_1 = \frac{(r - r_1)^{\alpha - 1/2}(r - r_2)^{\beta - 1/2}}{(r - ia)} e^{-\gamma r} f.$$

In the same way, we solve the system (3.15). Applying the change

$$R_3 = W/\sqrt{\Delta},\ R_2 = \sqrt{2}G/\sqrt{\Delta},$$

we get

$$W' - \frac{i\rho^2\omega}{\Delta} W + \frac{\Lambda_2}{\sqrt{\Delta}} G = 0,$$
$$G' - \left(-\frac{i\rho^2\omega}{\Delta} + \frac{\Delta'}{2\Delta} - \frac{2(r + ia)}{\rho^2} \right) G + \frac{\Lambda_2}{\sqrt{\Delta}} W = 0. \tag{5.6}$$

Expressing the function G from the first equation and substituting into the second, we get

$$W'' + \frac{2}{r - ia} W' + \Big(\omega^2 - \frac{\Lambda_2^2 - 2a\omega + 2ir_1\omega - 2r_1^2\omega^2}{(r - r_1)(r_1 - r_2)}$$
$$+ \frac{\Lambda_2^2 - 2a\omega + 2ir_2\omega - 2r_2^2\omega^2}{(r - r_2)(r_1 - r_2)} + \frac{ir_1\omega + r_1^2\omega^2}{(r - r_1)^2} + \frac{ir_2\omega + r_2^2\omega^2}{(r - r_2)^2} \Big) W = 0. \tag{5.7}$$

The solution of (5.7) is searched in the form $W = \frac{1}{(r-ia)}(r - r_1)^{\chi}(r - r_2)^{\xi} e^{-\gamma r} w$, we get

$$w'' - \Big(2\gamma - \frac{2\chi}{r - r_1} - \frac{2\xi}{r - r_2} \Big) w' + \Big(\frac{\Lambda_2^2 - 2\chi\xi - 2\chi\gamma\,(r_2 - r_1) + 2ir_1\omega - 2r_1^2\omega^2 - 2a\omega}{(r - r_1)\,(r_2 - r_1)}$$
$$- \frac{\Lambda_2^2 - 2\chi\xi + 2\xi\gamma\,(r_2 - r_1) + 2ir_2\omega - 2r_2^2\omega^2 - 2a\omega}{(r - r_2)\,(r_2 - r_1)} \Big) w = 0 \tag{5.8}$$

at $\gamma = \pm i\omega$; $\chi = ir_1\omega,\ 1 - ir_1\omega$; $\xi = ir_2\omega,\ 1 - ir_2\omega$. Taking into account the condition (3.18), the equation (5.8) reads

$$w'' - \Big(2\gamma - \frac{2\chi}{r - r_1} - \frac{2\xi}{r - r_2}\Big)w' + \Big(- \frac{-\Lambda_1^2 + 2\chi\xi + 2\chi\gamma\,(r_2 - r_1) - 2ir_1\omega + 2r_1^2\omega^2 - 2a\omega}{(r - r_1)\,(r_2 - r_1)} + \frac{-\Lambda_1^2 + 2\chi\xi - 2\xi\gamma\,(r_2 - r_1) - 2ir_2\omega + 2r_2^2\omega^2 - 2a\omega}{(r - r_2)\,(r_2 - r_1)}\Big)w = 0. \tag{5.9}$$

Comparing the equations (5.4) and (5.9), one can see that they are complex conjugated ones, so we have $w = f^*$ (a symbol * denotes complex conjugation). The function R_3 is determined by the expression

$$R_3 = \frac{(r + ia)}{(r - ia)} R_1^*.$$

Algebraic equation to find the function R_2 is derived from equations (3.14-3.15):

$$\Lambda_2 R_3 - \Lambda_1 R_1 + \frac{i\sqrt{2}\rho^2\omega}{\sqrt{\Delta}} R_2 = 0.$$

6. Behavior near horizon

To estimate the behavior of the functions R_1, R_2 in the vicinity of the outer horizon, one should consider the equations (5.2), (5.3) at $r \to r_2$. Preserving only the largest terms in these equations, one gets

$$F'' + \frac{2}{r - ia}F' + \frac{r_2\omega(-i + r_2\omega)}{(r - r_2)^2}F = 0,$$

$$G'' + \frac{2}{r - ia}G' + \frac{1 + 4r_2^2\omega^2}{4(r - r_2)^2}G = 0.$$

So, the solutions of the last equations in the vicinity of the horizon have the form

$$F \sim (r - r_2)^{-ir_2\omega},\ (r - r_2)^{1+ir_2\omega};$$

$$G \sim (r - r_2)^{\frac{1}{2}-ir_2\omega},\ (r - r_2)^{\frac{1}{2}+ir_2\omega}.$$

Then, for the original functions $R_1 = F/\sqrt{\Delta}$, $R_2 = \sqrt{2}G/\sqrt{\Delta}$, the solutions represent the incident and reflected waves

$$R_1 \sim (r - r_2)^{1/2+ir_2\omega},\ (r - r_2)^{-1/2-ir_2\omega};$$

$$R_2 \sim (r - r_2)^{ir_2\omega},\ (r - r_2)^{-ir_2\omega}.$$

According the procedure proposed in [25–27], the scattering probability

$$\Gamma = \left|\frac{\Psi_{out}(x > x_2)}{\Psi_{out}(x < x_2)}\right|^2 \tag{6.1}$$

is the probability of creating an outgoing particle outside the outer horizon and an ingoing antiparticle of negative energy inside the horizon. Then substituting the outgoing wave solutions into the formula (6.1), the probability of particle-antiparticle pair creating is

$$\Gamma = e^{-4\pi\omega r_2}. \tag{6.2}$$

the mean number $\bar{N}_\omega$ of photons emitted with a given frequency is determined by relation (ignoring the backscattering effect):

$$\bar{N}_\epsilon = \frac{\Gamma}{1 - \Gamma} = \frac{1}{e^{4\pi\omega r_2} - 1}. \tag{6.3}$$

We get the Bose-Einstein distribution

$$\bar{N}_\epsilon = \frac{1}{1 + e^{\omega/T}},\qquad T = \frac{1}{4\pi r_2} = \frac{1}{2\pi(r_g + \sqrt{r_g^2 + 4a^2})}, \tag{6.4}$$

where T determines the Hawking temperature. This expression for Hawking temperature coincides with the result obtained previously for the fermions production on the horizon.

7. Effective constitutive relations in NUT space

It is known that the Riemann geometry provides us with possibility to simulate special types of material media [24, 28]. For an arbitrary metric tensor $g_{\alpha\beta}$ one can obtain the effective constitutive relations due to the metric structure of the spacetime

$$H^{\alpha\beta}(x) = \varepsilon_0 \frac{\sqrt{-g(x)}}{\sqrt{-G(x)}} g^{\alpha\rho}(x) g^{\beta\sigma}(x) F_{\rho\sigma}(x), \tag{7.1}$$

where

$$F_{\alpha\beta} = \begin{vmatrix} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{vmatrix};$$

$$H^{\alpha\beta} = \begin{vmatrix} 0 & H^{01} & H^{02} & H^{03} \\ -H^{01} & 0 & H^{12} & H^{13} \\ -H^{02} & -H^{12} & 0 & H^{23} \\ -H^{03} & -H^{13} & -H^{23} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{vmatrix}.$$

Substituting expressions for $F_{\alpha\beta}$ and $H^{\alpha\beta}$ into the equation (7.1), we get the formulae for effective constitutive relations

$$D^i = \varepsilon_0 \varepsilon^{ik} E_k + \varepsilon_0 c \alpha^{ik} B_k,$$

$$H^i = \varepsilon_0 c \beta^{ik} E_k + \mu_0^{-1} (\mu^{-1})^{ik} B_k,$$

where the matrixes ε^{ik}, α^{ik}, β^{ik}, $(\mu^{-1})^{ik}$ are defined be the metric tensor as follows

$$\varepsilon^{ik} = \frac{\sqrt{-g}}{\sqrt{-G}} \left(g^{00} g^{ij} - g^{0i} g^{0k} \right),$$
$$(\mu^{-1})^{ik} = \frac{1}{2} \frac{\sqrt{-g}}{\sqrt{-G}} \epsilon_{imn} g^{ml} g^{nj} \epsilon_{ljk},$$

$$\alpha^{ik} = \frac{\sqrt{-g}}{\sqrt{-G}} g^{ij} g^{0l} \epsilon_{ljk}, \quad \beta^{ik} = -\frac{\sqrt{-g}}{\sqrt{-G}} g^{0j} \epsilon_{jil} g^{lk},$$

here G is the determinant of the Minkowski metric in the spherical coordinates. For relativistic non-diagonal metrics these effective constitutive relations entangle the vectors of electric and magnetic fields.

Substituting the explicit expressions for metric tensor components, we get the matrixes which determine the constitutive relations generated by NUT-spacetime:

$$\varepsilon^{ik} = \begin{vmatrix} -1 - \frac{a^2}{r^2}\left(1 - \frac{4\Delta}{r^2+a^2}\tan^2\frac{\theta}{2}\right) & 0 & 0 \\ 0 & -\frac{1}{\Delta} - \frac{a^2}{r^2}\left(\frac{1}{\Delta} - \frac{4}{(r^2+a^2)}\tan^2\frac{\theta}{2}\right) & 0 \\ 0 & 0 & -\frac{1}{\Delta\sin^2\theta} - \frac{a^2}{\Delta r^2 \sin^2\theta} \end{vmatrix},$$

$$\alpha^{ik} = \beta^{ki} = \begin{vmatrix} 0 & -\frac{a\Delta}{r^2(r^2+a^2)\cos^2\frac{\theta}{2}} & 0 \\ \frac{a}{r^2(r^2+a^2)\cos^2\frac{\theta}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad (\mu^{-1})^{ik} = \begin{vmatrix} \frac{1}{r^2(r^2+a^2)\sin^2\theta} & 0 & 0 \\ 0 & \frac{\Delta}{r^2(r^2+a^2)\sin^2\theta} & 0 \\ 0 & 0 & \frac{\Delta}{r^2(r^2+a^2)} \end{vmatrix}.$$

Let us compare the found expressions with those for Minkowski metric

$$\varepsilon_M^{ik} = \begin{vmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & -\frac{1}{r^2 \sin^2\theta} \end{vmatrix},$$

$$(\mu_M^{-1})^{ik} = \begin{vmatrix} \frac{1}{r^4 \sin^2\theta} & 0 & 0 \\ 0 & \frac{1}{r^2 \sin^2\theta} & 0 \\ 0 & 0 & \frac{1}{r^2} \end{vmatrix}, \quad \alpha_M^{ik} = \beta_M^{ki} = 0.$$

One can see that in the limiting case for $a = 0$, $\Delta = r^2$, the obtained matrixes for NUT-spacetime turn to the corresponding formulas for Minkowski space.

In Minkowski space, the coefficients in the matrix are determined the transition from Cartesian coordinates to spherical ones. Indeed, from the formula $F^{\alpha\beta} = G^{\alpha\gamma} G^{\beta\sigma} F_{\gamma\sigma}$, we get

$$\begin{vmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -E_1 & -\frac{E_2}{r^2} & -\frac{E_3}{r^2\sin^2\theta} \\ E_1 & 0 & -\frac{B_3}{r^2} & \frac{B_2}{r^2\sin^2\theta} \\ \frac{E_2}{r^2} & \frac{B_3}{r^2} & 0 & -\frac{B_1}{r^4\sin^2\theta} \\ \frac{E_3}{r^2\sin^2\theta} & -\frac{B_2}{r^2\sin^2\theta} & \frac{B_1}{r^4\sin^2\theta} & 0 \end{vmatrix}.$$

Then the effective constitutive relations generated by the NUT metric can be considered in Cartesian coordinates

$$D^i = \varepsilon_0 \varepsilon_{ik} E^k + \varepsilon_0 c \alpha_{ik} B^k, \quad H^i = \varepsilon_0 c \beta_{ik} E^k + \mu_0^{-1} (\mu^{-1})_{ik} B^k,$$

where

$$\varepsilon_{ik} = \begin{vmatrix} 1 + \frac{a^2}{r^2}\Big(1 - \frac{4\Delta}{r^2+a^2}\tan^2\frac{\theta}{2}\Big) & 0 & 0 \\ 0 & \frac{r^2}{\Delta} + a^2\Big(\frac{1}{\Delta} - \frac{4}{(r^2+a^2)}\tan^2\frac{\theta}{2}\Big) & 0 \\ 0 & 0 & \frac{r^2}{\Delta} + \frac{a^2}{\Delta} \end{vmatrix}, \quad \alpha_{ik} = \begin{vmatrix} 0 & -\frac{4a\Delta\sin^2\frac{\theta}{2}}{(r^2+a^2)} & 0 \\ \frac{4ar^2\sin^2\frac{\theta}{2}}{(r^2+a^2)} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$\beta_{ik} = \begin{vmatrix} 0 & \frac{a}{(r^2+a^2)\cos^2\frac{\theta}{2}} & 0 \\ -\frac{a\Delta}{r^2(r^2+a^2)\cos^2\frac{\theta}{2}} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad (\mu^{-1})_{ik} = \begin{vmatrix} \frac{r^2}{(r^2+a^2)} & 0 & 0 \\ 0 & \frac{\Delta}{(r^2+a^2)} & 0 \\ 0 & 0 & \frac{\Delta}{(r^2+a^2)} \end{vmatrix}.$$

Let us consider behavior of the constitutive relations near singular points: outer horizon and at infinity. For $r \to \infty$, the matrixes behave as

$$\varepsilon_{ik} = (\mu^{-1})_{ik} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$\alpha_{ik} = \begin{vmatrix} 0 & -4a\sin^2\frac{\theta}{2} & 0 \\ 4a\sin^2\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$\beta_{ik} = 0.$$

One can see that the matrixes ε^{ik} for dielectric and $(\mu^{-1})^{ik}$ for magnetic permittivity turn to ordinary expressions for Minkowski space and are unity matrixes for Cartesian coordinates.

The entangling matrixes α^{ik}, β^{ik} vanish identically at infinity everywhere except Misner string $\theta = \pm\pi$. In the vicinity of horizon we have

$$\Delta \equiv (r - r_1)(r - r_2) \to 0, \; r \to r_2,$$

and the matrixes can be rewritten approximately:

$$\varepsilon_{ik} = \begin{vmatrix} 1+\frac{a^2}{r_2^2} & 0 & 0 \\ 0 & \frac{r_2^2+a^2}{(r_2-r_1)(r-r_2)} & 0 \\ 0 & 0 & \frac{r_2^2+a^2}{(r_2-r_1)(r-r_2)} \end{vmatrix},$$

$$(\mu^{-1})_{ik} = \begin{vmatrix} \frac{r_2^2}{(r_2^2+a^2)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \ \alpha_{ik} = \begin{vmatrix} 0 & 0 & 0 \\ \frac{4ar_2^2\sin^2\frac{\theta}{2}}{(r_2^2+a^2)} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$\beta_{ik} = \begin{vmatrix} 0 & \frac{a}{(r_2^2+a^2)\cos^2\frac{\theta}{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

For small values of NUT parameter ($a \ll r_2$), one has $r_1 \to 0$, $r_2 \to r_g$, where r_g is the radius of horizon of Schwarzschild mass. Then we have

$$\varepsilon_{ik} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \frac{r_g}{r-r_g} & 0 \\ 0 & 0 & \frac{r_g}{r-r_g} \end{vmatrix}, \quad (\mu^{-1})_{ik} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$\alpha_{ik} = \begin{vmatrix} 0 & 0 & 0 \\ 4a\sin^2\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \beta_{ik} = \begin{vmatrix} 0 & \frac{a}{r_g\cos^2\frac{\theta}{2}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

The found result means that the presence of the NUT charge leads to arising non-vanishing component of the matrix which entangles the electric and magnetic fields. We can assume that this result correlates with some rotation effects related to the NUT metric.

8. Conclusion

Thus, we have studied the Maxwell equations in the NUT spacetime, by applying the conventional tetrad method. The variables have been separated and the exact solutions of the equations for angular and radial components have been found in terms of hypergeometric and confluent Heun functions, respectively. The quantization rule of separation constant has been obtained from the analysis of angular solutions. The behavior of the radial components with structure of outgoing and ingoing waves has been studied near the outer horizon, and the expression of the temperature of Hawking radiation of photons has been shown to coincide with that for the fermions production on the outer horizon. The effective constitutive relations which are generated by metric structure of NUT spacetime have been found. It has been shown that the existence of the NUT charge leads to entanglement of electric and magnetic field components in the constitutive relations.

References

[1] V. S. Manko and E. Ruiz, Class. Quantum Grav. **22**, 3555 (2005).

[2] E. Newman, L. Tamburino, and T. Unti., J.Math.Phys. **4**, 915 (1963).

[3] C. Chakraborty and B. Mukhopadhyay, Eur. Phys. J. C **83**, 937 (2023).

[4] D. V. Galtsov and K. Kobialko, Memoirs of the Faculty of Physics Lomonosov Moscow State University p. 1740901 (2017).

[5] A. Awad and S. Eissa, Phys. Rev. D p. 124034 (2022).

[6] W. Bonner, Math.Proc.Cambridge Phil.Soc. pp. 145–151 (1969).

[7] A. Al-Badawi and M. Halilsoy, Gen.Relativ.Gravit. **38**, 1729 (2006).

[8] G. Clément, D. Gal'tsov, and M. Guenouche, Phys. Rev. D **93**, 024048 (2016).

[9] C. Chakraborty and S. Bhattacharyya, Phys. Rev. D **106**, 103028 (2022).

[10] T. Frost, Phys. Rev. D **108**, 124019 (2023).

[11] M. Ghasemi-Nodehi, C. Chakraborty, Q. Yu, and Y. Lu, Eur. Phys. J. C **81**, 939 (2021).
[12] R. L. Zimmerman and B. Shahir, Gen. Relativ. Gravit. **21**, 821 (1989).
[13] M. Nouri-Zonoz and D. Lynden-Bell, Mon. Not. R. Astron. Soc. **292**, 714 (1997).
[14] V. P. Vandeev and A. N. Semenova, Int. J. Modern Phys. D **31**, 2250108 (2022).
[15] M. Nouri-Zonoz, Class. Quantum Grav. **21**, 471 (2004).
[16] H. Tetrode, Zeit. Phys. **50**, 336 (1928).
[17] H. Weyl, Proc. Nat. Acad. Sci. Amer. **15**, 323 (1929).
[18] V. Fock and D. Ivanenko, Zeit. Phys. **54**, 798 (1929).
[19] V. Fock, Zeit. Phys. **57**, 261 (1929).
[20] V. Red'kov, N. Tokarevskaya, E. Ovsiyuk, and G. Spix, J. Nonlinear Phen. Complex. Sys. **12**, 232 (2009).
[21] V. Red'kov, N. Tokarevskaya, and G. Spix, Advances in Applied Clifford Algebras **22**, 1129 (2012).
[22] V. Red'kov, N. Tokarevskaya, and G. Spix, Advances in Applied Clifford Algebras **23**, 165 (2013).
[23] L. Landau and E. Lifshitz, *The classical theory of fields* (Nauka, Moscow, 1973).
[24] E. Ovsiyuk, V. Kisel, and V. Red'kov, *Maxwell Electrodynamics and Boson Fields in Spaces of Constant Curvature* (Nova Science Publishers Inc., New York, 2014).
[25] S. Sannan, Gen. Relativ. Gravit. **20**, 239 (1988).
[26] C. Dariescu, M.-A. Dariescu, and C. Stelea, Advances in High Energy Phys. **2021**, 5512735 (2021).
[27] H. Vieira and V. Bezerra, Annals of Physics **373**, 28 (2016).
[28] V.M.Redkov, *Fields in Riemannian space and the Lorentz group (in Russian)* (Publishing House "Belarusian Science Minsk, 2009).