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ANALYTICAL PROPERTIES OF SOLUTIONS TO NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS ASSOCIATED WITH SOME RANDOM MATRIX TYPE MODELS

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We obtain new results, as well as those complementing already known ones, concerning the construction of solutions of systems of differential equations corresponding to certain models of random matrix type. These solutions are expressed in terms of solutions of Painlevé II–V *equations. We also show that solutions of systems of differential equations associated with random matrix type models having Laguerre and Hermitian kernels satisfy the formal Painlevé test. We obtain new formulas relating solutions of Painlevé* III *and Painlevé* V *equations under certain conditions imposed on the parameters entering these equations.*

Keywords: random matrix models, kernel, Painlevé equations, Painlevé test, Bäcklund transformation

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To the deeply respected Nikolai Alekseevich Kudryashov on his 80th birthday

1. Introduction

Over the past three decades, there has been strong interest in the study of certain classes of continuous and discrete probabilistic models called *random matrix type models*. Sources of such models are quite diverse: random matrix theory, developed by nuclear physicists about 50 years ago; directed percolation theory; stochastic crystal growth; random tilings; enumerative combinatorics; and asymptotic representation theory [1], [2].

Random matrix theory studies the following question [2]. Consider a large matrix whose entries are random variables with given distributions. What can we say about the distribution of a certain set of eigenvalues or eigenvectors of this matrix? This question is central to problems in quantum and classical statistical physics, number theory, etc.

One of the most important characteristics of such models is the so-called zero-probability, i.e., the probability of finding no particles in a given interval or combination of intervals. Usually, zero-probabilities can be represented as Fredholm determinants $\det(\mathbf{1}-K)|_J$, where K is some integral operator with a special-form kernel, and J is a set where there should be no particles. The only currently known method for calculating such determinants is to characterize them as solutions of some ordinary differential equation or system of partial differential equations. Usually, the kernel of the operator K has the form

$$K(x,y)=\frac{A(x)B(y)-B(x)A(y)}{x-y}\sqrt{\psi(x)\psi(y)} \tag{1.1}$$

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with appropriate functions ψ, A, and B. In [3], the authors solve the problem of deriving a differential equation for zero-probabilities in the case of a sine kernel, which is given by (1.1) with $\psi(x) = 1/x$, $A(x) = \sin x$, and $B(x) = \cos x$, and demonstrate that the value of the Fredholm determinant for the identity operator minus the sine kernel restricted to an interval of variable length t can be expressed through solution of the ordinary differential equation

$$\left(t\frac{d^2\sigma}{dt^2}\right)^2 = -4\left(t\frac{d\sigma}{dt} - 1 - \sigma\right)\left[t\frac{d\sigma}{dt} + \left(\frac{d\sigma}{dt}\right)^2 - \sigma\right].$$

Using successive transformations $\sigma(t) = t - d\rho(t)/dt$,

$$\rho(t) = \rho_0 \int_0^t \left\{\frac{\tau}{4w(1-w^2)}\left[\left(\frac{dw}{d\tau}\right)^2 + 4w^2\right] - \frac{(1+w)^2}{4\tau w}\right\} d\tau, \quad \rho_0 = \text{const},$$

the function $\sigma(t)$ can be expressed in terms of solution of the Painlevé V equation

$$\frac{d^2w}{dt^2} = \frac{3w-1}{2w(w-1)}\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\cdot\frac{dw}{dt} + \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1} \tag{P5}$$

with parameter values $\alpha = -\beta = 1/2$, $\gamma = -2i$, and $\delta = 2$.

We note that the relation between the Fredholm determinant for integral operators and solutions of Painlevé equations was first established in [4]. More exactly, the authors of that work showed that the one-parametric family of solutions of the equation

$$\frac{d^2\varphi}{d\tau^2} = \frac{1}{\tau}\frac{d\varphi}{d\tau} - \frac{1}{2}\sinh 2\varphi + 2a_0\tau^{-1}\sinh\varphi,$$

(where a_0 is a parameter) which is a particular case of the Painlevé III equation

$$\frac{d^2\lambda}{d\tau^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{d\tau}\right)^2 - \frac{1}{\tau}\cdot\frac{d\lambda}{d\tau} + \frac{1}{\tau}(a\lambda^2 + b) + c\lambda^3 + \frac{d}{\lambda} \tag{P3}$$

(with parameters a, b, c, and d), can be expressed in terms of Fredholm determinants of a special type.

It was established in [5]–[9] that the functions ("resolvents" [2]) associated with Fredholm determinants and corresponding to a certain type of kernel satisfy nonlinear second-order second-degree differential equations. It is significant that these equations are of the Painlevé type and that their solutions can be expressed in terms of solutions of equations for polynomial Hamiltonians associated with irreducible Painlevé equations [10], [11].

We also note the existence of nonpolynomial (in particular, rational) Hamiltonians associated with the Painlevé equations [12], [13].

The aim of our paper is constructing solutions for certain systems of ordinary differential equations associated with random matrix type models, as well as deriving new formulas relating the solutions of Eqs. (P5) and (P3).

2. Analysis of solutions of a system of two differential equations generated by differential–difference equations

2.1. The system of differential equations [14]

$$2S_n = \frac{1}{1-S_{n-1}^2}\left(\frac{n}{t}S_{n-1} - \frac{dS_{n-1}}{dt}\right), \quad 2S_{n-1} = \frac{1}{1-S_n^2}\left(\frac{n+1}{t}S_n + \frac{dS_n}{dt}\right), \tag{2.1}$$

where t is a continuous independent variable, n is an arbitrary parameter associated (in the case of natural n) with the equation [15]

$$(n+1)S_n = t(S_{n+1} + S_{n-1})(1 - S_n^2), \tag{2.2}$$

which is the discrete Painlevé II equation, and with the discrete modified Korteveg–de Vries equation [16]

$$\frac{dS_n}{dt} = -(S_{n+1} - S_{n-1})(1 - S_n^2). \tag{2.3}$$

The equations of system (2.1) can be obtain by subtracting Eq. (2.3) from (2.2) and replacing n with $n-1$ in the obtained expression, as well as by adding Eqs. (2.2) and (2.3).

In the notation $t = z$, $S_{n-1} = v = v(z)$, and $S_n = u = u(z)$, system (2.1) becomes

$$v = \frac{1}{2(1-u^2)}\left(\frac{n+1}{z}u + u'\right), \quad u = \frac{1}{2(1-v^2)}\left(\frac{n}{z}\cdot v - v'\right), \quad ' = \frac{d}{dz}. \tag{2.4}$$

For the unknown functions v, u, system (2.4) is equivalent to the respective equations

$$v'' = -\frac{v}{1-v^2}v'^2 - \frac{1}{z}v' + \frac{n^2}{z^2}\cdot\frac{v}{1-v^2} - 4v(1-v^2), \tag{2.5}$$

$$u'' = -\frac{u}{1-u^2}u'^2 - \frac{1}{z}u' + \frac{(n+1)^2}{z^2}\cdot\frac{u}{1-u^2} - 4u(1-u^2). \tag{2.6}$$

Thus, the following theorems hold true.

Theorem 1 (see [14]). *Let $v = v(z)$ be a solution of Eq. (2.5). Then the function $u(z)$ defined by the formula in (2.4) is a solution of Eq. (2.6).*

Theorem 2. *Let $u = u(z)$ be a solution of Eq. (2.6). Then the function $v(z)$ defined by the formula in (2.4) is a solution of Eq. (2.5).*

Equations (2.5), (2.6) can be reduced via the transformations

$$v = \frac{\varepsilon}{\sqrt{1-y}}, \quad u = \frac{\sigma}{\sqrt{1-q}}, \quad z^2 = \tau, \quad \varepsilon^2 = \sigma^2 = 1 \tag{2.7}$$

to the equations

$$y'' = \frac{3y-1}{2y(y-1)}y'^2 - \frac{y'}{\tau} - \frac{n^2}{2}\cdot\frac{(y-1)^2}{\tau^2 y} + 2\frac{y}{\tau}, \tag{2.8}$$

$$q'' = \frac{3q-1}{2q(q-1)}q'^2 - \frac{q'}{\tau} - \frac{(n+1)^2}{2}\cdot\frac{(q-1)^2}{\tau^2 q} + 2\frac{q}{\tau} \tag{2.9}$$

respectively. Each of Eqs. (2.8), (2.9) is the fifth Painlevé equation (P5) with the respective parameter values $\alpha = 0$, $\beta = -n^2/2$, $\gamma = 2$, $\delta = 0$ and $\alpha = 0$, $\beta = -(n+1)^2/2$, $\gamma = 2$, $\delta = 0$.

Using transformations (2.7), system (2.4) can be reduced to the following:

$$\frac{2\varepsilon}{\sqrt{1-q}} = -\frac{\sigma}{\sqrt{\tau}q(1-q)}[(n+1)(1-q) + \tau q'], \tag{2.10}$$

$$\frac{2\sigma}{\sqrt{1-q}} = -\frac{\varepsilon}{\sqrt{\tau}y\sqrt{1-y}}[n(1-y) - \tau y']. \tag{2.11}$$

Formulas (2.10), (2.11) with fixed ε and σ establish a one-to-one correspondence between solutions of Eqs. (2.8), (2.9). It follows from relations (2.10), (2.11) that

$$y = 1 + \frac{4\tau q^2(q-1)}{[(n+1)(1-q) + \tau q']^2}, \tag{2.12}$$

$$q = 1 + \frac{4\tau y^2(y-1)}{[n(y-1) + \tau y']^2}. \tag{2.13}$$

Formula (2.13) (Bäcklund transformation for Eq. (P5) with $\alpha = \delta = 0$, $\gamma \neq 0$) has been obtained in [17].

A generalization of formula (2.13) to the case of (P5) with $\alpha\gamma \neq 0$, $\delta = 0$ is given in [18]. It is shown in [19] that the system of differential equations (2.12), (2.13) is a Painlevé type system.

By a direct check, it is easy to verify that the following theorem is true.

Theorem 3. *Let $u = u(z)$, $v = v(z)$ be solutions of system* (2.4). *Then the function*

$$Y = \frac{u}{v} \tag{2.14}$$

is a solution of the equation

$$zYY'' = zY'^2 - YY' - 4nY^3 + 4(n+1)Y + 4zY^4 - 4z. \tag{2.15}$$

Equation (2.15) is the third Painlevé equation with parameters $a = -4n$, $b = 4(n+1)$, $c = -d = 4$. Using the substitution $Y \to 1/Y$, $n \to -n$, it reduces to Eq. (P3) with parameters $a = 4(n-1)$, $b = -4n$, $c = -d = 4$ [20].

Thus, formula (2.14) allows representing a solution of Eq. (2.15) as a nonlinear combination of solutions of Eqs. (2.8), (2.9). More precisely, the following theorem holds true.

Theorem 4. *Let $y = y(\tau)$ $(y(y-1) \not\equiv 0)$ and $q = q(\tau)$ $(q(q-1) \not\equiv 0)$ be two arbitrary functions satisfying system* (2.10), (2.11) *with fixed ε, σ. Then they are solutions of Eq.* (P5) *with the respective sets of parameters* $(0, -n^2/2, 2, 0)$ *and* $(0, -(n+1)^2/2, 2, 0)$, *and the equality*

$$Y(z) = \varepsilon\sigma\sqrt{\frac{1-y}{1-q}}, \quad z^2 = \tau, \tag{2.16}$$

holds true, where $Y(z)$ is a solution of Eq. (2.15).

Equations (2.5), (2.6) via the transformations

$$v = \frac{r+1}{r-1}, \quad u = \frac{p+1}{p-1} \tag{2.17}$$

reduce to the respective equations

$$r'' = \frac{3r-1}{2r(r-1)}r'^2 - \frac{r'}{z} + \frac{n^2}{8}\left(r - \frac{1}{r}\right)\frac{(r-1)^2}{z^2} - 8r^2(r+1), \tag{2.18}$$

$$p'' = \frac{3p-1}{2p(p-1)}p'^2 - \frac{p'}{z} + \frac{(n+1)^2}{8}\left(p - \frac{1}{p}\right)\frac{(p-1)^2}{z^2} - 8p^2(p+1), \tag{2.19}$$

i.e., to Eq. (P5) with the sets of parameters $(n^2/8, -n^2/8, 0, -8)$ and $((n+1)^2/8, -(n+1)^2/8, 0, -8)$.

In this case, system (2.4) transforms to the system

$$r = 1 + \frac{16zp}{2zp' - (n+1)(p^2-1) - 8zp}, \tag{2.20}$$

$$p = 1 - \frac{16zr}{2zr' + n(r^2-1) + 8zr}, \tag{2.21}$$

establishing a one-to-one correspondence (direct and inverse Bäcklund transformations for Eq. (P5) with $\delta \neq 0$ [21]) between the solutions of Eqs. (2.18), (2.19).

Theorem 5. *Let* $r = r(z)$ $((r-1)(r+1) \not\equiv 0)$, $p = p(z)$ $((p+1)(p-1) \not\equiv 0)$ *be arbitrary functions satisfying system* (2.20), (2.21), *and let*

$$2zp' - (n+1)(p^2-1) - 8\varepsilon_1 zp \not\equiv 0, \qquad 2zr' + n(r^2-1) + 8\varepsilon_2 zr \not\equiv 0, \qquad \varepsilon_1^2 = \varepsilon_2^2 = 1.$$

Then they are solutions of Eq. (P5) *with the respective sets of parameters*

$$(n^2/8, -n^2/8, 0, -8), \qquad ((n+1)^2/8, -(n+1)^2/8, 0, -8),$$

and the following equality holds true:

$$Y(z) = \left(\frac{r+1}{r-1}\right)^{-1}\left(\frac{p+1}{p-1}\right), \tag{2.22}$$

where $Y(z)$ *is a solution of Eq.* (2.15).

Corollary 1. *Under the conditions of Theorems 4 and 5, the following equality holds true*:

$$\left(\frac{r(z)+1}{r(z)-1}\right)^{-1}\left(\frac{p(z)+1}{p(z)-1}\right) = \varepsilon\sigma\sqrt{\frac{1-y(\tau)}{1-q(\tau)}}, \quad z^2 = \tau. \tag{2.23}$$

Remark 1. Formula (2.22) relating solutions of Eqs. (P5) ($\delta \neq 0$) and (P3) ($cd \neq 0$) with other sets of parameters was obtained in [22].

After substituting the values of v from (2.7) and (2.17) into system (2.4), it takes the form

$$y(\tau) = 1 - \frac{64zp^2(z)}{[2zp'(z) - (n+1)(p^2(z)-1)]^2}, \quad \tau = z^2, \tag{2.24}$$

$$p(z) = 1 - \frac{4\varepsilon\sqrt{\tau}\cdot y(\tau)\cdot\sqrt{1-y(\tau)}}{2\varepsilon\sqrt{\tau}\cdot y(\tau)\sqrt{1-y(\tau)} + n(1-y(\tau)) - \tau y'(\tau)}. \tag{2.25}$$

Formulas (2.24), (2.25) with fixed ε establish a one-to-one correspondence between solutions of Eq. (P5) with the set of parameters $(0, -n^2/2, 2, 0)$ and solution of the same equation with the set of parameters $((n+1)^2/2, -(n+1)^2/2, 0, -8)$.

Since Eq. (P5) is invariant under the transformation [23]

$$S\colon w(z,\alpha,\beta,\gamma,\delta) \to w^{-1}(z, -\beta, -\alpha, -\gamma, \delta)$$

formula (2.24) can be written as

$$\begin{gathered} y_1(\tau) = \frac{[2zp'(z) - (n+1)(p^2(z)-1)]^2}{[2zp'(z) - (n+1)(p^2(z)-1)]^2 - 64z^2\cdot p^2(z)}, \\ p(z) \not\equiv 0, \quad [2zp'(z) - (n+1)(p^2(z)-1)]^2 - 64z^2p^2(z) \not\equiv 0, \quad z^2 = \tau. \end{gathered} \tag{2.26}$$

Formula (2.26) is given in [24] for the case $n^2 = -4$.

Theorem 6 (see [25]). *If* $\alpha = \kappa^2/8$, $\beta = 0$, $\gamma = m^2/2$, $\delta = 0$ *in Eq.* (P5), *then the substitution* $w(z) = (\tilde{w}+1)^2(4\tilde{w})^{-1}$, $z = \rho^2$, *results in Eq.* (P5) *for* $\tilde{w}$, *where* $\tilde{\alpha} = -\tilde{\beta} = \kappa^2/32$, $\tilde{\gamma} = 0$, $\tilde{\delta} = 2m^2$.

2.2. We consider the system of differential equations equivalent to Eq. (P5) [23]:

$$zu' = -\mu - (\mu+\nu)u - uv - u^2 v, \tag{2.27}$$

$$zv' = \delta z^2 - \gamma z + (\mu+\nu)v + 2\delta z^2 u + \frac{1}{2}v^2 + uv^2, \tag{2.28}$$

where $\nu^2 = 2\alpha$, $\mu^2 = -2\beta$, and the function $u(z)$ is related to $w(z)$ via the transformation

$$u = \frac{w}{1-w}. \tag{2.29}$$

We assume $\mu + \nu = 1$, $\gamma = 0$. Then, in view of (2.29), system (2.27), (2.28) in the unknowns v, w takes the form

$$-\frac{zw'}{w} - \mu\frac{(w-1)^2}{w} + w - 1 = v, \tag{2.30}$$

$$\frac{2(v - zv')}{v^2 + 2\delta z^2} = \frac{w+1}{w-1}. \tag{2.31}$$

We also assume $\delta \neq 0$, because Eq. (P5) with $\gamma = \delta = 0$ can be integrated in elementary functions [23].

System (2.30), (2.31) with $\mu = 0$, $\delta = 8$ is given in [2]. System (2.30), (2.31) for the function v is equivalent to the equation

$$v'' = \frac{v}{v^2 + 2\delta z^2}v'^2 - \frac{1}{z}\cdot\frac{v^2 - 2\delta z^2}{v^2 + 2\delta z^2}\cdot v' - \frac{2\delta v}{v^2 + 2\delta z^2} + \frac{2 - 4\mu + v}{4z^2}(v^2 + 2\delta z^2). \tag{2.32}$$

By substituting $v = \bar{\kappa} z V$, $\bar{\kappa}^2 + 2\delta = 0$, we obtain the following equation for V:

$$V'' = \frac{VV'^2}{V^2 - 1} - \frac{V'}{z} + \frac{\bar{\kappa}(1-2\mu)}{2z}(V^2 - 1) + \frac{\bar{\kappa}^2}{4}V(V^2 - 1). \tag{2.33}$$

The substitution [2]

$$V = -\frac{i}{2}\left(T - \frac{1}{T}\right), \quad i^2 + 1 = 0, \tag{2.34}$$

transforms Eq. (2.33) into the equation

$$T'' = \frac{T'^2}{T} - \frac{T'}{z} + \frac{\bar{\kappa}(1-2\mu)}{4iz}(T^2 + 1) + \frac{\delta}{8}\left(T^3 - \frac{1}{T}\right), \tag{2.35}$$

i.e., into Eq. (P3) with the set of parameters $(\bar{\kappa}(1-2\mu)/4i, \bar{\kappa}(1-2\mu)/4i, \delta/8, -\delta/8)$, $\delta \neq 0$. We note that the inverse substitution

$$T = iV + \varepsilon\sqrt{1 - V^2}, \quad \varepsilon^2 = 1, \tag{2.36}$$

leads to Eq. (2.33). In view of the introduced transformations, system (2.30), (2.31) takes the form

$$T - \frac{1}{T} = \frac{2i}{\bar{\kappa} z}\left(w - 1 - \frac{\mu(w-1)^2}{w} - \frac{zw'}{w}\right), \tag{2.37}$$

$$\frac{w+1}{w-1} = -\frac{4i}{\bar{\kappa}}\cdot\frac{T'}{T^2 + 1}. \tag{2.38}$$

According to [23], all solutions of the equation

$$R(z, \bar{\kappa}) = zw' - (1-\mu)w^2 + (1 - 2\mu + \bar{\kappa} z)w + \mu = 0 \tag{2.39}$$

or $R(z,-\bar{\kappa})=0$ are also solutions of Eq. (P5) if $\mu+\nu=1$, $\gamma=0$. Here, if $w(z)\not\equiv 0$ is a solution of Eq. (P5) with $\nu+\mu=1$, $\gamma=0$, and also a solution of the equation $R(z,\bar{\kappa})=0$ ($R(z,-\bar{\kappa})=0$), then, by formula (2.37), it generates the solutions $T=i$ and $T=-i$ of Eq. (2.35). We consider Eqs. (2.37), (2.38) with

$$T^2(z)+1\not\equiv 0,\quad w(z)\not\equiv 0,\quad R(z,\bar{\kappa})\not\equiv 0\quad (R(z,-\bar{\kappa})\not\equiv 0). \tag{2.40}$$

Thus, formulas (2.37), (2.38) establish the relation between solutions of Eq. (P5) with $\mu+\nu=1$, $\nu^2=2\alpha$, $\mu^2=-2\beta$, $\gamma=0$, $\delta\neq 0$ and solutions of Eq. (2.35) under conditions (2.40).

Example 1. The solution $w=-1$ of Eq. (P5) with $\alpha=\nu^2/2=1/8$, $\beta=-\mu^2/2=-1/8$, $\gamma=0$, $\delta\neq 0$ generates two solutions $T=1$ and $T=-1$ of Eq. (2.35) with $\mu=1/2$ using formula (2.37) with $\nu=\mu=1/2$. Vice versa, Eq. (2.38) with $T=1$, $T=-1$ yields $w=-1$.

Other formulas relating the solutions of Eqs. (P3) and (P5) (under certain constraints on the parameters in these equations) are given in [23], [24].

Remark 2. Solutions of Eq. (2.5) define a class of separable solutions of the complex sine-Gordon equation, also known as the Pohlmeyer–Lund–Regge–Getmanov model. More detailed information about other applications can be found in [26]. The system of two differential equations associated with the strongly shunted Josephson model [27] can also be reduced to system (2.4) [22]. System (2.4) can be derived from the system of two differential equations (describing the class of axially symmetric stationary solutions of the Einstein equations) [28] (see also [29]) by performing in it scale transformations of the unknown functions and the independent variable.

3. Analysis of solutions of the system of differential equations corresponding to random matrix type models with Bessel kernel

It was found in [30] that the distribution of distances between levels that arise under scaling the Laguerre and Jacobi ensembles of Hermitian matrices can be expressed in terms of the Fredholm determinant of an integral operator with a kernel expressed trough the second-order Bessel functions. We obtain a system of partial differential equations related to the logarithmic derivative of this Fredholm determinant when the base domain is a union of intervals. In the case of a single interval, the system degenerates to the system of ordinary differential equations

$$sq'=p+\frac{1}{4}qu, \tag{3.1}$$

$$sp'=\frac{1}{4}(\alpha^2-s)q+\frac{1}{2}qv-\frac{1}{4}pq, \tag{3.2}$$

$$u'=q^2, \tag{3.3}$$

$$v'=pq \tag{3.4}$$

with unknown functions p, q, u, v of the independent variable s and with a constant parameter α. System (3.1)–(3.4) has two first integrals

$$u^2+8v=4sq^2-4u+C_1, \tag{3.5}$$

$$u=4p^2-(\alpha^2-s+2v)q^2+2pqu+C_2, \tag{3.6}$$

where C_1, C_2 are arbitrary constants. Multiplying both sides of Eq. (3.1) by $s\frac{d}{ds}$ and using Eqs. (3.2), (3.3), we obtain the following relation:

$$s(sq')'=\frac{1}{4}(\alpha^2-s)q+\frac{1}{16}(u^2+8v)q+\frac{1}{4}sq^3, \tag{3.7}$$

which, in view of Eq. (3.5), takes the form

$$s(sq')' = \frac{1}{4}(\alpha^2 - s)q + \frac{q}{16}(8sq^2 - 4u + C_1). \tag{3.8}$$

Eliminating the unknown function p from (3.1) and (3.6), we obtain

$$(sq')^2 = \frac{1}{4}u + \frac{1}{4}(\alpha^2 - s)q^2 + \frac{q^2}{16}(u^2 + 8v) - \frac{C_2}{4}. \tag{3.9}$$

Substituting the expression $\frac{1}{16}(u^2 + 8v)q$ from (3.7) into (3.9), we find

$$sq(sq')' = (sq')^2 - \frac{1}{4}u + \frac{1}{4}sq^4 + \frac{C_2}{4}. \tag{3.10}$$

Eliminating the unknown function u from (3.8), (3.10), we obtain the equation to determine q:

$$s(q^2 - 1)(sq')' = q(sq')^2 + \frac{1}{4}sq^3(q^2 - 2) - \frac{1}{4}\left(\alpha^2 + \frac{C_1}{4} - C_2 - s\right)q. \tag{3.11}$$

Equation (3.11) with $C_1 = C_2 = 0$ was obtained in [30]. By introducing in (3.11) the transformation (according to [31])

$$q = \frac{w(x) + 1}{w(x) - 1}, \quad x^2 = s,$$

defining the function $w(x)$, we obtain Eq. (P5) with the parameters

$$\alpha' = -\beta' = \frac{1}{8}\left(\alpha^2 + \frac{C_1}{4} - C_2\right), \quad \gamma' = 0, \quad \delta' = -2.$$

Theorem 7. *System (3.1)–(3.4) is a Painlevé type system.*

The proof of Theorem 7 is based on the fact that the general solution of Eq. (3.11) can be expressed rationally in terms of the general solution of Eq. (P5), which has no movable critical singular points. The other elements u, v, p of system (3.1)–(3.4) also has no movable critical singular points, because

$$-\frac{1}{4}u = sq(sq')' - (sq')^2 - \frac{1}{4}sq^2 - \frac{C_2}{4}, \quad 8v = 4sq^2 - 4u - u^2 + C_1, \quad p = sq' - \frac{1}{4}qu.$$

4. Analysis of solutions of two systems of differential equations related to random matrix type models

4.1. We consider the system of differential equations

$$q' = p - qu + \alpha s, \tag{4.1}$$

$$v' = -pq - \alpha sq, \tag{4.2}$$

$$p' = sq - 2qv + pu + \alpha su, \tag{4.3}$$

$$u' = -q^2, \tag{4.4}$$

where s is an independent variable and α is an arbitrary constant parameter. If $\alpha = 0$, system (4.1)–(4.4) corresponds to a random matrix type model with the Airy kernel [32]. A peculiar feature of system (4.1)–(4.4) is that it is a Hamiltonian system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{sq^2}{2} + q^2v - pqu + \alpha sp - \alpha squ$$

and has the first integral $u^2 - 2v - q^2 = C$, where C is an arbitrary constant. The following theorem holds true.

Theorem 8. *System (4.1)–(4.4) is a Painlevé type system. Its solutions can be expressed in terms of solutions of the second Painlevé equation*

$$q'' = 2q^3 + (s + C) + \alpha. \tag{4.5}$$

Proof. Taking into account the first integral, we pass from system (4.1)–(4.4) to the following one:

$$q' = p - qu + \alpha s, \tag{4.6}$$
$$p' = (s + C)q + q^3 - qu^2 + pu + \alpha su, \tag{4.7}$$
$$u' = -q^2. \tag{4.8}$$

By differentiating both sides of Eq. (4.6) taking into account Eqs. (4.7), (4.8), we arrive at Eq. (4.5). Setting $z = s + C$ in (4.5), we obtain the second Painlevé equation

$$q'' = 2q^3 + zq + \alpha. \tag{P2}$$

According to [33], any solution of Eq. (P2) is a meromorphic function, and it can have only simple poles as singular points. The expansion of a solution in a neighborhood of a pole can be represented as [23]

$$q(z) = \frac{a_{-1}}{z - z_0} + \sum_{k=1}^{\infty} a_k (z - z_0)^k, \tag{4.9}$$

where $a_{-1}^2 = 1$ and a_3 is an arbitrary coefficient. All the other coefficients are uniquely determined as polynomials in z_0, a_{-1}, a_3, and α.

In view of expansion (4.9) with $z = s + C$ and Eqs. (4.8), (4.1), $v = \frac{1}{2}(u^2 - q^2 - C)$, we conclude that the components u, p, v have no movable critical singular points too.

Example 2. System (4.1)–(4.4) with $\alpha = -1$ has a two-parametric solution family

$$q = \frac{1}{s + C}, \quad u = \frac{1}{s + C} + C_1, \quad v = \frac{C_1}{s + C} + \frac{C_1^2 - C}{2}, \quad p = \frac{C_1}{s + C} + s.$$

4.2. The system of differential equations [34]

$$r' = -pu, \tag{4.10}$$
$$u'' = (x^2 - 2n - 1)u + 2u^2 p, \tag{4.11}$$
$$p'' = (x^2 - 2n + 1)p + 2p^2 u \tag{4.12}$$

corresponds to a random matrix type model related to the Dyson processes. In Eqs. (4.10)–(4.12), p, r, u are unknown functions of the independent variable x, and n is an arbitrary constant parameter.

It follows from Eqs. (4.11), (4.12) that

$$pu'' - p''u = -2up, \tag{4.13}$$

hence, in view of (4.1), we obtain

$$pu' - p'u = 2r + C, \tag{4.14}$$

where C is an arbitrary constant. By substituting $p = [u'' - (x^2 - 2n - 1)u]/2u^2$ from (4.11) into (4.13) and setting

$$u = \exp\left[\int w(x)\,dx\right],$$

we obtain the equation to determine w:

$$w''' - 6w^2w' + 2(x^2 - 2n - 2)w' - 2w^2 + 4wx + 2x^2 - 2(2n+2) = 0. \tag{4.15}$$

This equation, by the subsequent transformation $x \to ix$, $w \to -iw$, $w = q(x) + x$, $i^2 = -1$, reduces to the equation [35]

$$q''' - 6q^2q' - 12qq'x - 4(x^2 - n - 1)q' - 4qx - 4q^2 = 0,$$

whose first integral is the fourth Painlevé equation

$$q'' = \frac{q'^2}{2q} + \frac{3}{2}q^3 + 4xq^2 + 2(x^2 - n - 1)q + \frac{\beta}{q}, \tag{P4}$$

where $\alpha = n + 1$ and β is an arbitrary constant. Thus, the following theorem holds true.

Theorem 9. *Let $w = w(x)$ be a solution of Eq.* (4.15) *with a fixed value of the parameter n. Then the functions*

$$u = \exp\left[\int w(x)\,dx\right], \quad p = [w' + w^2 - x^2 + 2n + 1]\left\{2\exp\left[\int w(x)\,dx\right]\right\}^{-1},$$
$$r = \frac{1}{4}[-w'' + 2w^3 - 2w(x^2 - 2n - 1) + 2x - 2C]$$

are solutions of system (4.10)–(4.12) *and have no movable critical singular points.*

We note that the third-order differential equation for the function r has been obtained in [34], and the first integral of that equation is the equation for the polynomial Hamiltonian [11], [36] associated with Eq. (P4) under particular values of α, β. However, the formulas expressing the components p, u of system (4.10)–(4.12) in terms of solutions of Eq. (P4) have not been provided.

5. Analysis of solutions of two systems of differential equations via the Painlevé test

5.1. Using the Painlevé test [37]–[39] to analyze the solutions of the system of equations [6]

$$\begin{aligned} q' &= -sq + (\sqrt{2N} - 2u)p, \\ w' &= -p^2, \\ p' &= sp - (\sqrt{2N} + 2w)q, \\ u' &= -q^2 \end{aligned} \tag{5.1}$$

with unknown functions q, w, p, u of the independent variable s and an arbitrary parameter N, we can prove the following theorem.

Theorem 10. *System* (5.1) *passes the formal Painlevé test.*

System (5.1) corresponds to the random matrix type model with the Hermitian kernel.

We have also studied the convergence problem for the formal Laurent series (satisfying system (5.1))

$$\begin{aligned}
q &= a_{-1}\tau^{-1} + a_1\tau + a_2\tau^2 + \cdots, \quad \tau = s - s_0, \\
w &= (4a_{-1}^2\tau)^{-1} + b_0 + b_1\tau + b_2\tau^2 + \cdots, \\
p &= (2a_{-1}\tau)^{-1} + c_1\tau + c_2\tau^2 + \cdots, \\
u &= a_{-1}^2\tau^{-1} + u_0 + u_1\tau + u_2\tau^2 + \cdots,
\end{aligned} \tag{5.2}$$

containing four arbitrary parameters s_0, $a_{-1} \neq 0$, c_1, c_2.

Theorem 11. *Expansions (5.2) converge in some neighborhood* $0 < |s - s_0| < \rho$, $\rho > 0$.

Indeed, by virtue of Theorem 10, according to [40], system (5.1) reduces by the transformations

$$tq = a_{-1} + \bar{q}(t), \;\; tu = a_{-1}^2 + \bar{u}(t), \;\; tp = (2a_{-1})^{-1} + \bar{p}(t), \;\; tw = (4a_{-1}^2)^{-1} + \bar{w}(t), \;\; t = s - s_0$$

to the Briot–Bouquet system [41], [42], having the four-parametric holomorphic solution [43] vanishing at zero.

Remark 3. 1. System (5.1) has the first integral [6] $\sqrt{2N}(u - w) + 2uw = pq + C$, where C is an arbitrary constant.

2. System (5.1) is a Hamiltonian system with the Hamiltonian

$$H = -spq + (\sqrt{2N} - 2u)\frac{p^2}{2} + (\sqrt{2N} + 2w)\frac{q^2}{2}.$$

3. The function $R(s) = -2spq + (\sqrt{2N} - 2u)p^2 + (\sqrt{2N} + 2w)q^2$, according to [6], is a solution of the third-order differential equation [6] whose first integral (with zero integration constant) is an equation for the polynomial Hamiltonian [11], [36] associated with Eq. (P4). However, obtaining explicit formulas expressing the components of system (5.1) in terms of the function $R(s)$ seems to be impossible.

5.2. The system of differential equations

$$\begin{aligned}
sq' &= \left(\frac{s}{2} - \frac{\alpha}{2} - N\right)q + (\sqrt{N(N+\alpha)} + u)p, \\
w' &= p^2, \\
sp' &= -(\sqrt{N(N+\alpha)} - w) - \left(\frac{s}{2} - \frac{\alpha}{2} - N\right)p, \\
u' &= q^2,
\end{aligned} \tag{5.3}$$

with unknown functions q, w, p, u of the independent variable s and arbitrary constant parameters N, α, is a particular case of the random matrix type model with the Laguerre kernel [6].

Similarly to the case of system (5.1), the following theorem holds true.

Theorem 12. *System (5.3) passes the formal Painlevé test.*

We have also studied the convergence problem for the formal Laurent series (satisfying system (5.3))

$$\begin{aligned}
q &= a_{-1}\tau^{-1} + a_1\tau + a_2\tau^2 + \cdots, \quad \tau = s - s_0, \\
w &= -s_0^2(a_{-1}^2\tau)^{-1} + d_0 + d_1\tau + d_2\tau^2 + \cdots, \\
p &= s_0(a_{-1}\tau)^{-1} + b_1\tau + b_2\tau^2 + \cdots, \\
u &= -a_{-1}^2\tau^{-1} + c_0 + c_1\tau + c_2\tau^2 + \cdots,
\end{aligned} \tag{5.4}$$

containing four arbitrary parameters $s_0 \neq 0$, $a_{-1} \neq 0$, a_1, b_2.

In view of expansions (5.4), the following theorem holds true.

Theorem 13. *Expansions* (5.4) *converge in some neighborhood* $0 < |s - s_0| < \rho$, $\rho > 0$, $s_0 \neq 0$.

Remark 4. 1. System (5.3) has the first integral [6]

$$(\sqrt{N(N+\alpha)} + u)p^2 + (\sqrt{N(N+\alpha)} - w)q^2 - (\alpha + 2N)pq + wu + \sqrt{N(N+\alpha)}(w-u) = C,$$

where C is an arbitrary constant.

2. The function $sR(s) = (s - \alpha - 2N)pq + (\sqrt{N(N+\alpha)} + u)p^2 + (\sqrt{N(N+\alpha)} - w)q^2$, according to [6], is a solution of a third-order differential equation whose first integral (with zero integration constant) is a solution of the second-order second degree differential equation for the polynomial Hamiltonian [11], [36] associated with Eq. (P5) under special parameter values. However, it is impossible to obtain the explicit formulas allowing to express the components of system (5.3) in terms of the function $R(s)$.

6. On reducing a bilinear differential equation to a Painlevé type equation

In [44]–[46], hierarchies of bilinear partial differential equations for the Fredholm determinants (corresponding to random matrix type models) of integral operators with given kernels were obtained. The first equation in one of the hierarchies is

$$\left(\mathcal{A}_1^3 - 4\left(\mathcal{A}_3 - \frac{1}{2}\right)\right) f + 6(\mathcal{A}_1 f)^2 = 0, \tag{6.1}$$

where

$$\mathcal{A}_n = \sum_{i=1}^{2r} x_i^{(n-1)/2} \cdot \frac{\partial}{\partial x_i}, \qquad n = 1, 3,$$

and f is an unknown function of the independent variables $x_1, x_2, \ldots, x_{2r}$.

Eq. (6.1) is associated with the set

$$E = \bigcup_{i=1}^{r} [x_{2i-1}, x_{2i}] \subset \mathbb{R}.$$

The transformation $q_1 = f(\tau)$, $\tau = x_1 + x_2 + \cdots + x_{2r}$, reduces Eq. (6.1) to the equation [47]

$$\tau q'''_1 - 2\tau q'_1 + q_1 + 6\tau q'^2_1 = 0, \quad q'_1 = \frac{dq_1}{d\tau}, \quad q'''_1 = \frac{d^3 q_1}{d\tau^3},$$

having the first integral

$$q''^2_1 + 4q'^3_1 - \frac{2}{\tau}\tau q'^2_1 + \frac{2}{\tau} q_1 q'_1 = C, \tag{6.2}$$

where C is an arbitrary constant. The transformation $q_1 = \lambda q$, $\tau = \mu z$, $\lambda\mu = 1$, $\mu^3 = \tau$, allows writing Eq. (6.2) in the form [48]

$$q''^2 + 4q'^3 - 2zq'^2 + 2qq' - \left(\alpha - \frac{\varepsilon}{2}\right)^2 = 0, \tag{6.3}$$

where α is an arbitrary parameter, $\varepsilon^2 = 1$.

Equation (6.3) is an equation for the polynomial Hamiltonian associated with Eq. (P2).

7. Conclusions

In this paper we have addressed the problem of constructing solutions of systems of differential equations associated with certain random matrix type models. For such systems, we have obtain new classes of solutions, which are expressed in terms of solutions of the Painlevé II–V equations. We have shown that the solutions of two systems of ordinary differential equations associated with the random matrix type models with Laguerre and Hermitian kernels pass the formal Painlevé test. We have obtained new formulas relating the solutions of the fifth and third Painlevé equations with respective parameters

$$\alpha = \frac{\nu^2}{2}, \quad \beta = -\frac{\mu^2}{2}, \quad \gamma = 0, \quad \delta, \nu + \mu = 1, \quad \delta \neq 0$$

and

$$a = \frac{(1-2\mu)\bar{\kappa}}{4i}, \quad b = \frac{(1-2\mu)\bar{\kappa}}{4i}, \quad c = \frac{\delta}{8}, \quad d = -\frac{\delta}{8}, \quad i^2 + 1 = 0, \quad \bar{\kappa}^2 + 2\delta = 0.$$

A vital problem is studying the solutions of systems of ordinary differential equations corresponding to random matrix type models with different kernels via the Painlevé test.

Of interest is the question of the existence of self-similar reductions of Painlevé type partial differential equations of orders higher than that of Eq. (6.1)) from the hierarchy of bilinear differential equations.

Among numerous publications devoted to the study of asymptotic properties of solutions of differential equations for random matrix type models, we note the studies [49]–[51], which are based on the Riemann problem, as well as [52], which is related to the study of the asymptotic properties of solutions of the second discrete Painlevé equation.

As regards the applications of random matrices mentioned at the beginning of this paper, we especially emphasize the important role of them and Painlevé equations in statistical physics [53].

The subject and possible directions for further studies closely related to the random matrix type models are outlined in the survey [54], which contains a rather comprehensive reference list.

Remark 5. System (2.27), (2.28) coincides with the system from [55] up to notation if we replace the independent variable x with $-z$ or the parameter γ with $-\gamma$ in that system.

In [21], the Bäcklund transformation for solutions of Eq. (P5) with $\delta \neq 0$ was obtained using system (2.27), (2.28).

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