

Parametric instability of plasma waves in graphene excited by a high-frequency electric field

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The system of Vlasov equations for electrons with a linear Dirac spectrum is solved, and a dispersion equation is derived that describes the influence of a uniform high-frequency electric field on the spectrum and damping rate of two-dimensional terahertz (THz) plasmons in single-layer doped graphene. Our analysis demonstrates that a THz electric field of several tens of kV/cm, applied parallel to the graphene plane, can induce parametric instability of the long-wavelength two-dimensional plasmons. This parametric instability arises from the resonant coupling of plasmon modes, which occurs in a high-frequency electric field due to third-order electrodynamic nonlinearity.

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I. INTRODUCTION

Graphene is a single-atom layer of carbon atoms in a hexagonal lattice. This material exhibits a number of unusual electronic properties arising from its specific band structure: the absence of a band gap and a linear massless dispersion law for electrons and holes near the Dirac points [1]. One of the remarkable features of graphene, resulting from its electronic spectrum, is its strong nonlinear electromagnetic response over a wide frequency range, which can significantly exceed the nonlinearity of conventional materials with a standard (parabolic) dispersion law of current carriers [2–13]. Graphene is also known to support weakly damped two-dimensional (2D) plasmon modes, the parameters of which can be tuned by doping or electrical gating. This tunability, combined with large nonlinear susceptibilities, makes graphene a very promising material for practical applications in optoelectronics, photonics, and plasmonics [14–21].

Recently, considerable attention has been given to the study of nonlinear properties of doped single-layer graphene in the terahertz (THz) spectral region, when the nonlinear electromagnetic response is caused mainly by intraband electron transitions. Two main mechanisms leading to anomalously large values of nonlinear susceptibilities were considered. The first mechanism is of a dynamic nature: the nonlinear dependence of the current on the THz field arises due to the linear electron dispersion law, which leads to a nonlinear dependence of the electron velocity on the momentum [5–11]. It is worth noting that a similar mechanism of optical nonlinearity occurs in narrow-band semiconductors with a nonparabolic energy spectrum [22,23]. The second mechanism of nonlinear electrical conductivity at THz frequencies is caused by the heating of electrons in a THz electric field and their ultrafast thermalization, which occurs in graphene in a time (~ 100 fs), significantly shorter than the characteristic time of change of the THz field [24–26].

The pronounced nonlinear electrodynamics response of graphene enables the conversion of free-space electromagnetic radiation into 2D plasmons. Previous studies [27,28] have demonstrated that 2D plasmons can be efficiently generated through the second-order nonlinear frequency wave mixing. A generation mechanism consisting of parametric decay of an incident photon into a photon with a lower frequency and a 2D plasmon was also considered [29]. Both of these effects are caused by the second-order nonlinearity and arise in graphene only when spatial dispersion is taken into account. It has recently been shown that Floquet engineering of the band structure of gapped Dirac materials in an electromagnetic wave periodically modulated in amplitude can lead to the parametric instability of plasmons [30].

In the present paper, the possibility of parametric excitation of 2D plasma waves in graphene subjected to a uniform high-frequency electric field is theoretically studied. In this case the parametric instability responsible for the plasmon generation stems from the third-order nonlinearity inherent to graphene due to the linear energy spectrum of charge carriers. Our analysis reveals that, in the second order in the electric field amplitude, the parametric instability of 2D plasmons emerges when the frequency of the applied field is close to the plasmon frequency. It should be emphasized that the effect under consideration is due to third-order nonlinearity, as the nonlinear response leading to the parametric instability is quadratic in the amplitude of the high-frequency field and linear in the electric field of the plasma wave. From a quantum-mechanical perspective, this process can be interpreted as the conversion of two photons into a pair of terahertz plasmons having the same frequency but opposite wave vectors.

Parametric instabilities in gaseous and solid-state plasmas are known to arise from the resonant coupling of collective modes, which is due to the periodic modulation of plasma parameters in an alternating electric field. These phenomena have been extensively studied in electron-ion plasma and semiconductors, and it has been shown that instabilities stem from oscillations of the electron velocity relative to heavier ions or holes [31,32]. In a single-component plasma, such

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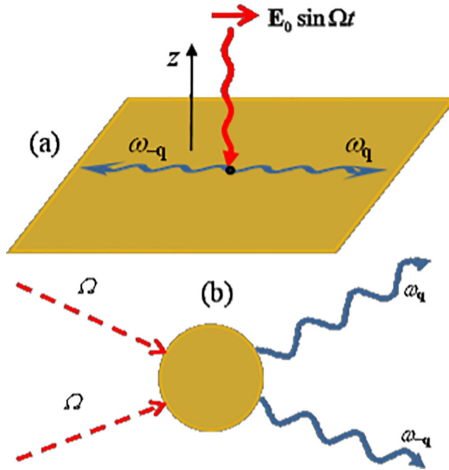


FIG. 1. (a) Sketch of the single-layer graphene (yellow plane) in a uniform high-frequency electric field $\mathbf{E}_0 \sin \Omega t$. The generated 2D plasmons are marked by blue wavy lines. (b) Conversion of two photons (dashed lines) into two 2D plasmons (wavy lines) with the same frequency and oppositely directed wave vectors.

as an electron plasma with a quadratic dispersion law, the parametric instability typically does not occur, since all electrons oscillate collectively as a whole. However, parametric resonance can emerge in a single-component plasma with a nonparabolic energy spectrum of charge carriers, for example, in a relativistic gas plasma [33,34], or in narrow-gap semiconductors with the Kane dispersion law [35,36]. In such systems, the instability is caused by oscillations of the electron effective mass under the action of an alternating electric field. A similar mechanism of parametric instability can also be expected in single-layer graphene with a linear electron dispersion law.

II. DISPERSION EQUATION FOR 2D PLASMONS

We consider single-layer graphene in a uniform high-frequency electric field $\mathbf{E}(t) = \mathbf{E}_0 \sin \Omega t$ with the amplitude $\mathbf{E}_0$ and the frequency Ω (see Fig. 1). The field frequency is assumed to satisfy the condition $\hbar\Omega < 2\varepsilon_F$ (ε_F is the Fermi energy of electrons), which means that interband transitions are suppressed, and the interaction of electrons with a high-frequency electric field is governed by the Drude mechanism. For a typical electron density of about 10^{11} – 10^{12} cm $^{-2}$ the above condition is fulfilled in the THz frequency range (0.1–10 THz). Furthermore, it is assumed that the frequency ω_q and the wave vector $\mathbf{q}$ of 2D plasmons satisfy the conditions $v_F \ll \omega_q/q \ll c$ ($v_F \simeq 10^8$ cm/s is the Fermi velocity of electrons in graphene, and c is the speed of light), which indicate that 2D plasmons are considered within the long-wavelength approximation and retardation effects are insignificant.

In the collisionless approximation used here, the kinetic equation for the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ of electrons in graphene layer is given by

$$\frac{\partial f}{\partial t} + \mathbf{v}_p \frac{\partial f}{\partial \mathbf{r}} - e(\mathbf{E}_0 \sin \Omega t - \nabla \varphi|_{z=0}) \frac{\partial f}{\partial \mathbf{p}} = 0, \quad (1)$$

where $\varphi(\mathbf{r}, z, t)$ is the potential of the self-consistent electric field generated by a 2D plasma wave, which is determined

by the Poisson equation. The two-dimensional vectors $\mathbf{r}$ and $\mathbf{p}$ denote the position and momentum of an electron in the graphene plane at $z = 0$; $\mathbf{v}_p = \partial \varepsilon_p / \partial \mathbf{p}$ is the velocity of an electron with momentum $\mathbf{p}$, and $\varepsilon_p = v_F |\mathbf{p}|$ is the electron spectrum in single-layer graphene.

We seek the solution of Eq. (1) in the form $f = f^{(0)} + \delta f(\mathbf{p}, t) \exp(i\mathbf{q}\mathbf{r})$, where the second term describes a small perturbation caused by the self-consistent field of the plasma wave. The function $f^{(0)}$ obeys the equation

$$\frac{\partial f^{(0)}}{\partial t} - e\mathbf{E}_0 \sin \Omega t \frac{\partial f^{(0)}}{\partial \mathbf{p}} = 0, \quad (2)$$

the solution of which is given by

$$f^{(0)}(\mathbf{p}) = f_0(\mathbf{p} - \mathbf{p}_0(t)), \quad (3)$$

where

$$\mathbf{p}_0(t) = -e \int_{-\infty}^t \mathbf{E}_0 \sin \Omega t' dt' = \frac{e\mathbf{E}_0}{\Omega} \cos \Omega t,$$

and $f_0(\varepsilon_p) = \{1 + \exp[(\varepsilon_p - \varepsilon_F)/T]\}^{-1}$ is the Fermi-Dirac function. Equation (3) is a shifted in momentum space Fermi-Dirac distribution function, describing the nonequilibrium state of electrons in a high-frequency uniform electric field.

In the linear approximation in the small perturbation δf , Eq. (1) takes the form

$$\frac{\partial \delta f}{\partial t} + i\mathbf{q}\mathbf{v}_p \delta f - e\mathbf{E}_0 \sin \Omega t \frac{\partial \delta f}{\partial \mathbf{p}} + ie\varphi_q(t)\mathbf{q} \frac{\partial f^{(0)}}{\partial \mathbf{p}} = 0. \quad (4)$$

Here, $\varphi_q(t)$ is the spatial Fourier component of the self-consistent electric potential in the graphene plane, which is determined from the Poisson equation

$$\frac{\partial^2 \varphi_q}{\partial z^2} - q^2 \varphi_q = 4\pi e \delta(z) \int \delta f d^2 p, \quad (5)$$

where $\delta(z)$ is the Dirac delta function. The solution of this equation is given by

$$\varphi_q(z, t) = -\frac{2\pi e}{q} e^{-q|z|} \int \delta f d^2 p. \quad (6)$$

Next, it is convenient to introduce a new function

$$\psi(\mathbf{p}, t) = \delta f(\mathbf{p} + \mathbf{p}_0(t)) \exp \left[i\mathbf{q} \int_{-\infty}^t (\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(t')} - \mathbf{v}_{\mathbf{p}}) dt' \right]. \quad (7)$$

Using Eqs. (6) and (7), the self-consistent potential in the graphene plane can be written as

$$\begin{aligned} \varphi_q(t) = & -\frac{2\pi e}{q} \int d^2 p \psi(\mathbf{p}, t) \\ & \times \exp \left[-i\mathbf{q} \int_{-\infty}^t (\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(t')} - \mathbf{v}_{\mathbf{p}}) dt' \right]. \end{aligned} \quad (8)$$

The equation for the newly introduced function $\psi(\mathbf{p}, t)$ is obtained by substituting $\delta f(\mathbf{p}, t)$ from Eq. (7) into Eq. (4), and reads as

$$\begin{aligned} & \frac{\partial \psi}{\partial t} + i\mathbf{q}\mathbf{v}_p \psi \\ & = -ie\varphi_q \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \exp \left[i\mathbf{q} \int_{-\infty}^t (\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(t')} - \mathbf{v}_{\mathbf{p}}) dt' \right]. \end{aligned} \quad (9)$$

The solution of this equation is given by

$$\psi(t, \mathbf{p}) = -ie\mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \int_{-\infty}^t dt' \varphi_{\mathbf{q}}(t') e^{-i\mathbf{q}\mathbf{v}_{\mathbf{p}}(t-t')} \exp \left[i\mathbf{q} \int_{-\infty}^{t'} d\tau (\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(\tau)} - \mathbf{v}_{\mathbf{p}}) \right]. \quad (10)$$

Substituting Eq. (10) into Eq. (8), after simple transformations we obtain

$$\varphi_{\mathbf{q}}(t) = \frac{2\pi ie^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \int_{-\infty}^t dt' \varphi_{\mathbf{q}}(t') \exp \left(-i\mathbf{q} \int_{t'}^t d\tau \mathbf{v}_{\mathbf{p}+\mathbf{p}_0(\tau)} \right), \quad (11)$$

where the electron velocity $\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(\tau)}$ appearing in Eq. (11) is given by

$$\mathbf{v}_{\mathbf{p}+\mathbf{p}_0(\tau)} = v_F \frac{\mathbf{p} + \frac{e\mathbf{E}_0}{\Omega} \cos \Omega\tau}{\left| \mathbf{p} + \frac{e\mathbf{E}_0}{\Omega} \cos \Omega\tau \right|} = v_F \frac{\mathbf{p} + \frac{e\mathbf{E}_0}{\Omega} \cos \Omega\tau}{\sqrt{p^2 + \frac{2e\mathbf{E}_0\mathbf{p}}{\Omega} \cos \Omega\tau + \frac{e^2 E_0^2}{\Omega^2} \cos^2 \Omega\tau}}. \quad (12)$$

The integral of the velocity (12) over time, which is included in Eq. (11), cannot be expressed in closed analytical form. Therefore, in the following analysis, we restrict ourselves to the quadratic approximation with respect to the amplitude of the high-frequency field. Expanding Eq. (12) in powers of the field and substituting the resulting expression into Eq. (11), we perform the integration over τ in the exponential factor, leading to

$$\varphi_{\mathbf{q}}(t) = \frac{2\pi ie^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \int_{-\infty}^t dt' \varphi_{\mathbf{q}}(t') \exp \left\{ -iv_F \frac{(\mathbf{q}\mathbf{p})}{p} \left[(1-a_0)(t-t') + \frac{a_1}{\Omega} (\sin \Omega t - \sin \Omega t') - \frac{a_0}{2\Omega} (\sin 2\Omega t - \sin 2\Omega t') \right] \right\}, \quad (13)$$

where the coefficients a_0 and a_1 are determined by the expressions

$$a_0 = \frac{e^2 E_0^2}{4p^2 \Omega^2} + \frac{e^2 (\mathbf{E}_0 \mathbf{q})(\mathbf{E}_0 \mathbf{p})}{2p^2 \Omega^2 (\mathbf{q}\mathbf{p})} - \frac{3e^2 (\mathbf{E}_0 \mathbf{p})^2}{4p^4 \Omega^2}, \quad a_1 = \frac{e(\mathbf{E}_0 \mathbf{q})}{(\mathbf{q}\mathbf{p})\Omega} - \frac{e(\mathbf{E}_0 \mathbf{p})}{p^2 \Omega}. \quad (14)$$

In the approximation used here, Eq. (13), after expanding the exponential factor in powers of the electric field, takes the form

$$\varphi_{\mathbf{q}}(t) = \frac{2\pi ie^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \int_{-\infty}^t dt' \varphi_{\mathbf{q}}(t') \exp \left[-iv_F \frac{(\mathbf{q}\mathbf{p})}{p} (1-a_0)(t-t') \right] \times \left\{ 1 - iv_F \frac{(\mathbf{q}\mathbf{p})}{p} \left[\frac{a_1}{\Omega} (\sin \Omega t - \sin \Omega t') - \frac{a_0}{2\Omega} (\sin 2\Omega t - \sin 2\Omega t') \right] + \frac{a_1^2}{2\Omega^2} (\sin \Omega t - \sin \Omega t')^2 \right\}. \quad (15)$$

Then, applying the Fourier transform with respect to time to Eq. (15), we obtain the following equation for the Fourier components of $\varphi_{\mathbf{q}}(\omega)$,

$$\begin{aligned} \varphi_{\mathbf{q}}(\omega) & \left\{ 1 + \frac{2\pi e^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \left[\frac{1}{\omega - \mathbf{q}\mathbf{v}_{\mathbf{p}}(1-a_0)} + \frac{a_1^2 (\mathbf{q}\mathbf{v}_{\mathbf{p}})^2}{4\Omega^2} [\Delta_{\mathbf{q}}(\omega, \Omega) + \Delta_{\mathbf{q}}(\omega, -\Omega)] \right] \right\} \\ & - \frac{2\pi e^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \frac{\mathbf{q}\mathbf{v}_{\mathbf{p}}}{2\Omega} [\varphi_{\mathbf{q}}(\omega + \Omega) \Delta_{\mathbf{q}}(\omega, \Omega) - \varphi_{\mathbf{q}}(\omega - \Omega) \Delta_{\mathbf{q}}(\omega, -\Omega)] \\ & + \frac{2\pi e^2}{q} \int d^2 p \mathbf{q} \frac{\partial f_0}{\partial \mathbf{p}} \frac{\mathbf{q}\mathbf{v}_{\mathbf{p}}}{4\Omega} \left\{ \varphi_{\mathbf{q}}(\omega + 2\Omega) \left[a_0 \Delta_{\mathbf{q}}(\omega, 2\Omega) + \frac{a_1^2 \mathbf{q}\mathbf{v}_{\mathbf{p}}}{2\Omega} [\Delta_{\mathbf{q}}(\omega, 2\Omega) - 2\Delta_{\mathbf{q}}(\omega, \Omega)] \right] \right. \\ & \left. - \varphi_{\mathbf{q}}(\omega - 2\Omega) \left[a_0 \Delta_{\mathbf{q}}(\omega, -2\Omega) - \frac{a_1^2 \mathbf{q}\mathbf{v}_{\mathbf{p}}}{2\Omega} [\Delta_{\mathbf{q}}(\omega, -2\Omega) - 2\Delta_{\mathbf{q}}(\omega, -\Omega)] \right] \right\}, \end{aligned} \quad (16)$$

where

$$\Delta_{\mathbf{q}}(\omega, \nu) = \frac{1}{\omega + \nu - \mathbf{q}\mathbf{v}_{\mathbf{p}}} - \frac{1}{\omega - \mathbf{q}\mathbf{v}_{\mathbf{p}}}. \quad (17)$$

Equation (16) is quite complicated, so we further assume that the electric field $\mathbf{E}_0$ is directed parallel to the wave vector $\mathbf{q}$. In addition, we restrict our analysis to the case of degenerate electron statistics. Under these conditions, after integration over the electron momenta Eq. (16) reduces to the following form,

$$A_0(\omega) \varphi_{\mathbf{q}}(\omega) = \alpha \omega^2 \frac{\varepsilon_F}{\hbar \Omega} \sum_{\substack{n=-2 \\ n \neq 0}}^{+2} \beta^{|n|} A_n(\omega) \varphi_{\mathbf{q}}(\omega + n\Omega), \quad (18)$$

where

$$\begin{aligned}
 A_0(\omega) &= \omega^2 - \omega_q^2 \left\{ f_0\left(\frac{\omega}{qv_F}\right) + \beta^2 \frac{\omega^2}{\Omega^2} \left[f_1\left(\frac{\omega + \Omega}{qv_F}\right) + f_1\left(\frac{\omega - \Omega}{qv_F}\right) - 2f_1\left(\frac{\omega}{qv_F}\right) \right] \right\} \\
 A_{\pm 1}(\omega) &= \mp \left\{ f_2\left(\frac{\omega \pm \Omega}{qv_F}\right) - f_2\left(\frac{\omega}{qv_F}\right) \right\}, \\
 A_{\pm 2}(\omega) &= \pm \left\{ f_3\left(\frac{\omega \pm 2\Omega}{qv_F}\right) - f_3\left(\frac{\omega}{qv_F}\right) \pm \frac{qv_F}{\Omega} \left[f_1\left(\frac{\omega \pm 2\Omega}{qv_F}\right) + f_1\left(\frac{\omega}{qv_F}\right) - 2f_1\left(\frac{\omega \pm \Omega}{qv_F}\right) \right] \right\}.
 \end{aligned} \tag{19}$$

Here, we have introduced the following notations: $\alpha = \frac{e^2}{\hbar v_F}$, $\beta = \frac{eE_0}{p_F \Omega}$ ($p_F = \hbar \sqrt{\pi n}$ is the Fermi momentum, n is the electron density), and $\omega_q = (2\alpha \varepsilon_F v_F q / \hbar)^{1/2}$ is the frequency of 2D plasma wave. The parameter β introduced above can be represented as

$$\beta = \frac{eE_0}{p_F \Omega} = \frac{eE_0 v_F / \Omega}{\varepsilon_F},$$

from which it is clear that it is the ratio of the energy acquired by an electron in the high-frequency electric field over one period to the Fermi energy. The dimensionless parameter β can be expressed as the ratio of the high-frequency electric field amplitude to the characteristic field $E_c = p_F \Omega / e$, which is determined by the graphene doping and the frequency of an external field. The quadratic approximation used here in the parameter β is valid provided that $E_0 < E_c$. The functions $f_n(x)$ in (19) after integration take the form

$$\begin{aligned}
 f_0(x) &= \frac{x^2}{\pi} \int_0^{2\pi} \frac{\cos \varphi}{x - [1 - (3\beta^2/4)\sin^2 \varphi]} d\varphi \simeq 2x^2 \left\{ (1 - x^{-2})^{-1/2} \theta(|x| - 1) \right. \\
 &\quad \left. - 1 - \frac{3}{8} \beta^2 [1 - 6x^2 + (6x^2 - 4)(1 - x^{-2})^{-1/2} \theta(|x| - 1)] \right\}, \\
 f_1(x) &= \frac{1}{4\pi} \int_0^{2\pi} \frac{\cos \varphi \sin^4 \varphi}{x - \cos \varphi} d\varphi = \frac{3}{4} x^2 + \frac{1}{2} x^4 [(1 - x^{-2})^{3/2} \theta(|x| - 1) - 1] - \frac{3}{16}, \\
 f_2(x) &= \frac{1}{\pi} \int_0^{2\pi} \frac{\cos \varphi \sin^2 \varphi}{x - \cos \varphi} d\varphi = -2x^2 [(1 - x^{-2})^{1/2} \theta(|x| - 1) - 1] - 1, \\
 f_3(x) &= \frac{3}{8\pi} \int_0^{2\pi} \frac{\cos^2 \varphi \sin^2 \varphi}{x - \cos \varphi} d\varphi = -\frac{3}{4} \left\{ x^3 [(1 - x^{-2})^{1/2} \theta(|x| - 1) - 1] + \frac{x}{2} \right\},
 \end{aligned} \tag{20}$$

where $\theta(x)$ is the Heaviside step function. In the first expression of Eq. (20) we retained only the quadratic terms in the parameter β .

Equation (18) represents a system of homogeneous linear equations for the Fourier components of the electric field potential of plasmon. The solvability condition for this system of equations is that its determinant be zero. Using the smallness of the parameter β on the right-hand side of Eq. (18), we will seek its approximate solution by perturbation method. Replacing the argument ω in the left-hand side of Eq. (18) with $\omega + n\Omega$ and substituting the resulting expression for $\varphi(\omega + n\Omega)$ into the right-hand side, we obtain

$$\begin{aligned}
 &\left\{ A_0(\omega) - \alpha^2 \omega^2 \left(\frac{\varepsilon_F}{\hbar \Omega} \right)^2 \sum_{\substack{n=-2 \\ n \neq 0}}^{+2} \beta^{2|n|} (\omega + n\Omega)^2 \frac{A_n(\omega) A_{-n}(\omega + n\Omega)}{A_0(\omega + n\Omega)} \right\} \varphi_q(\omega) \\
 &= \alpha^2 \omega^2 \left(\frac{\varepsilon_F}{\hbar \Omega} \right)^2 \sum_{\substack{n=-2 \\ n \neq 0, -k}}^{+2} \sum_{\substack{k=-2 \\ k \neq 0}}^{+2} \beta^{|n|+|k|} (\omega + n\Omega)^2 \frac{A_n(\omega) A_k(\omega + n\Omega)}{A_0(\omega + n\Omega)} \varphi_q(\omega + k\Omega + n\Omega).
 \end{aligned} \tag{21}$$

Continuing the iteration process, we derive the dispersion equation expressed as a power series in β . In the quadratic approximation with respect to the electric field, it is sufficient to limit ourselves to two iterations, and we obtain

$$\begin{aligned}
 &A_0(\omega) - \alpha^2 \omega^2 \left(\frac{\varepsilon_F}{\hbar \Omega} \right)^2 \sum_{\substack{n=-2 \\ n \neq 0}}^{+2} \beta^{2|n|} (\omega + n\Omega)^2 \frac{A_n(\omega) A_{-n}(\omega + n\Omega)}{A_0(\omega + n\Omega)} - \alpha^4 \beta^4 \omega^2 \left(\frac{\varepsilon_F}{\hbar \Omega} \right)^4 \\
 &\times \left[(\omega + \Omega)^4 (\omega + 2\Omega)^2 \frac{A_1(\omega) A_1(\omega + \Omega) A_{-1}(\omega + 2\Omega) A_{-1}(\omega + \Omega)}{A_0^2(\omega + \Omega) A_0(\omega + 2\Omega)} \right. \\
 &\left. + (\omega - \Omega)^4 (\omega - 2\Omega)^2 \frac{A_{-1}(\omega) A_1(\omega - \Omega) A_{-1}(\omega - \Omega) A_1(\omega - 2\Omega)}{A_0^2(\omega - \Omega) A_0(\omega - 2\Omega)} \right] = 0.
 \end{aligned} \tag{22}$$

III. RESULTS AND DISCUSSION

When the condition $\Omega \simeq 2\omega_q$ is satisfied, the term with $n = -1$ in the sum and the last term in the square brackets in Eq. (22) become resonant. However, these terms are proportional to the coefficients $A_{-1}(\omega)$, $A_1(\omega - \Omega)$, and $A_{-1}(\omega - \Omega)$, which, as follows from Eqs. (19) and (20), vanish. Consequently, the parametric resonance does not occur at the frequency $\Omega \simeq 2\omega_q$, and the electric field leads only to a small shift of the plasmon frequency.

Next, we consider the possibility of parametric resonance at the frequency $\Omega \simeq \omega_q$. By retaining in Eq. (22) solely the last term in the square brackets and the term with $n = -2$ in the sum, both containing resonant denominators, while disregarding all nonresonant contributions, the dispersion equation reads as

$$A_0(\omega)A_0(\omega - 2\Omega) - \alpha^2\omega^2\left(\frac{\varepsilon_F}{\hbar\Omega}\right)^2\beta^4(\omega - 2\Omega)^2A_{-2}^2(\omega) - \alpha^4\beta^4\omega^2\left(\frac{\varepsilon_F}{\hbar\Omega}\right)^4(\omega - \Omega)^4(\omega - 2\Omega)^2\frac{A_{-1}^2(\omega)A_{-1}^2(\omega - \Omega)}{A_0^2(\omega - \Omega)} = 0. \quad (23)$$

In deriving Eq. (23), we used the relations $A_1(\omega - \Omega) = A_{-1}(\omega)$, $A_1(\omega - 2\Omega) = A_{-1}(\omega - \Omega)$, $A_2(\omega - 2\Omega) = A_{-2}(\omega)$, which follow directly from (19). Further, we will assume that the frequencies Ω and ω satisfy the following two conditions: $|\omega - \Omega| \ll qv_F$ and $qv_F/|\omega + n\Omega| \ll 1$ ($n \neq -1$). The first inequality is satisfied for a sufficiently small detuning between the electric field and 2D plasmon frequencies. The second condition corresponds to the long-wavelength approximation, which is applicable when the phase velocity ω/q of the plasmon is much greater than the electron velocity v_F . This regime is usually realized in the THz frequency range. Indeed, using the dispersion law for 2D plasmons $\omega_q = (2\alpha\varepsilon_F v_F q / \hbar)^{1/2}$, we obtain the relation $qv_F/\omega_q = \hbar\omega_q/2\alpha\varepsilon_F$, from which it follows that in the frequency range (0.1–10 THz) the long-wavelength approximation is valid for electron densities $n > 10^{11} \text{ cm}^{-2}$.

Expanding the functions $f_n(x)$ [Eq. (20)] in powers of the small parameters $|\omega - \Omega|/qv_F \ll 1$ and $qv_F/|\omega + n\Omega| \ll 1$, and substituting the results into Eq. (19), we obtain

$$A_0(\omega) \simeq \omega^2 - \omega_q^2 \left[1 - \frac{3}{16}\beta^2 \left(1 + \frac{\omega^2}{\Omega^2} \right) \right], \quad A_0(\omega - \Omega) \simeq 2(\omega - \Omega)^2 \frac{\omega_q^2}{q^2 v_F^2},$$

$$A_{-1}(\omega) \simeq -A_{-1}(\omega - \Omega) \simeq -1, \quad A_{-2}(\omega) \simeq \frac{3}{32}qv_F \left(\frac{1}{\omega} - \frac{1}{\omega - 2\Omega} - \frac{4}{\Omega} \right). \quad (24)$$

After substituting the expressions (24) into Eq. (23) we obtain the dispersion equation

$$\left\{ \omega^2 - \omega_q^2 \left[1 - \frac{3}{16}\beta^2 \left(1 + \frac{\omega^2}{\Omega^2} \right) \right] \right\} \left\{ (\omega - 2\Omega)^2 - \omega_q^2 \left[1 - \frac{3}{16}\beta^2 \left(1 + \frac{(\omega - 2\Omega)^2}{\Omega^2} \right) \right] \right\}$$

$$= \frac{1}{16}\alpha^2\beta^4\left(\frac{\varepsilon_F}{\hbar\Omega}\right)^2\omega^2(\omega - 2\Omega)^2\left(\frac{qv_F}{\Omega}\right)^2\left[1 + \frac{9}{64}\left(4 - \frac{\Omega}{\omega} + \frac{\Omega}{\omega - 2\Omega} \right)^2 \right], \quad (25)$$

which for the small detuning ($|\Omega - \omega_q| \ll \omega_q$) can be rewritten as

$$(\omega^2 - \omega_q^2)[(\omega - 2\Omega)^2 - \omega_q^2] = \frac{1}{16}\alpha^2\beta^4\left(\frac{\varepsilon_F}{\hbar\Omega}\right)^2\omega^2(\omega - 2\Omega)^2\left(\frac{qv_F}{\Omega}\right)^2\left[1 + \frac{9}{64}\left(4 - \frac{\Omega}{\omega} + \frac{\Omega}{\omega - 2\Omega} \right)^2 \right],$$

$$\omega_q^2 = \omega_q^2 \left(1 - \frac{3}{8}\beta^2 \right). \quad (26)$$

We seek the solution of the dispersion Eq. (26) in the form $\omega = \omega'_q + \delta_q$, with $|\delta_q| \ll \omega'_q$. Considering that the correction term $\delta_q \sim \beta^2$, it is sufficient to replace Ω and ω on the right-hand side of Eq. (26) with the 2D plasmon frequency ω_q , which does not take into account the influence of the high-frequency electric field. In the lowest-order approximation in the small parameters $|\delta_q|/\omega_q$ and $|\Delta|/\omega_q$ ($\Delta = \Omega - \omega'_q$ denotes the detuning) Eq. (26) reduces to

$$\delta_q^2 - 2\Delta\delta_q = -\left(\frac{5}{64}\right)^2\alpha^2\beta^4\left(\frac{\varepsilon_F}{\hbar\Omega}\right)^2(qv_F)^2$$

$$= -\left(\frac{5}{64}\right)^2\beta^4\omega_q^2. \quad (27)$$

The solution of this equation yields the following expression for the correction term,

$$\delta_q = \Delta \pm \sqrt{\Delta^2 - \left(\frac{5}{64}\right)^2\beta^4\omega_q^2}. \quad (28)$$

When the condition $\beta^2 > 64|\Delta|/(5\omega_q)$ is satisfied, the expression under the square root in Eq. (28) becomes negative, indicating the possibility of parametric instability of 2D plasma wave in graphene. At zero detuning ($\Delta = 0$) the growth rate is given by

$$\Gamma = \text{Im } \delta_q = \frac{5}{64}\beta^2\omega_q. \quad (29)$$

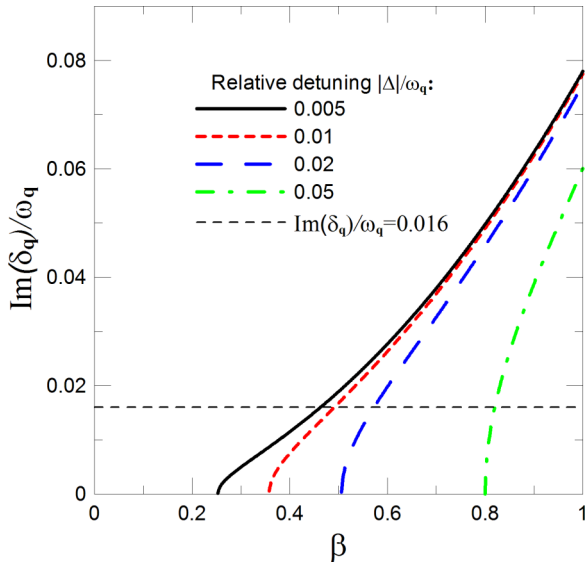


FIG. 2. The ratio of the plasmon growth rate to its frequency $[\text{Im}(\delta_q)/\omega_q]$ as a function of the parameter β for different values of the relative detuning $|\Delta|/\omega_q$. The Thin dashed line determines the collisional damping of plasmon: $\gamma_q/\omega_q = (2\tau_p\omega_q)^{-1} = 0.16$.

The parametric instability manifests, when the growth rate of the plasma wave exceeds its damping rate γ_q , i.e., when the following condition

$$\sqrt{\left(\frac{5}{64}\right)^2 \beta^4 \omega_q^2 - \Delta^2} > \gamma_q \quad (30)$$

will be satisfied. From this condition one can obtain the expression for the threshold amplitude of the high-frequency electric field:

$$E_0^{\text{th}} = \frac{8}{\sqrt{5}} \frac{\hbar \sqrt{\pi n} \Omega}{e} \left(\frac{\Delta^2 + \gamma_q^2}{\omega_q^2} \right)^{1/4} = \frac{8}{\sqrt{5}} E_c \left(\frac{\Delta^2 + \gamma_q^2}{\omega_q^2} \right)^{1/4}. \quad (31)$$

The second expression in Eq. (31) for the threshold amplitude is obtained after substitution $E_c = p_F \Omega / e = \hbar \sqrt{\pi n} \Omega / e$.

For the intraband electronic transitions considered here, the phase velocity of the 2D plasma wave exceeds the electron velocity v_F , and hence the collisionless Landau damping is absent. The contribution of electron scattering into the damping rate of plasmons can be estimated as $\gamma_q \approx 1/2\tau_p$, where τ_p is the momentum relaxation time of electrons. The dependencies of the plasmon growth rate on the high-frequency electric field amplitude and detuning for $n \simeq 10^{11} \text{ cm}^{-2}$ and $\Omega \approx \omega_q = 2\pi \times 10^{13} \text{ s}^{-1}$ (10 THz) are shown in Fig. 2. The region of the parametric instability in Fig. 2(a) is located above the dashed horizontal line showing the collisional damping of plasmons at $\tau_p \simeq 0.5 \text{ ps}$. The minimum threshold amplitude of the high-frequency electric field is achieved at zero detuning and for

the selected parameter values is approximately equal to 10 kV/cm that lower than the characteristic electric field $E_c = 23.2 \text{ kV/cm}$. At present, THz fields with such strength are quite achievable.

The parametric instability of 2D plasmons in a high-frequency electric field, considered here, arises from a third-order electrodynamic nonlinearity. This nonlinearity gives rise to a current at the combination frequency $2\Omega - \omega_q$, which acts as a driving force for the 2D plasma wave and, under the condition $\Omega \simeq \omega_q$, leads to its resonant amplification. The parametric instability can be interpreted as a direct nonlinear process in which two photons are converted into two plasmons (Fig. 1) In this scenario, the nonlinear coupling between the external electric field and the plasma wave originates from the linear spectrum of electrons in single-layer graphene.

IV. CONCLUSION

We have theoretically studied the influence of a homogeneous high-frequency electric field on the propagation and damping of long-wavelength 2D plasmons in single-layer doped graphene. Within the framework of the collisionless approximation and using the Vlasov equations, we have derived the dispersion equation that determines the plasmon spectrum and damping. Our analysis reveals that the nonlinear dependence of the electron velocity on the alternating electric field, originating from the massless Dirac electron spectrum in graphene, can give rise to a parametric instability of 2D plasmons. This instability arises due to the resonant parametric coupling of two plasmon modes and manifests itself when the frequencies of the two-dimensional plasmon and the external electric field closely coincide.

The parametric instability of plasma waves leads to an exponential increase in fluctuations in the electric potential and electron density. Therefore, the primary method for detecting plasma instabilities is probing the plasma with an electromagnetic wave. The onset of plasma instability can be detected by a sharp increase in the scattering and absorption of the probing electromagnetic wave. This method can also be used to detect the parametric instability of plasmons in graphene. A near-field electric probe can also be used to detect electric field fluctuations in graphene.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available. The data are available from the author upon reasonable request.

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