Chapter 1

Non-relativistic description of the Dirac-Kähler particle on the background of curved space-time

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In 16-component relativistic wave equation for the Dirac–Kähler particle, the procedure of the non-relativistic approximation in presence of external electromagnetic field is performed. An eight-component quantum mechanical Pauli-like equation is constructed, the wave function includes scalar, pseudoscalar, 3-vector and 3-pseudovector. The Pauli equation is invariant with respect to spatial P-reflection4 it consists of two disconnected sub-systems for scalar-pseudovector and pseudoscalar-vector respectively. In presence of only electric field, the Pauli equation reduces to more simple form of four disconnected wave equations for scalar, pseudoscalar, vector, and pseudovector. These features are substantial for physical interpretation of the Dirac–Kähler particle: it interacts in very different manner with magnetic and electric fields. This approach is generalized to a Riemannian space-time structure. Curved geometry substantially influences the structure of the nonrelativistic equation. This theory is considered in more detail on the background of the spherical Riemann space.

1.1 Intoduction

Mathematical description of the elementary particles as certain relativistically invariant objects was introduced firstly in the frames of 4-dimensional Minkowski space-time. A common view is that generalization of a wave equation on Riemannian space-time is substantially determined by what a particle is – boson or fermion. As a rule, they say that tensor equations for bosons are extended in a simpler way then spinor equations for fermions. This believing evidently correlates with the fact:

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concepts of both flat and curved space model are based on the notion of a vector. In that context, a very interesting problem is of extension a wave equation for Dirac–Kähler field (there are used other terms as well: Ivanenko–Landau field, or a vector field of a general type). Scientific literature concerned with this field is enormous, it started early in the development of quantum mechanical wave equations theory, just after the concept of a particle with spin 1/2 arises – see the review and references in [1, 2]. Three most interesting points in connection of general covariant extension of the wave equation for this field are: in flat Minkowski space there exist tensor and spinor formulations of the theory; in the initial tensor form there are presented tensors with different intrinsic parities; there exist different views about physical interpretation of the object: whether it is a composite boson or a set of four fermions. In this paper we will examine the problem of non-relativistic approximation in the theory of this field.

1.2 Pauli approximation for Dirac–Kähler field in Riemannian space-time, spinor approach

Let us start with the Dirac–Kähler equation in an arbitrary curved space-time in 4-spinor form (we adhere the designation from [1, 2]; note that $D_\alpha = \partial_\alpha + ieA_\alpha$)

$$[i\gamma^\alpha(x)(D_\alpha + \Gamma_\alpha(x) \otimes I + I \otimes \Gamma_\alpha(x)) - m]\, U = 0 \; , \tag{1.1}$$

where the generalized Dirac matrices and 4-spinor connections are defined as

$$\gamma^\alpha(x) = \gamma^a e^\alpha_{(a)}(x) \; , \quad \Gamma_\alpha(x) = \frac{1}{4}\gamma^\beta(x)\gamma_{\beta;\alpha}(x) \; . \tag{1.2}$$

As shown in [1, 2], non-relativistic approximation in relativistic wave equations is possible in space models with metric of the form $dS^2 = (dx^0)^2 + g_{ij}(x)dx^i dx^j$. Restrictions on tetrads and Christoffel symbols in those models look as

$$e_{(0)0}(x) = +1 \; , \quad e_{(0)i}(x) = 0 \; , \quad e_{(l)0}(x) = 0 \; , \quad e_{(l)i}(x)e_{(m)j}(x)\eta^{lm} = g_{ij}(x) \; ;$$
$$\Gamma^0_{00} = 0 \; , \quad \Gamma^0_{ij} = -\frac{1}{2}\, g_{ij,0} \; , \quad \Gamma^i_{00} = 0 \; , \quad \Gamma^i_{0j} = \frac{1}{2}\, g^{ik} g_{kj,0} \; , \quad \Gamma^i_{kl} = g^{im}\, \Gamma_{m,kl} \; . \tag{1.3}$$

We will need $(3+1)$-splitting in Dirac matrices and connections:

$$\gamma^0(x) = \gamma^0 \; , \quad \gamma^j(x) = \gamma^k e^j_{(k)}(x) \; , \quad \Gamma_0(x) = \frac{1}{4}\gamma^k(x)\gamma_{k;0}(x) \; ,$$
$$\Gamma_j(x) = \frac{1}{4}\gamma^0\gamma_{0;j}(x) + \frac{1}{4}\gamma^k(x)\gamma_{k;j}(x) \; . \tag{1.4}$$

In accordance with general theory, big and small constituents of the wave function, respectively U_+ and U_-, are separated with the help of two projective operators as follows

$$U_\pm = P_\pm U = \frac{I \otimes I \pm \gamma^0 \otimes I}{2} U \; , \qquad I = P_+ + P_- \; . \tag{1.5}$$

One can readily derives relations:

$$P_+[\gamma^0 \otimes I] = +P_+ \; , \quad P_-[\gamma^0 \otimes I] = -P_- \; ,$$
$$P_+[\gamma^j(x) \otimes I] = [\gamma^j(x) \otimes I]P_- \; , \quad P_-[\gamma^j(x) \otimes I] = [\gamma^j(x) \otimes I]P_+ \; ,$$
$$P_\pm[\Gamma_0(x) \otimes I] = [\Gamma_0(x) \otimes I]P_\pm \; , \quad P_\pm[I \otimes \Gamma_0(x)] = [I \otimes \Gamma_0(x)]P_\pm \; ,$$
$$P_\pm[\Gamma_j(x) \otimes I] = [\Gamma_j(x) \otimes I]P_\pm \; , \quad P_\pm[I \otimes \Gamma_j(x)] = [I \otimes \Gamma_j(x)]P_\pm \; ; \tag{1.6}$$

with the help of which , acting on the main equation (1.1) written in the form

$$\begin{aligned} &\big\{ i\gamma^0 \otimes I \, [(I \otimes I) D_0 + \Gamma_0(x) \otimes I + I \otimes \Gamma_0(x)] \\ &+ i\gamma^j(x) \otimes I \, [(I \otimes I) D_j + \Gamma_j(x) \otimes I + I \otimes \Gamma_j(x)] - m(I \otimes I) \big\} \, U = 0 \, , \end{aligned} \tag{1.7}$$

by operators P_+ and P_-, we obtain

$$\begin{aligned} &i \, [D_0 + \Gamma_0 \otimes I + I \otimes \Gamma_0] \, U_+ + i\gamma^j(x) \, [D_j + \Gamma_j \otimes I + I \otimes \Gamma_j] \, U_- - mU_+ = 0 \, , \\ &-i \, [D_0 + \Gamma_0 \otimes I + I \otimes \Gamma_0] \, U_- + i\gamma^j(x) \, [D_j + \Gamma_j \otimes I + I \otimes \Gamma_j] \, U_+ - mU_- = 0 \, . \end{aligned} \tag{1.8}$$

Now , separate the rest energy by the formal change $D_0 \implies (D_0 - im)$:

$$i \, [D_0 + \Gamma_0 \otimes I + I \otimes \Gamma_0] \, U_+ + i\gamma^j(x) \, [D_j + \Gamma_j \otimes I + I \otimes \Gamma_j] \, U_- = 0 \, , \tag{1.9}$$

$$-i \, [D_0 + \Gamma_0 \otimes I + I \otimes \Gamma_0] \, U_- + i\gamma^j(x) \, [D_j + \Gamma_j \otimes I + I \otimes \Gamma_j] \, U_+ - 2mU_- = 0 \, . \tag{1.10}$$

Changing eq. (1.10) by its approximate form

$$U_- \approx \frac{1}{2m} \left\{ i\gamma^j(x) \, [D_j + \Gamma_j \otimes I + I \otimes \Gamma_j] \right\} U_+ \, ,$$

from (1.9) we derive a needed Pauli-like equation for the non-relativistic wave function U_+

$$\begin{aligned} &i \, (D_0 + \Gamma_0 \otimes I + I \otimes \Gamma_0) \, U_+ \\ &= \frac{1}{2m} \left[\gamma^j(x) \, (D_j + \Gamma_j \otimes I + I \otimes \Gamma_j) \right] \Big[\gamma^k(x) \, (D_k + \Gamma_k \otimes I + I \otimes \Gamma_k) \Big] \, U_+ \, . \end{aligned} \tag{1.11}$$

In a sense, the derived equation is only formal, because the non-relativistic function has only 8 independent components. Indeed, in Weyl 2-spinor basis, the wave function U_+ reads explicitly as

$$U_+ = \frac{1 + \gamma^0}{2} U = \frac{1}{2} \left| \begin{array}{cc} I & I \\ I & I \end{array} \right| \left| \begin{array}{cc} \xi & \Delta \\ H & \eta \end{array} \right| = \frac{1}{2} \left| \begin{array}{cc} \xi + H & \Delta + \eta \\ \xi + H & \Delta + \eta \end{array} \right| \, . \tag{1.12}$$

1.3 2-spinor Weyl approach

One can reach more clarity when using the 2-spinor formalism. The relativistic Dirac-Kähler wave function consists of four spinor blocks

$$U(x) = \left| \begin{array}{cc} \xi^{\alpha\beta}(x) & \Delta^{\alpha}{}_{\dot{\beta}}(x) \\ H_{\dot{\alpha}}{}^{\beta}(x) & \eta_{\dot{\alpha}\dot{\beta}}(x) \end{array} \right| \, , \tag{1.13}$$

these spinors are parameterized by tetrad tensor according to the rules (ϵ, $\dot{\epsilon}$ stand for spinor metrical matrices – see notation in [1]

$$\begin{aligned} &\Delta = (\Psi_l + i\tilde{\Psi}_l) \, \bar{\sigma}^l \dot{\epsilon} \, , \quad H = (\Psi_l - i\tilde{\Psi}_l) \sigma^l \epsilon^{-1} , \\ &\xi = (-i\Psi - \tilde{\Psi} + i\Sigma^{mn} \Psi_{mn}) \, \epsilon^{-1} \, , \quad \eta = (-i\Psi + \tilde{\Psi} + i\bar{\Sigma}^{mn} \Psi_{mn}) \dot{\epsilon} \, . \end{aligned} \tag{1.14}$$

In 2-dimensional spinor form, the matrices and connections look as

$$
\gamma^{\alpha}(x) = \left(\begin{array}{cc} 0 & \bar{\sigma}^{\alpha}(x) \\ \sigma^{\alpha}(x) & 0 \end{array} \right), \qquad \Gamma_{\alpha} = \left(\begin{array}{cc} \Sigma_{\alpha}(x) & 0 \\ 0 & \bar{\Sigma}_{\alpha}(x) \end{array} \right),
$$
$$
\sigma^{\alpha}(x) = e^{\alpha}_{(0)} + e^{\alpha}_{(j)} \sigma^{j} \; , \qquad \bar{\sigma}^{\alpha}(x) = e^{\alpha}_{(0)} - e^{\alpha}_{(j)} \sigma^{j} \; ,
$$
$$
\Sigma_{\alpha}(x) = \frac{1}{2}\, \Sigma^{ab}\, e^{\beta}_{(a)}\, \nabla_{\alpha}(e_{(b)\beta}) \; , \qquad \bar{\Sigma}_{\alpha}(x) = \frac{1}{2}\, \bar{\Sigma}^{ab}\, e^{\beta}_{(x)}\, \nabla_{\alpha}(e_{(b)\beta}) \; ,
$$
$$
\Sigma^{ab} = \frac{1}{4}\, (\; \bar{\sigma}^{a}\, \sigma^{b} \;-\; \bar{\sigma}^{b}\, \sigma^{a}\;) \; , \;\; \bar{\Sigma}^{ab} = \frac{1}{4}\, (\; \sigma^{a}\, \bar{\sigma}^{b} \;-\; \sigma^{b}\, \bar{\sigma}^{a}\;) \; ,
$$
$$
\Sigma^{0j} = \frac{1}{2}\sigma^{j}, \quad \bar{\Sigma}^{0j} = -\frac{1}{2}\sigma^{j}, \quad \Sigma^{kl} = \bar{\Sigma}^{kl} = -\frac{i}{2}\epsilon_{klj}\sigma^{j} \; , \tag{1.15}
$$

and the Dirac-Kähler equation in 2-spinor form reads (we perform $(3+1)$-splitting)

$$
i\bar{\sigma}^{0}(x)\; [\; D_{0} \;+\; \bar{\Sigma}_{0}(x) \otimes I + I \otimes \Sigma_{0}(x)\;]\; H(x)
$$
$$
+ i\bar{\sigma}^{n}(x)\; [\; D_{n} \;+\; \bar{\Sigma}_{n}(x) \otimes I + I \otimes \Sigma_{n}(x)\;]\; H(x) = m\; \xi(x) \; ,
$$
$$
i\sigma^{0}(x)\; [\; D_{0} \;+\; \Sigma_{0}(x) \otimes I + I \otimes \Sigma_{0}(x)\;]\; \xi(x)
$$
$$
+ i\sigma^{n}(x)\; [\; D_{n} \;+\; \Sigma_{n}(x) \otimes I + I \otimes \Sigma_{n}(x)\;]\; \xi(x) = m\; H(x) \; ;
$$
$$
i\sigma^{0}(x)\; [\; D_{0} \;+\; \Sigma_{0}(x) \otimes I + I \otimes \bar{\Sigma}_{0}(x)\;]\; \Delta(x)
$$
$$
+ i\sigma^{n}(x)\; [\; D_{n} \;+\; \Sigma_{n}(x) \otimes I + I \otimes \bar{\Sigma}_{n}(x)\;]\; \Delta(x) = m\; \eta(x) \; ,
$$
$$
i\bar{\sigma}^{0}(x)\; [\; D_{0} \;+\; \bar{\Sigma}_{0}(x) \otimes I + I \otimes \bar{\Sigma}_{0}(x)\;]\; \eta(x)
$$
$$
+ i\bar{\sigma}^{n}(x)\; [\; D_{n} \;+\; \bar{\Sigma}_{n}(x) \otimes I + I \otimes \bar{\Sigma}_{n}(x)\;]\; \eta(x) = m\; \Delta(x) \; .
$$

Taking into account structure of the matrices and connections for models specified by (1.3):

$$
\sigma^{0}(x) = \bar{\sigma}^{0}(x) = 1, \quad \sigma^{k}(x) = +e^{k}_{(j)}\sigma^{j} \; , \quad \bar{\sigma}^{k}(x) = -e^{k}_{(j)}\sigma^{j} \; ,
$$
$$
\Sigma_{0}(x) = \bar{\Sigma}_{0}(x) = \frac{1}{2}\sigma^{j}(-\frac{i}{2}\; \epsilon_{jkl}\; e^{n}_{(k)}\; e_{(l)n;0}) \; ,
$$
$$
\Sigma_{n}(x) = \frac{1}{2}\sigma^{j}(+e_{(j)0;n} - \frac{i}{2}\; \epsilon_{jkl}\; e^{m}_{(k)}\; e_{(l)m;n}) = +K_{n} + L_{n} \; ,
$$
$$
\bar{\Sigma}_{n}(x) = \frac{1}{2}\sigma^{j}\; (-e_{(j)0;n} - \frac{i}{2}\; \epsilon_{jkl}\; e^{m}_{(k)}\; e_{(l)m;n}) = -K_{n} + L_{n} \; , \tag{1.16}
$$

the wave equation can be presented as follows

$$
i[\; D_{0} \;+\; \Sigma_{0} \otimes I + I \otimes \Sigma_{0}\;]\; H
$$
$$
- i\sigma^{n}(x)\; [\; D_{n} \;+\; (-K_{n} + L_{n}) \otimes I + I \otimes (K_{n} + L_{n})\;]\; H = m\; \xi \; ,
$$
$$
i[\; D_{0} \;+\; \Sigma_{0} \otimes I + I \otimes \Sigma_{0}\;]\; \xi
$$
$$
+ i\sigma^{n}(x)\; [\; D_{n} \;+ (K_{n} + L_{n}) \otimes I + I \otimes (K_{n} + L_{n})\;]\; \xi = m\; H \; ,
$$
$$
i[\; D_{0} \;+\; \Sigma_{0} \otimes I + I \otimes \Sigma_{0}\;]\; \Delta(x)
$$
$$
+ i\sigma^{n}(x)\; [\; D_{n} + (K_{n} + L_{n}) \otimes I + I \otimes (-K_{n} + L_{n})\;]\; \Delta = m\; \eta \; ,
$$
$$
i[\; D_{0} \;+\; \Sigma_{0} \otimes I + I \otimes \Sigma_{0}\;]\; \eta
$$
$$
- i\sigma^{n}\; [\; D_{n} + (-K_{n} + L_{n}) \otimes I + I \otimes (-K_{n} + L_{n})\;]\; \eta = m\; \Delta \; . \tag{1.17}
$$

Translating eqs. (1.17) to big B_1, B_2 and small M_1, M_2 components defined according to the rules

$$\frac{1}{2}(\xi + H) = B_1 \; , \; \frac{1}{2}(\xi - H) = M_1 \; , \quad \frac{1}{2}(\eta + \Delta) = B_2 \; , \; \frac{1}{2}(\eta - \Delta) = M_2 \; , \tag{1.18}$$

and separating the rest energy by the formal change $D_0 \Longrightarrow (D_0 - im)$, we arrive at

$$\begin{gathered}
i[\, D_0 + \Sigma_0 \otimes I + I \otimes \Sigma_0 \,]\, (B_1 - M_1) \\
-i\sigma^n(x)\, [\, D_n \; + \; (-K_n + L_n) \otimes I + I \otimes (K_n + L_n) \,]\, (B_1 - M_1) = 2m\, M_1 \; , \\
i[\, D_0 + \Sigma_0 \otimes I + I \otimes \Sigma_0 \,]\, (B_1 + M_1) \\
+i\sigma^n(x)\, [\, D_n \; + (K_n + L_n) \otimes I + I \otimes (K_n + L_n) \,]\, (B_1 + M_1) = -2m\, M_1 \; ;
\end{gathered}$$

$$\begin{gathered}
i[\, D_0 + \Sigma_0 \otimes I + I \otimes \Sigma_0 \,]\, (B_2 - M_2) \\
+i\sigma^n(x)\, [\, D_n + (K_n + L_n) \otimes I + I \otimes (-K_n + L_n) \,]\, (B_2 - M_2) = 2m\, M_2 \; , \\
i[\, D_0 + \Sigma_0 \otimes I + I \otimes \Sigma_0 \,]\, (B_2 + M_2) \\
-i\sigma^n \; [\, D_n + (-K_n + L_n) \otimes I + I \otimes (-K_n + L_n) \,]\, (B_2 + M_2) = -2m\, M_2 \; .
\end{gathered}$$

Summing and subtracting equations in each pair, we derive

$$\begin{aligned}
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_1 \\
+i\sigma^n(x)[\, D_n + L_n \otimes I + I \otimes (K_n + L_n)]\; M_1 = 0 \; , \\
-i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; M_1 \\
-i\sigma^n(x)[\, D_n + L_n \otimes I + I \otimes (K_n + L_n)]\; B_1 = 2m\, M_1 \; ; \\
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_2 \\
-i\sigma^n(x)\, [D_n + L_n \otimes I + I \otimes (-K_n + L_n)]\;\; M_2 = 0 \; , \\
-i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; M_2 \\
+i\sigma^n(x)\, [D_n + L_n \otimes I + I \otimes (-K_n + L_n)]\;\; B_2 = 2m\, M_2 \; .
\end{aligned}$$

Then neglecting by small terms in comparison with big ones, we get

$$\begin{aligned}
-i \frac{1}{2m} \sigma^n(x)[\, D_n + L_n \otimes I + I \otimes (K_n + L_n)]\; B_1 = M_1 \; , \\
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_1 \\
+i\sigma^n(x)[\, D_n + L_n \otimes I + I \otimes (K_n + L_n)]\; M_1 = 0 \, ; \\
+i \frac{1}{2m} \sigma^n(x)\, [D_n + L_n \otimes I + I \otimes (-K_n + L_n)]\;\; B_2 = M_2 \; , \\
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_2 \\
-i\sigma^n(x)\, [D_n + L_n \otimes I + I \otimes (-K_n + L_n)]\;\; M_2 = 0 \; .
\end{aligned}$$

Excluding small components, we obtain two equations where only the big components enter

$$\begin{aligned}
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_1 \\
= -\frac{1}{2m}\, \{\sigma^n(x)[\, D_n + L_n \otimes I + I \otimes L_n + I \otimes K_n]\}^2\, B_1 \; , \\
i\, [D_0 + \sigma^n(x) K_n + \Sigma_0 \otimes I + I \otimes \Sigma_0]\;\; B_2 \\
= -\frac{1}{2m}\, \{\sigma^n(x)\, [D_n + L_n \otimes I + I \otimes L_n - I \otimes K_n]\}^2\;\; B_2 \; .
\end{aligned}$$

These are the system of Dirac–Kähler equations in Pauli approximation. Expressions for spinors B_1 are B_2 van be presented in terms of tetrad tensors as follows (we use expressions for more simple linear combinations)

$$B_1 + B_2 = \Big\{ -(\tilde{\Psi} + i\tilde{\Psi}_0) + \sigma^1(\Psi_1 + i\Psi_{01}) + \sigma^2(\Psi_2 + i\Psi_{02}) + \sigma^3(\Psi_3 + i\Psi_{03}) \Big\} (-i\sigma^2) \, ,$$

$$B_1 - B_2 = i \Big\{ -(\Psi + i\Psi_0) - \sigma^1(\tilde{\Psi}_1 + i\Psi_{23}) - \sigma^2(\tilde{\Psi}_2 + i\Psi_{32}) - \sigma^3(\tilde{\Psi}_3 + i\Psi_{12}) \Big\} (-i\sigma^2) \, .$$

Using the following notation

$$\Phi = \Psi + i\Psi_0 \ , \ \tilde{\Phi} = \tilde{\Psi} + i\tilde{\Psi}_0 \ ,$$

$$\Phi_j = \Psi_j + iE_j \ , \ \tilde{\Phi}_j = \tilde{\Psi}_j + iH_h, \quad (E_j = \Psi_{0j} \ , \ H_i = \frac{1}{2}\epsilon_{ijk}\Psi_{jk}). \tag{1.19}$$

the nonrelativistic wave functions van be written as

$$\Psi_+ = B_1 + B_2 = \Big\{ -\tilde{\Phi} + \sigma^j \Phi_j \Big\} (-i\sigma^2) \, ,$$

$$\Psi_- = B_1 - B_2 = i \Big\{ -\Phi - \sigma^j \tilde{\Phi}_j \Big\} (-i\sigma^2) \, . \tag{1.20}$$

1.4 Pauli equation for Dirac-Kähler particle, Minkowski space

Let us examine the most simple case of the flat Minkowski space. The couple of Pauli equations reads

$$iD_0 \, B_1 = -\frac{1}{2m}(\sigma^n D_n)^2 B_1 \, , \qquad iD_0 B_2 = -\frac{1}{2m}(\sigma^n D_n)^2 \, B_2 \, . \tag{1.21}$$

Summing and subtracting them, we get yet another representation for the system

$$iD_0 \Psi_+ = -\frac{1}{2m}(\sigma^n D_n)^2 \Psi_+ \, , \qquad iD_0 \Psi_- = -\frac{1}{2m}(\sigma^n D_n)^2 \, \Psi_- \, . \tag{1.22}$$

In contrast to spinors B_1, B_2 from (1.21), spinors Ψ_+ and Ψ_- are equivalent to tensor sets which are independent representations of the 3-rotation group:

$$\Psi_+(x) \quad <--> \quad \tilde{\Phi}(x), \ \Phi_j(x) \ , \quad \Psi_-(x) \quad <--> \quad \Phi(x), \ \tilde{\Phi}_j(x) \ .$$

Let us specify the operator

$$(\sigma^j D_j)(\sigma^k D_k) = D_j D_j - ie\sigma^l \frac{1}{2}\epsilon_{ljk} F_{jk} = D_j D_j - ie\sigma^l B_l \ ,$$

where $B_l = \frac{1}{2}\epsilon_{ljk}F_{jk}$. Therefore. eqs. (1.22) take the form

$$i(\partial_0 + ieA_0)\Psi_+ = -\frac{1}{2m}(\vec{D}^{\,2} - ie\sigma^l B_l)\Psi_+ \ ,$$

$$i(\partial_0 + ieA_0)\Psi_- = -\frac{1}{2m}(\vec{D}^{\,2} - ie\sigma^l B_l) \ \Psi_- \ . \tag{1.23}$$

From whence, allowing for (1.20), we get

$$i(\partial_0 + ieA_0)(\tilde{\Phi} - \sigma^k \Phi_k)$$

$$= -\frac{1}{2m}\left[\vec{D}^{\,2}\tilde{\Phi} + ieB_k\Phi_k - \left(\vec{D}^{\,2}\Phi_k + e\epsilon_{klj}B_l\Phi_j + ieB_k\tilde{\Phi}\right)\sigma^k\right] ,$$

$$i(\partial_0 + ieA_0)(\Phi + \sigma^k\tilde{\Phi}_k) =$$

$$-\frac{1}{2m}\left[\vec{D}^{\,2}\Phi - ieB_k\tilde{\Phi}_k - \left(-\vec{D}^{\,2}\tilde{\Phi}_k - e\epsilon_{klj}B_l\tilde{\Phi}_j + ieB_k\Phi\right)\sigma^k\right] .$$

Further, taking into account linear independence of the Pauli matrices, we arrive at tensor equations:

$$i(\partial_0 + ieA_0)\tilde{\Phi} = -\frac{1}{2m}\left(\vec{D}^{\,2}\tilde{\Phi} + ieB_k\Phi_k\right) ,$$

$$i(\partial_0 + ieA_0)\Phi_k = -\frac{1}{2m}\left(\vec{D}^{\,2}\Phi_k + e\epsilon_{klj}B_l\Phi_j + ieB_k\tilde{\Phi}\right) ;$$

$$i(\partial_0 + ieA_0)\Phi = -\frac{1}{2m}\left(\vec{D}^{\,2}\Phi - ieB_k\tilde{\Phi}_k\right) ,$$

$$i(\partial_0 + ieA_0)\tilde{\Phi}_k = -\frac{1}{2m}\left(\vec{D}^{\,2}\tilde{\Phi}_k + e\epsilon_{klj}B_l\tilde{\Phi}_j - ieB_k\Phi\right) . \qquad (1.24)$$

The Pauli equation consists of two disconnected sub-systems for scalar-pseudovector and pseudoscalar-vector respectively. In presence of only electric field, the Pauli equation becomes much more simple: it reduces to four disconnected wave equations for scalar, pseudoscalar, vector, and pseudovector. These features are substantial for physical interpretation of the Dirac–Kähler particle: it interacts in very different manner with magnetic and electric fields.

1.5 Spherical waves in Minkowski space, relativistic case

Let us start with the relativistic Dirac–Kähler equation, written in spherical tetrad in Minkowski space [2]

$$\left[i\gamma^0\,\partial_t + i\,(\gamma^3\,\partial_r + \frac{\gamma^1 J^{31} + \gamma^2 J^{32}}{r}) + \frac{1}{r}\,\Sigma_{\theta,\phi} - m \right] U(x) = 0 , \qquad (1.25)$$

$$\Sigma_{\theta,\phi} = i\gamma^1\,\partial_\theta + \gamma^2\,\frac{i\partial_\phi + iJ^{12}\,\cos\theta}{\sin\theta} , \qquad J^{12} = (\sigma^{12}\otimes I + I\otimes\sigma^{12}) . \qquad (1.26)$$

Diagonalizing operators of the total angular momentum

$$J_1 = l_1 + \frac{iJ^{12}\cos\phi}{\sin\theta} , \quad J_2 = l_2 + \frac{iJ^{12}\sin\phi}{\sin\theta} , \quad J_3 = l_3 , \qquad (1.27)$$

for wave function we have substitution

$$U_{\epsilon JM}(t,r,\theta,\phi) = \frac{e^{-i\epsilon t}}{r}\left|\begin{array}{llll} f_{11}\,D_{-1} & f_{12}\,D_0 & f_{13}\,D_{-1} & f_{14}\,D_0 \\ f_{21}\,D_0 & f_{22}\,D_{+1} & f_{23}\,D_0 & f_{24}\,D_{+1} \\ f_{31}\,D_{-1} & f_{32}\,D_0 & f_{33}\,D_{-1} & f_{34}\,D_0 \\ f_{41}\,D_0 & f_{42}\,D_{+1} & f_{43}\,D_0 & f_{44}\,D_{+1} \end{array}\right| , \qquad (1.28)$$

where $f_{ab} = f_{ab}(r)$, $D_\sigma = D^j_{-m,\sigma}(\phi,\theta,0)$ stands for Wigner functions, j takes on the values $0,\ 1,\ 2,\ldots$ In order to simplify the task, let us diagonalize additionally the space reflection operator.

In the spherical tetrad basis it has the form

$$\hat{\Pi} = \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| \otimes \left| \begin{array}{cccc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right| \otimes \hat{P} \; . \tag{1.29}$$

Eigenvalue equation $\hat{\Pi} U_{\epsilon jm} = \Pi \; U_{\epsilon jm}$ provides us with the followings restrictions:

$$\begin{array}{c} f_{31} = \pm f_{24} \;\; , \;\; f_{32} = \pm f_{23} \;\; , \;\; f_{33} = \pm f_{22} \;\; , \;\; f_{34} = \pm f_{21} \;\; , \\ f_{41} = \pm f_{14} \;\; , \;\; f_{42} = \pm f_{13} \;\; , \;\; f_{43} = \pm f_{12} \;\; , \;\; f_{44} = \pm f_{11} \;\; ; \end{array} \tag{1.30}$$

upper sign refers to eigenvalue $\Pi = (-1)^{j+1}$, lower sign refers to $\Pi = (-1)^{j}$. For brevity, below we use the following convention the case $\Pi = (-1)^{j}$ relates to $\delta = +1$, and $\Pi = (-1)^{j+1}$ relates to $\delta = -1$. Thus, the main substitution takes the form

$$U_{\epsilon jm\delta}(t,r,\theta,\phi) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{cccc} f_{11}\, D_{-1} & f_{12}\, D_{0} & f_{13}\, D_{-1} & f_{14}\, D_{0} \\ f_{21}\, D_{0} & f_{22}\, D_{+1} & f_{23}\, D_{0} & f_{24}\, D_{+1} \\ \delta f_{24}\, D_{-1} & \delta f_{23}\, D_{0} & \delta f_{22}\, D_{-1} & \delta f_{21}\, D_{0} \\ \delta f_{14}\, D_{0} & \delta f_{13}\, D_{+1} & \delta f_{12}\, D_{0} & \delta f_{11}\, D_{+1} \end{array} \right| . \tag{1.31}$$

The system of radial equations for the case $\delta = +1$ reads

$$\begin{array}{ll} \epsilon f_{24} - i\frac{d}{dr} f_{24} + \frac{i}{r} 0 - \frac{ia}{r} f_{14} - m f_{11} = 0 \; , & \epsilon f_{23} - i\frac{d}{dr} f_{23} - \frac{i}{r} f_{14} - \frac{ia}{r} f_{13} - m f_{12} = 0 \; , \\ \epsilon f_{14} + i\frac{d}{dr} f_{14} + \frac{i}{r} f_{23} + \frac{ia}{r} f_{24} - m f_{21} = 0 \; , & \epsilon f_{13} + i\frac{d}{dr} f_{13} + \frac{i}{r} 0 + \frac{ia}{r} f_{23} - m f_{22} = 0 \; , \\ \epsilon f_{22} - i\frac{d}{dr} f_{22} + \frac{i}{r} 0 - \frac{ia}{r} f_{12} - m f_{13} = 0 \; , & \epsilon f_{21} - i\frac{d}{dr}{}_{21} - \frac{i}{r} f_{12} - \frac{ia}{r} f_{11} - m f_{14} = 0 \; , \\ \epsilon f_{12} + i\frac{d}{dr} f_{12} + \frac{i}{r} f_{21} + \frac{ia}{r} f_{22} - m f_{23} = 0 \; , & \epsilon f_{11} + i\frac{d}{dr} f_{11} + \frac{i}{r} 0 + \frac{ia}{r} f_{21} - m f_{24} = 0 \; . \end{array}$$

To get equation for the case $\delta = -1$ it is enough to make one formal change: m into $-m$.

In order to obtain from equations (1.32) the system without complex coefficient, let us introduce new functions

$$\begin{array}{llll} A = (f_{11} + f_{22}) \; , & iB = (f_{11} - f_{22}) \; , & C = (f_{12} + f_{21}) \; , & iD = (f_{12} - f_{21}) \; , \\ K = (f_{13} + f_{24}) \; , & iL = (f_{13} - f_{24}) \; , & M = (f_{14} + f_{23}) \; , & iN = (f_{14} - f_{23}) \; . \end{array}$$

Thus, we get new system

$$\begin{array}{ll} \epsilon K - \frac{dL}{dr} + \frac{a}{r} N - mA = 0 \; , & \epsilon A - \frac{dB}{dr} + \frac{a}{r} D - mK = 0 \; , \\ \epsilon L + \frac{dK}{dr} + \frac{a}{r} M + mB = 0 \; , & \epsilon B + \frac{dA}{dr} + \frac{a}{r} C + mL = 0 \; , \\ \epsilon M - \frac{dN}{dr} + \frac{1}{r} N + \frac{a}{r} L - mC = 0 \; , & \epsilon C - \frac{dD}{dr} + \frac{1}{r} D + \frac{a}{r} B - mM = 0 \; , \\ \epsilon N + \frac{dM}{dr} + \frac{1}{r} M + \frac{a}{r} K + mD = 0 \; , & \epsilon D + \frac{dC}{dr} + \frac{1}{r} C + \frac{a}{r} A + mN = 0 \; . \end{array} \tag{1.32}$$

Eqs. (1.32) permit the following linear constraints ($\lambda = \pm 1$):

$$A = \lambda\, K \;, \quad B = \lambda\, L \;, \quad C = \lambda\, M \;, \quad D = \lambda\, N \quad . \tag{1.33}$$

For the case $\lambda = +1$, instead of (1.32) we derive four equations

$$\frac{dK}{dr} + \frac{a}{r} M + (\epsilon + m) L = 0 \;, \quad \frac{dL}{dr} - \frac{a}{r} N - (\epsilon - m) K = 0 \;,$$
$$(\frac{d}{dr} + \frac{1}{r}) M + \frac{a}{r} K + (\epsilon + m) N = 0 \;, \quad (\frac{d}{dr} - \frac{1}{r}) N - \frac{a}{r} L - (\epsilon - m) M = 0 \;. \tag{1.34}$$

If one change m into $-m$, one gets equations for the case $\lambda = -1$. Eqs. (1.34) can be solved trough the use of two different substitutions:

$$I. \;\; \sqrt{j+1}\, K(r) = f(r) \;\;, \; \sqrt{j+1}\;\; L(r) = g(r) \;\;,$$
$$\sqrt{j}\; M(r) = f(r) \;, \; \sqrt{j}\; N(r) = g(r) \;,$$

$$(\frac{d}{dr} + \frac{j+1}{r}) f + (\epsilon + m) g = 0 \;, \qquad (\frac{d}{dr} - \frac{j+1}{r}) g - (\epsilon - m) f = 0 \;; \tag{1.35}$$

$$II. \;\; \sqrt{j}\; K(r) = f(r) \;\;, \; \sqrt{j}\; L(r) = g(r) \;,$$
$$\sqrt{j+1}\; M(r) = -f(r) \;, \; \sqrt{j+1}\; N(r) = -g(r) \;,$$

$$(\frac{d}{dr} - \frac{j}{r}) f + (\epsilon + m) g = 0 \;, \qquad (\frac{d}{dr} + \frac{j}{r}) g - (\epsilon - m) f = 0 \;. \tag{1.36}$$

The case with $j = 0$ needs separate treatment. Here, initial substitution is simpler

$$U_{\epsilon 00}(t,r) = \frac{e^{-i\epsilon t}}{r} \left| \begin{array}{cccc} 0 & f_{12} & 0 & f_{14} \\ f_{21} & 0 & f_{23} & 0 \\ 0 & f_{32} & 0 & f_{34} \\ f_{41} & 0 & f_{43} & 0 \end{array} \right| \;. \tag{1.37}$$

Space reflection operator divides solutions into two classes:

$\Pi = +1 \;\; (\delta = +1)$

$$f_{32} = +f_{23} \;, \quad f_{34} = +f_{21} \;, \quad f_{41} = +f_{14} \;, \quad f_{43} = +f_{12} \quad ; \tag{1.38}$$

$\Pi = -1 \;\; (\delta = -1)$

$$(\delta = -1) \;\; : \;\; f_{32} = -f_{23} \;, \quad f_{34} = -f_{21} \;, \quad f_{41} = -f_{14} \;, \quad f_{43} = -f_{12} \quad . \tag{1.39}$$

The angular operators acts as follows: $\Sigma_{\theta,\phi}\, U_{\epsilon 00} = 0$. We get the radial system (the above functions $A,\; B,\; K,\; L$ vanish identically)

$$\epsilon\, M - \frac{dN}{dr} + \frac{N}{r} - m\, C = 0 \;, \quad \epsilon\, N + \frac{dM}{dr} + \frac{M}{r} + m\, D = 0 \;,$$
$$\epsilon\, C - \frac{dD}{dr} + \frac{D}{r} - m\, M = 0 \;, \quad \epsilon\, D + \frac{dC}{dr} + \frac{C}{r} - m\, N = 0 \;. \tag{1.40}$$

This system is solved through the use of two different substitutions:

$C = +M \ , \ D = +N \ (\lambda = +1)$

$$(\frac{d}{dr} + \frac{1}{r})M + (\epsilon + m)N = 0 \ , \qquad (\frac{d}{dr} - \frac{1}{r})N - (\epsilon - m)M = 0 \ ; \tag{1.41}$$

$C = -M, \ D = -N \ (\lambda = -1)$

$$(\frac{d}{dr} + \frac{1}{r})M + (\epsilon - m)N = 0 \ , \qquad (\frac{d}{dr} - \frac{1}{r})N - (\epsilon + m)M = 0 \ . \tag{1.42}$$

Note that the above analysis preserves its applicability in presence of any external spherically symmetrical potential; this requires only one formal change $\epsilon \implies \epsilon - U(r)$.

1.6 Spherical solution in nonrelativistic approximation

In spherically symmetric case, big and small components are (we separate states with opposite parity)

$\delta = +1$:

$$B_1 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{11} + f_{24}) \ D_{-1} & (f_{12} + f_{23}) \ D_{\ 0} \\ (f_{21} + f_{14}) \ D_{\ 0} & (f_{22} + f_{13}) \ D_{+1} \end{array} \right),$$
$$B_2 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{22} + f_{13}) D_{-1} & (f_{21} + f_{14}) D_{\ 0} \\ (f_{12} + f_{23}) D_{\ 0} & (f_{11} + f_{24}) D_{+1} \end{array} \right),$$
$$M_1 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{11} - f_{24}) \ D_{-1} & (f_{12} - f_{23}) \ D_{\ 0} \\ (f_{21} - f_{14}) \ D_{\ 0} & (f_{22} - f_{13}) \ D_{+1} \end{array} \right),$$
$$M_2 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{22} - f_{13}) D_{-1} & (f_{21} - f_{14}) D_{\ 0} \\ (f_{12} - f_{23}) D_{\ 0} & (f_{11} - f_{24}) D_{+1} \end{array} \right); \tag{1.43}$$

$\delta = -1$:

$$B_1 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{11} - f_{24}) \ D_{-1} & (f_{12} - f_{23}) \ D_{\ 0} \\ (f_{21} - f_{14}) \ D_{\ 0} & (f_{22} - f_{13}) \ D_{+1} \end{array} \right),$$
$$B_2 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (-f_{22} + f_{13}) D_{-1} & (-f_{21} + f_{14}) D_{\ 0} \\ (-f_{12} + f_{23}) D_{\ 0} & (-f_{11} + f_{24}) D_{+1} \end{array} \right),$$
$$M_1 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (f_{11} + f_{24}) \ D_{-1} & (f_{12} + f_{23}) \ D_{\ 0} \\ (f_{21} + f_{14}) \ D_{\ 0} & (f_{22} + f_{13}) \ D_{+1} \end{array} \right),$$
$$M_2 = \frac{e^{-i\epsilon t}}{r} \left(\begin{array}{cc} (-f_{22} - f_{13}) D_{-1} & (-f_{21} - f_{14}) D_{\ 0} \\ (-f_{12} - f_{23}) D_{\ 0} & (-f_{11} - f_{24}) D_{+1} \end{array} \right). \tag{1.44}$$

It should be noted that for both δ, big spinors B_1, B_2 contains only four different radial function (as we might expect in advance); similarly behave small spinors M_1, M_2.

Let us examine in detail the case $\delta = +1$. It is convenient to group radial equations in pairs:

$$\epsilon f_{24} - i\frac{d}{dr}f_{24} - \frac{ia}{r}f_{14} - mf_{11} = 0 \ , \quad \epsilon f_{11} + i\frac{d}{dr}f_{11} + \frac{ia}{r}f_{21} - mf_{24} = 0 \ ;$$
$$\epsilon f_{13} + i\frac{d}{dr}f_{13} + \frac{ia}{r}f_{23} - mf_{22} = 0 \ , \quad \epsilon f_{22} - i\frac{d}{dr}f_{22} - \frac{ia}{r}f_{12} - mf_{13} = 0 \ ;$$
$$\epsilon f_{23} - i\frac{d}{dr}f_{23} - \frac{i}{r}f_{14} - \frac{ia}{r}f_{13} - mf_{12} = 0 \ , \quad \epsilon f_{12} + i\frac{d}{dr}f_{12} + \frac{i}{r}f_{21} + \frac{ia}{r}f_{22} - mf_{23} = 0 \ ;$$
$$\epsilon f_{14} + i\frac{d}{dr}f_{14} + \frac{i}{r}f_{23} + \frac{ia}{r}f_{24} - mf_{21} = 0 \ , \quad \epsilon f_{21} - i\frac{d}{dr}f_{21} - \frac{i}{r}f_{12} - \frac{ia}{r}f_{11} - mf_{14} = 0 \ .$$

Combining equations within pairs and introducing the following functions

$$F_+ = f_{11} + f_{24} \ , \quad G_+ = f_{22} + f_{13} \ , \quad K_+ = f_{12} + f_{23} \ , \quad N_+ = f_{21} + f_{14} \ ,$$
$$iF_- = f_{11} - f_{24} \ , \quad iG_- = f_{22} - f_{13} \ , \quad iK_- = f_{12} - f_{23} \ , \quad iN_- = f_{21} - f_{14} \ , \qquad (1.45)$$

we arrive at the system

$$\epsilon F_+ - \frac{d}{dr}F_- - \frac{a}{r}N_- - mF_+ = 0 \ , \quad -\epsilon F_- - \frac{d}{dr}F_+ - \frac{a}{r}N_+ - mF_- = 0 \ ;$$
$$\epsilon G_+ + \frac{d}{dr}G_- + \frac{a}{r}K_- - mG_+ = 0 \ , \quad -\epsilon G_- + \frac{d}{dr}G_+ + \frac{a}{r}K_+ - mG_- = 0 \ ;$$

$$\epsilon K_+ - \frac{d}{dr}K_- - \frac{1}{r}N_- - \frac{a}{r}G_- - mK_+ = 0 \ , -\epsilon K_- - \frac{d}{dr}K_+ - \frac{1}{r}N_+ - \frac{a}{r}G_+ - mK_- = 0 \ ;$$
$$\epsilon f_{14} + i\frac{d}{dr}f_{14} + \frac{i}{r}f_{23} + \frac{ia}{r}f_{24} - mf_{21} = 0 \ , \ \epsilon f_{21} - i\frac{d}{dr}f_{21} - \frac{i}{r}f_{12} - \frac{ia}{r}f_{11} - mf_{14} = 0 \ ;$$
$$\epsilon N_+ + \frac{d}{dr}N_- + \frac{1}{r}K_- + \frac{a}{r}F_- - mN_+ = 0 \ , \ -\epsilon N_- + \frac{d}{dr}N_+ + \frac{1}{r}K_+ + \frac{a}{r}F_+ - mN_- = 0 \ .$$

Now, in equations we should perform the formal change $\epsilon \Longrightarrow m + E$, this leads to

$$EF_+ - \frac{d}{dr}F_- - \frac{a}{r}N_- = 0 \ , \quad -EF_- - \frac{d}{dr}F_+ - \frac{a}{r}N_+ - 2mF_- = 0 \ ;$$
$$EG_+ + \frac{d}{dr}G_- + \frac{a}{r}K_- = 0 \ , \quad -EG_- + \frac{d}{dr}G_+ + \frac{a}{r}K_+ - 2mG_- = 0 \ ;$$
$$EK_+ - \frac{d}{dr}K_- - \frac{1}{r}N_- - \frac{a}{r}G_- = 0 \ , \quad -EK_- - \frac{d}{dr}K_+ - \frac{1}{r}N_+ - \frac{a}{r}G_+ - 2mK_- = 0 \ ;$$
$$EN_+ + \frac{d}{dr}N_- + \frac{1}{r}K_- + \frac{a}{r}F_- = 0 \ , \quad -EN_- + \frac{d}{dr}N_+ + \frac{1}{r}K_+ + \frac{a}{r}F_+ - 2mN_- = 0 \ .$$

Their approximate (nonrelativistic) form reads

$$EF_+ = \frac{d}{dr}F_- + \frac{a}{r}N_- \ , \qquad F_- = -\frac{1}{2m}\left(\frac{d}{dr}F_+ + \frac{a}{r}N_+\right) ;$$
$$EG_+ = -\frac{d}{dr}G_- - \frac{a}{r}K_- \ , \qquad G_- = \frac{1}{2m}\left(\frac{d}{dr}G_+ + \frac{a}{r}K_+\right) ;$$
$$EK_+ = \frac{d}{dr}K_- + \frac{1}{r}N_- + \frac{a}{r}G_- \ , \qquad K_- = -\frac{1}{2m}\left(\frac{d}{dr}K_+ + \frac{1}{r}N_+ + \frac{a}{r}G_+\right) ;$$
$$EN_+ = -\frac{d}{dr}N_- - \frac{1}{r}K_- - \frac{a}{r}F_- \ , \qquad N_- = \frac{1}{2m}\left(\frac{d}{dr}N_+ + \frac{1}{r}K_+ + \frac{a}{r}F_+\right) . \qquad (1.46)$$

From whence we derive four equations that contain only big constituents:

$$\left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) F_+ = \frac{a}{r^2}(N_+ + K_+) \; ,$$
$$\left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) G_+ = \frac{a}{r^2}(N_+ + K_+) \; ,$$
$$\left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) K_+ = \frac{1}{r^2}(aF_+ + aG_+ + N_+ + K_+) \; ,$$
$$\left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) N_+ = \frac{1}{r^2}(aF_+ + aG_+ + N_+ + K_+) \; . \qquad (1.47)$$

Let us introduce notation

$$F_+ + G_+ = f_+ \; , \; F_+ - G_+ = f_- \; , \quad K_+ + N_+ = g_+ \; , \; K_+ - N_+ = g_- \; ; \qquad (1.48)$$

then the last system reads

$$\left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) f_- = 0 \; , \quad \left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) g_- = 0 \; ,$$
$$\frac{1}{2}r^2 \left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) f_+ = ag_+ \; , \quad \frac{1}{2}r^2 \left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) g_+ = af_+ + g_+ \; . \qquad (1.49)$$

In fact, sub-system of two last equations in (1.49) should be considered separately. It can be presented in a matrix form

$$\frac{1}{2}r^2 \left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{r^2}\right) = \hat{\Delta} \; , \quad \hat{\Delta} \left| \begin{array}{c} f_+ \\ g_+ \end{array} \right| = \left| \begin{array}{cc} 0 & a \\ a & 1 \end{array} \right| \left| \begin{array}{c} f_+ \\ g_+ \end{array} \right| , \qquad \hat{\Delta} f = Af \; . \qquad (1.50)$$

Let us construct a transformation reducing the matrix A to a diagonal form, $f' = Sf$, $\hat{\Delta} f' = SAS^{-1} f'$; which gives the task

$$\left| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right| \left| \begin{array}{cc} 0 & a \\ a & 1 \end{array} \right| = \left| \begin{array}{cc} A_1 & a \\ 0 & A_2 \end{array} \right| \left| \begin{array}{cc} s_{11} & s_{12} \\ s_{21} & s_{22} \end{array} \right| .$$

It is equivalent to two homogeneous linear systems

$$\left| \begin{array}{cc} -A_1 & a \\ a & (1-A_1) \end{array} \right| \left| \begin{array}{c} s_{11} \\ s_{12} \end{array} \right| = 0, \qquad \left| \begin{array}{cc} -A_2 & a \\ a & (1-A_2) \end{array} \right| \left| \begin{array}{c} s_{21} \\ s_{22} \end{array} \right| = 0 \; .$$

Non-trivial solutions exist if A_1, A_2 equal to

$$A_1 = j+1, (-j) \; , \qquad A_2 = -j, (j+1) \; .$$

In fact, it is enough to consider only one variant, $A_1 = j + 1$, $A_2 = -j$:

$$S = \left| \begin{array}{cc} \sqrt{j} & \sqrt{j+1} \\ -\sqrt{j+1} & \sqrt{j} \end{array} \right| , \qquad S^{-1} = \frac{1}{2j+1} \left| \begin{array}{cc} \sqrt{j} & -\sqrt{j+1} \\ \sqrt{j+1} & \sqrt{j} \end{array} \right| ,$$

$$f_+ = \frac{1}{2j+1}(\sqrt{j}\; f'_+ - \sqrt{j+1}\; g'_+) \; , \quad g_+ = \frac{1}{2j+1}(\sqrt{j+1}\; f'_+ + \sqrt{j}\; g'_+) \; . \tag{1.51}$$

Thus, after linear transformation, the system of four radial equations reads as straightforwardly solvable four disconnected equations

$$\left(\frac{d^2}{dr^2} + 2mE - \frac{j(j+1)^2}{r^2}\right) f_- = 0 \; , \quad \left(\frac{d^2}{dr^2} + 2mE - \frac{j(j+1)}{r^2}\right) g_- = 0 \; ,$$
$$\left(\frac{d^2}{dr^2} + 2mE - \frac{(j+1)(j+2)}{r^2}\right) f'_+ = 0 \; , \quad \left(\frac{d^2}{dr^2} + 2mE - \frac{(j-1)j}{r^2}\right) g'_+ = 0 \; . \tag{1.52}$$

Let us shortly discus the case of $j = 0$. Here we have

$$F_+ = 0 \; , \quad G_+ = 0 \; , \quad K_+ = f_{12} + f_{23} \; , \quad N_+ = f_{21} + f_{14} \; ;$$

and the final radial equations are

$$\left(\frac{d^2}{dr^2} + 2mE\right) g_- = 0 \; , \qquad \left(\frac{d^2}{dr^2} + 2mE\right) g_+ = \frac{2}{r^2} g_+ \; . \tag{1.53}$$

It is evident that one can take into account any additional potential with spherical symmetry by the formal change $E \implies E - U(r)$.

1.7 Dirac–Kähler particle in spherical Riemann space

Let us detail the case of the Riemann spherical space:

$$dS^2 = dt^2 - dr^2 - \sin^2 r (\, d\theta^2 + \sin^2\theta d\phi^2 \,) \; ,$$
$$e^{\alpha}_{(0)} = (1,\; 0,\; 0,\; 0) \; , \;\; e^{\alpha}_{(1)} = (0,\; 0,\; \sin^{-1} r,\; 0) \; ,$$
$$e^{\alpha}_{(2)} = (0,\; 0,\; 0,\; \sin^{-1} r \sin^{-1}\theta) \; , \;\; e^{\alpha}_{(3)} = (0,1,0,0) \; , \tag{1.54}$$

the Dirac–Kähler equation looks

$$\left[i\gamma^0 \,\partial_t \; + \; i\left(\gamma^3 \,\partial_r \; + \; \frac{\gamma^1 J^{31} + \gamma^2 J^{32}}{\tan r}\right) \; + \; \frac{1}{\sin r}\; \Sigma_{\theta,\phi} - m \right] \; U(x) = 0 \; . \tag{1.55}$$

The most of above calculation are much the same as above. Instead of (1.32) here we will obtain $(\delta = +1, \; \lambda = +1)$

$$\frac{dK}{dr} + \frac{a}{\sin r} M + (\epsilon + m) L = 0 \; , \quad \frac{dL}{dr} - \frac{a}{\sin r} N - (\epsilon - m) K = 0 \; ,$$

$$(\frac{d}{dr} + \frac{1}{\tan r}) M + \frac{a}{\sin r} K + (\epsilon + m) N = 0 \; , \quad (\frac{d}{dr} - \frac{1}{\tan r}) N - \frac{a}{\sin r} L - (\epsilon - m) M = 0 \; .$$

This system cannot be solved with the help of substitutions (1.35) or (1.36) – they turn to be inconsistent these equations. This correlate with one simple fact: in any curved space-time, the Dirac–Kähler equation cannon be reduced to four disconnected Dirac equations.

Let shortly consider the nonrelativistic approximation for this system in spherical space model. After performing needed calculation, we derive the following non-relativistic radial system

$$\hat{\Delta} f_- = 0 \; , \qquad (\hat{\Delta} + 1) g_- = 0 \; , \tag{1.56}$$

$$\hat{\Delta} f_+ = 2a \frac{\cos r}{\sin^2 r} g_+ \; , \qquad \hat{\Delta} g_+ = \frac{1 + \cos^2 r}{\sin^2 r} g_+ + 2a \frac{\cos r}{\sin^2 r} f_+ \tag{1.57}$$

where

$$\hat{\Delta} = \left(\frac{d^2}{dr^2} + 2mE - \frac{a^2}{\sin^2 r} \right), \quad a = \sqrt{j(j+1)} \; .$$

In the case of $j = 0$, we have more simple and exactly solvable equations

$$(\frac{d^2}{dr^2} + 2mE) g_+ = \frac{1 + \cos^2 r}{\sin^2 r} g_+ \; , \quad (\frac{d^2}{dr^2} + 2mE + 1) g_- = 0 \; . \tag{1.58}$$

Equations (1.56) and (1.58) are readily solvable – see [3]. The system of two second order interrelated equations (1.57) is equivalent to a 4-th order differential equation, solutions of this problem will be constructed in next section (we use the method applied in [4]).

1.8 Conclusion

The general covariant Pauli-like equation for Dirac–Kähler particle in curved space time and in present of external electromagnetic field has been derived In the flat Minkowski space it becomes more simple and consists of two disconnected sub-systems for scalar-pseudovector and pseudoscalar-vector respectively. In presence of only electric field, the Pauli equation becomes more simple and reduce to four disconnected wave equations for scalar, pseudoscalar, vector, and pseudovector. These features are substantial for physical interpretation of the Dirac–Kähler particle: it interacts in very different manner with magnetic and electric fields. In the case of Minkowski space, exact solutions of relativistic and nonrelativisric Dirac–Kähler equations can be constructed – the problem reduces to differential equations of 2-nd order with the structure arising in the theory of ordinary Dirac or Pauli particle. The case of any curved space model is much more complicated. In simple spherical Riemann space, the problem reduces in general to a 4-th order differential equations, in fact in relativistic case these equations are solved in recent paper [4]. Obtained result are similar. The derived solutions for the Dirac–Kähler particle reveal significant differences of energy spectra as compared with the result for the ordinary Dirac or Pauli particle in spherical space.

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